Three Paths
in complete geometric graphs

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(joint work with Philipp Kindermann, Giuseppe Liotta, and Pavel Valtr)

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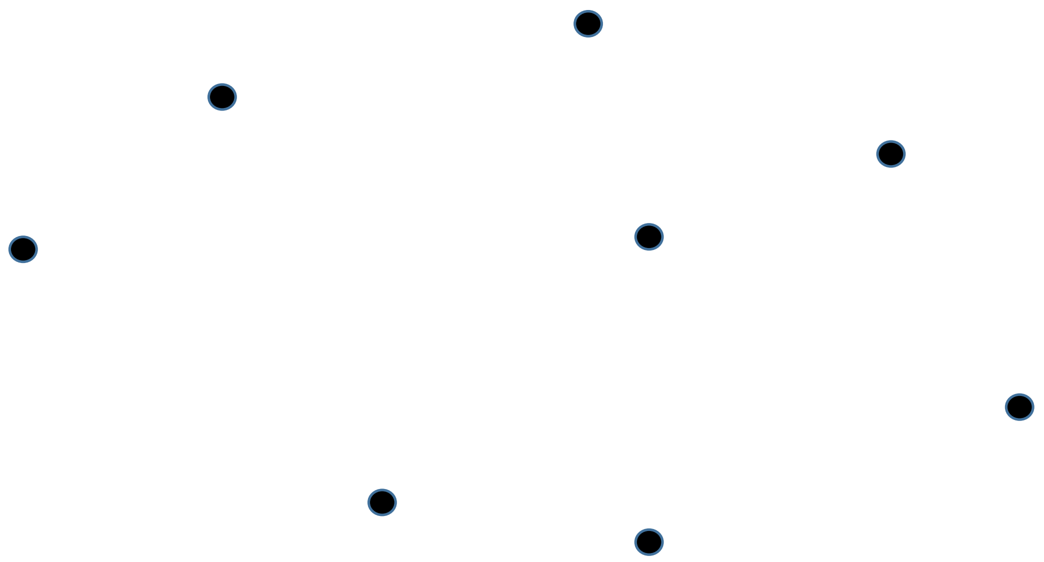
Setting: n points in the plane, no three in a line (general position)



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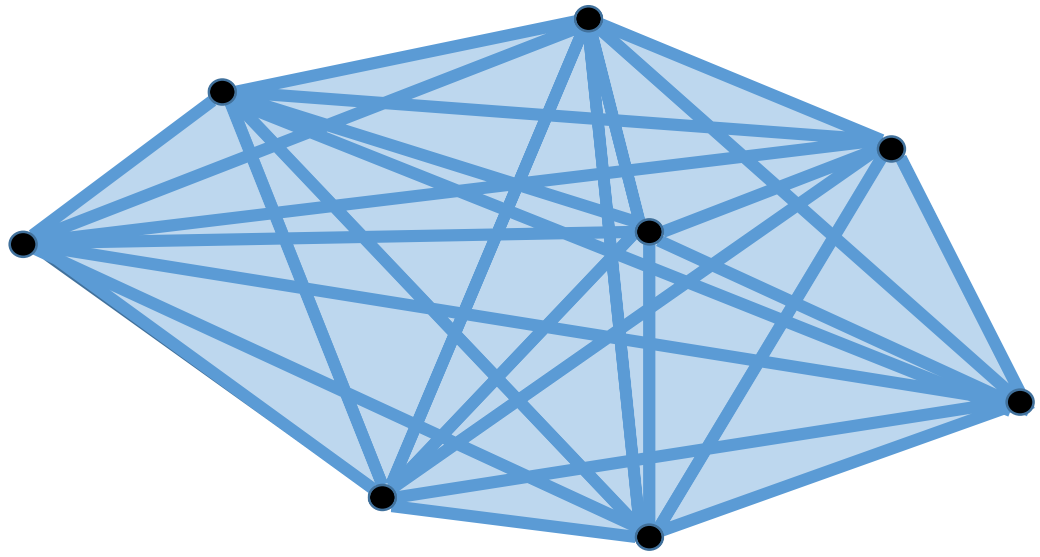
define a complete geometric graph (edges are straight-line segments)



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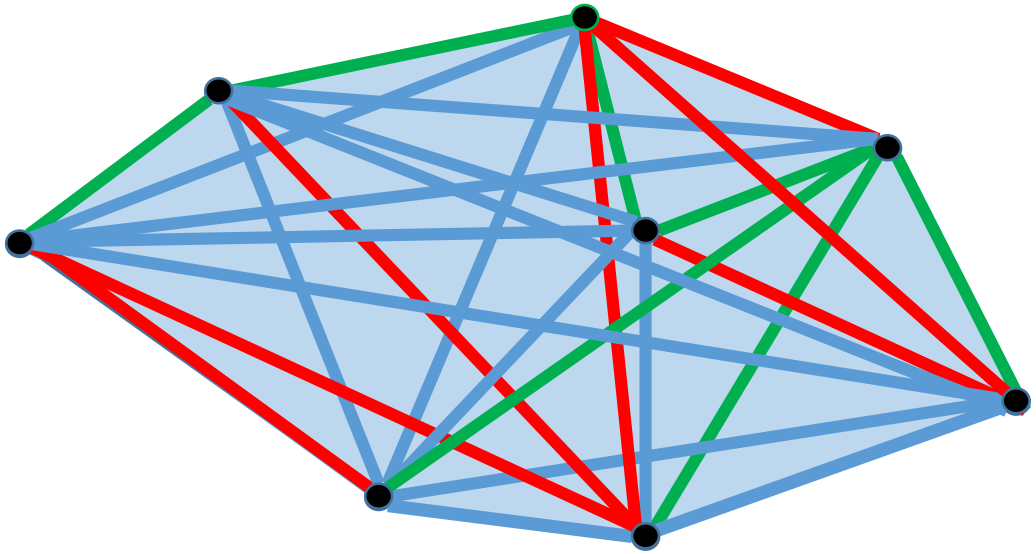
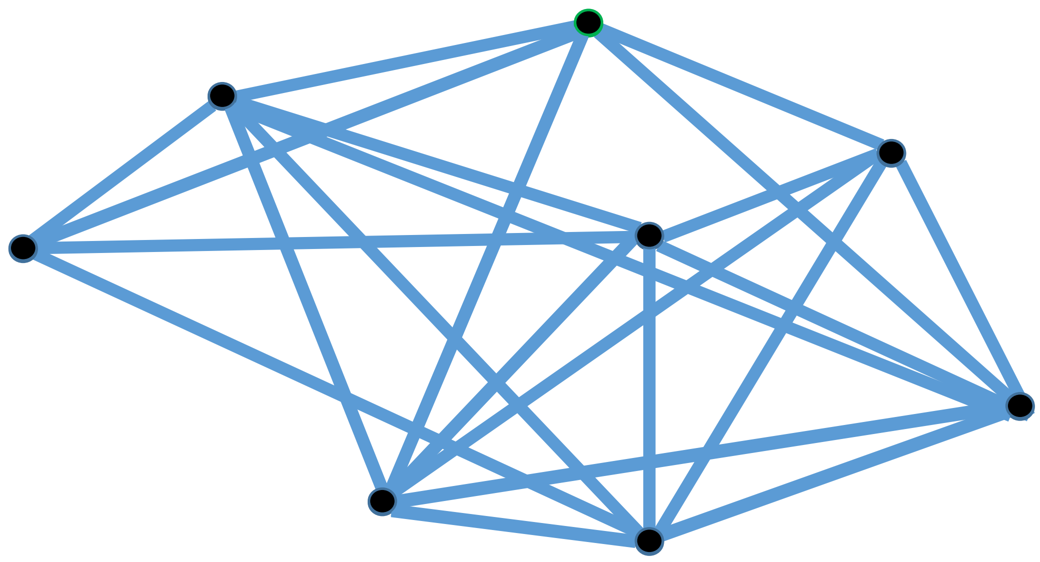


Objective: Packing (edge-disjoint) plane (non-crossing) spanning (Hamiltonian) subgraphs

Optimization question: How many edge-disjoint non-crossing Hamiltonian paths can wee pack in a given geometric graph?



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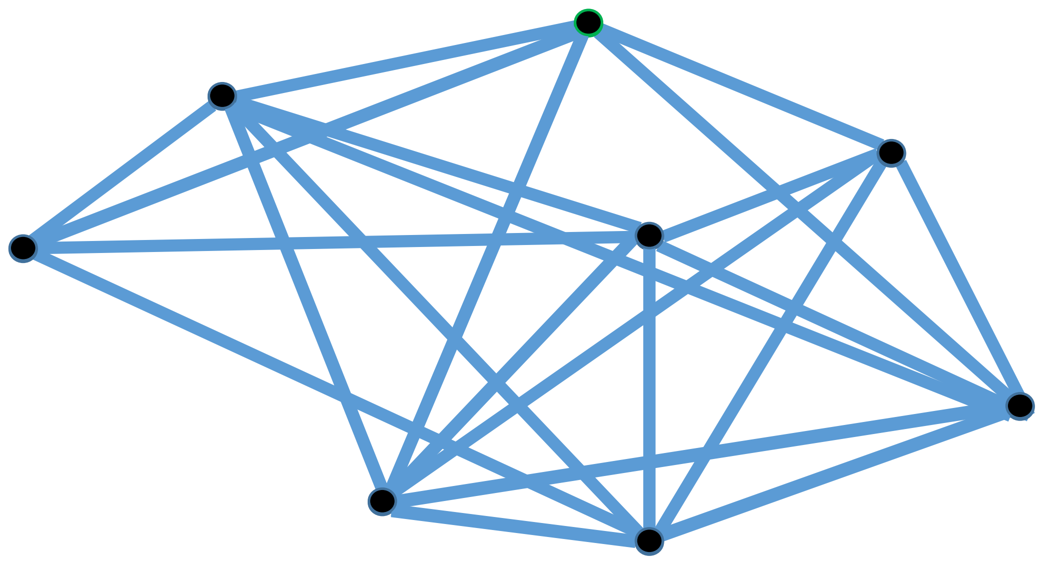
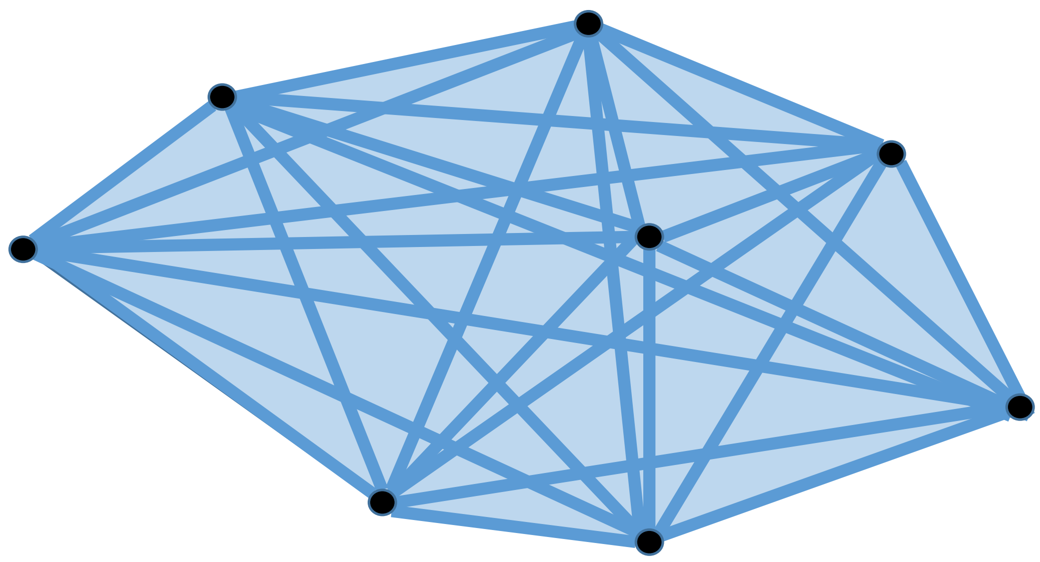


NP-complete (Hamilton path is NP-complete in planar graphs)

Another question: How many edge-disjoint non-crossing Hamiltonian paths can wee pack in a given complete geometric graph?



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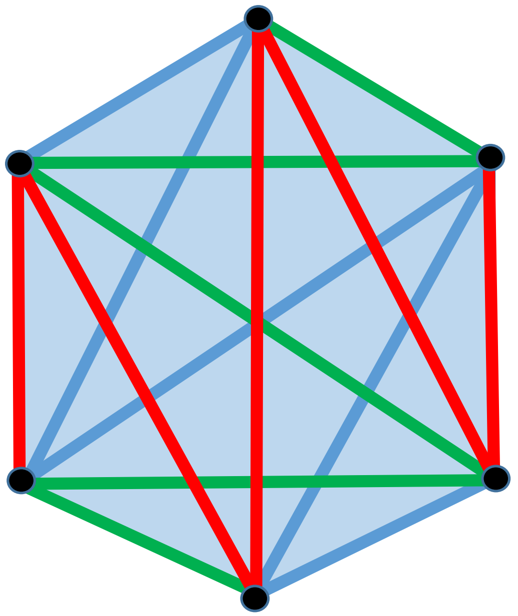

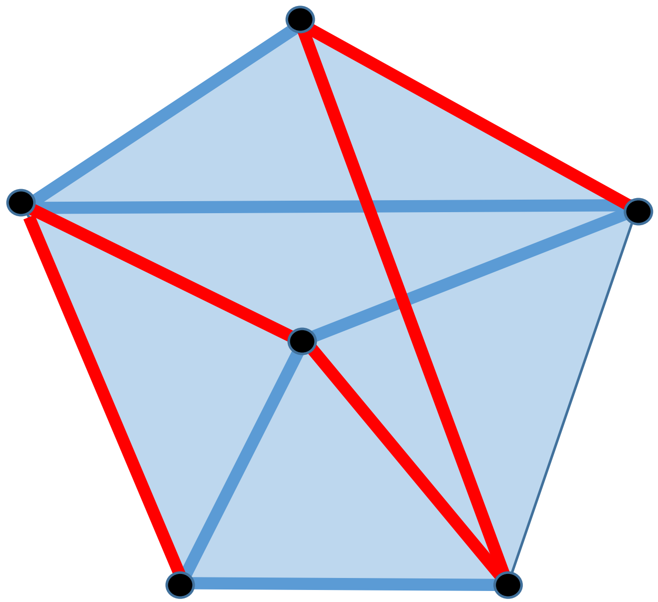

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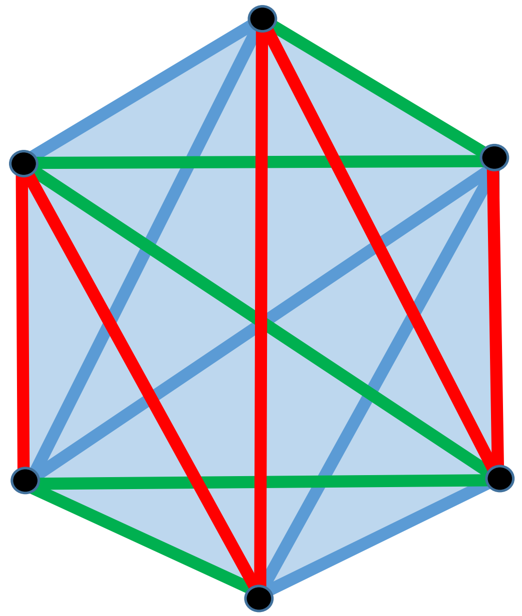

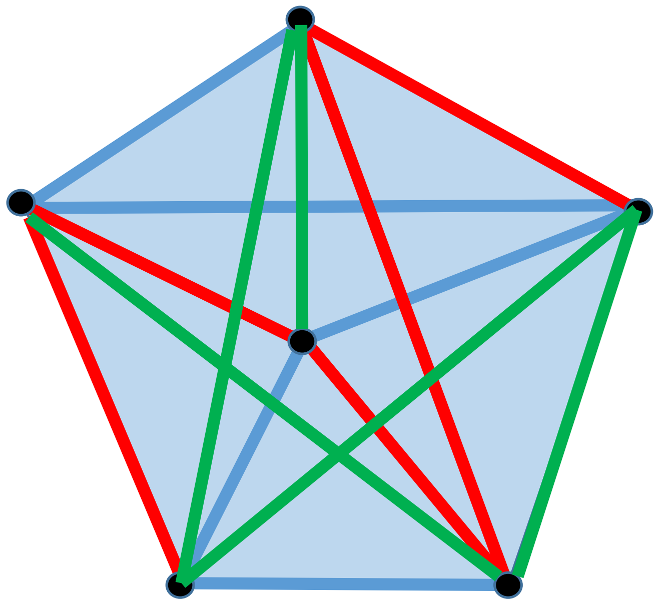

The question: How many edge-disjoint non-crossing Hamiltonian paths can we ALWAYS find in ANY given complete geometric graph with n points?

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Advertisement: We improve the best known lower bound by 50%.

Packing (edge-disjoint) plane (non-crossing) spanning (Hamiltonian) subgraphs in geometric graphs

Known:

Folklore – 1 path

Abellanas et al. [1999] – zig-zag path

Aichholzer et al. [2017] – $\sqrt{n}$ trees (types not prescribed)

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Our results:

- 2 paths with prescribed starting vertices (on the boundary of conv(S))
- 3 paths

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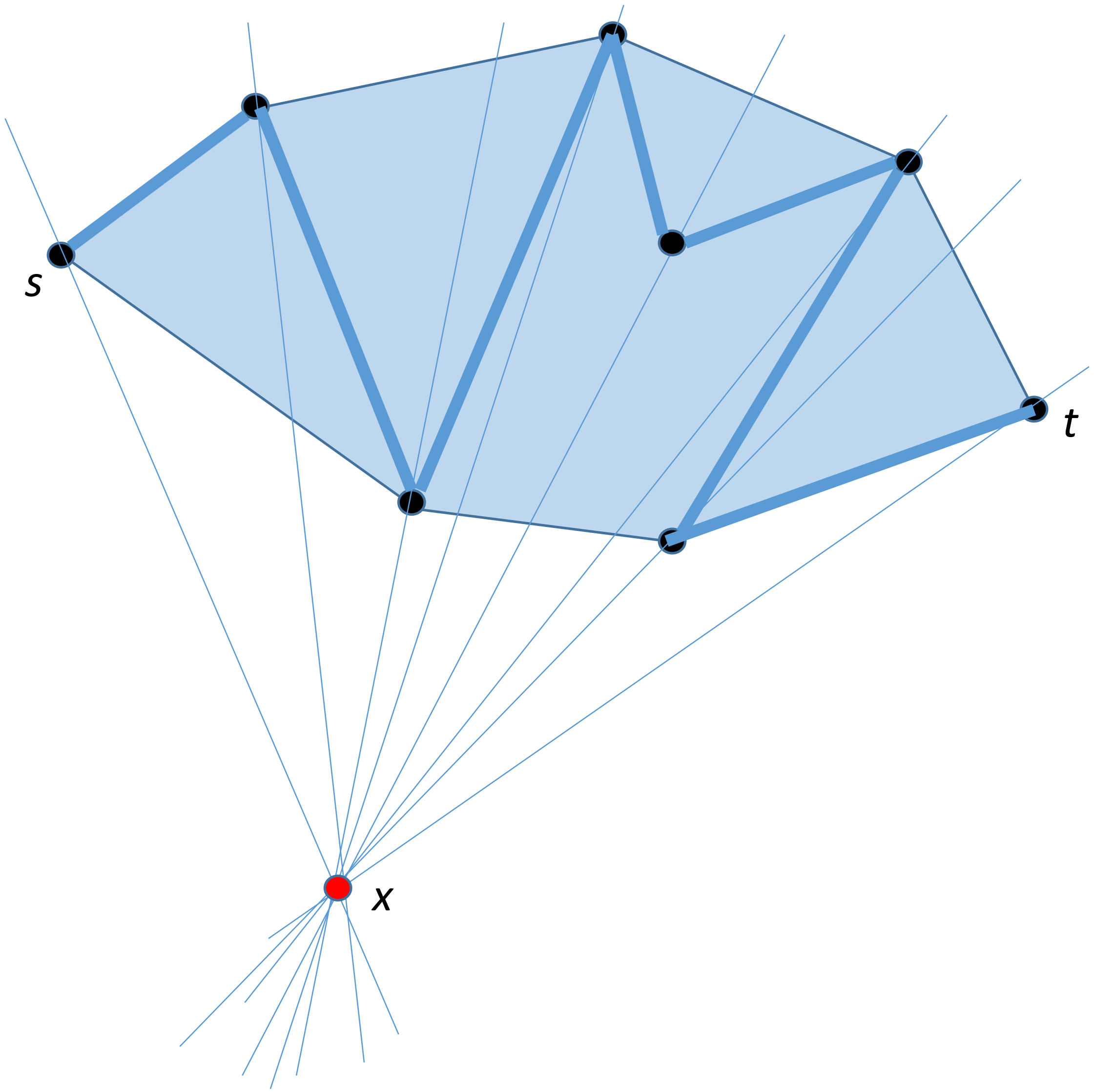
Proof: Case A, *s* and *t* on the boundary of conv(*S*)



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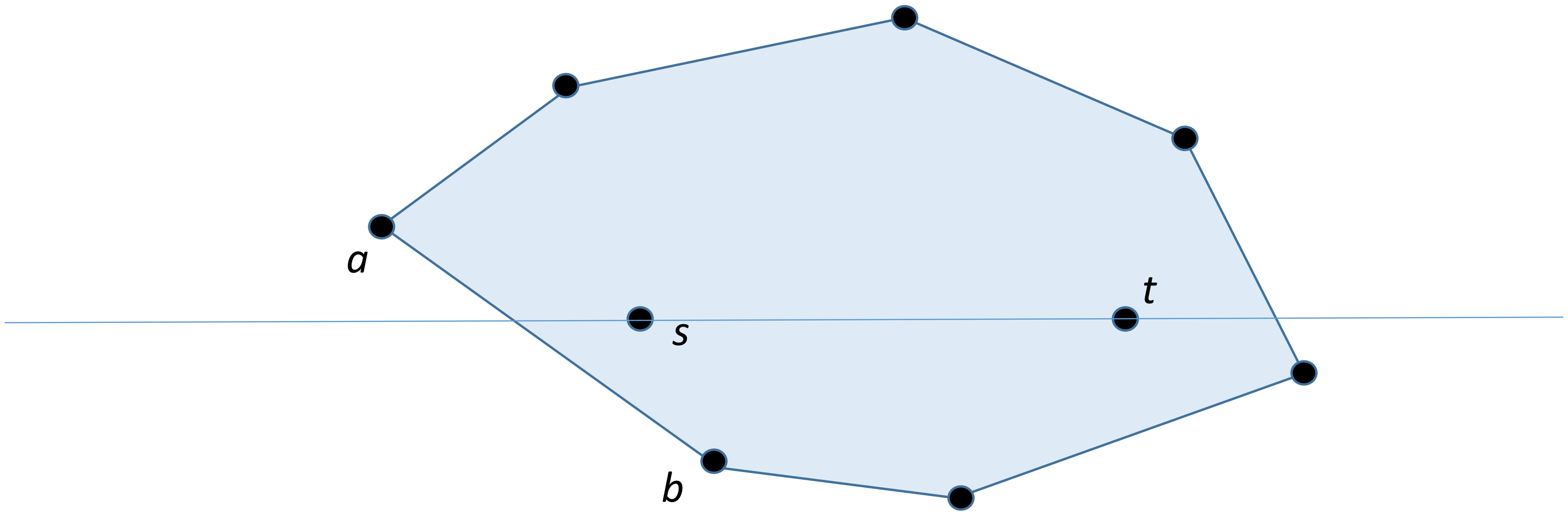

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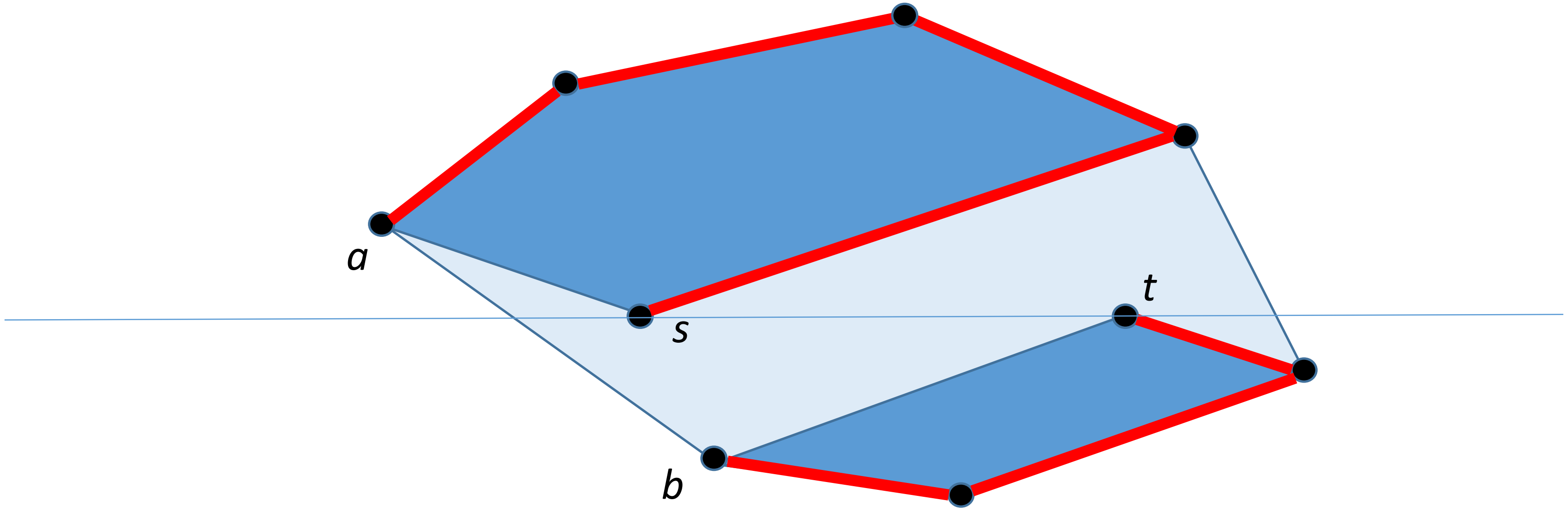

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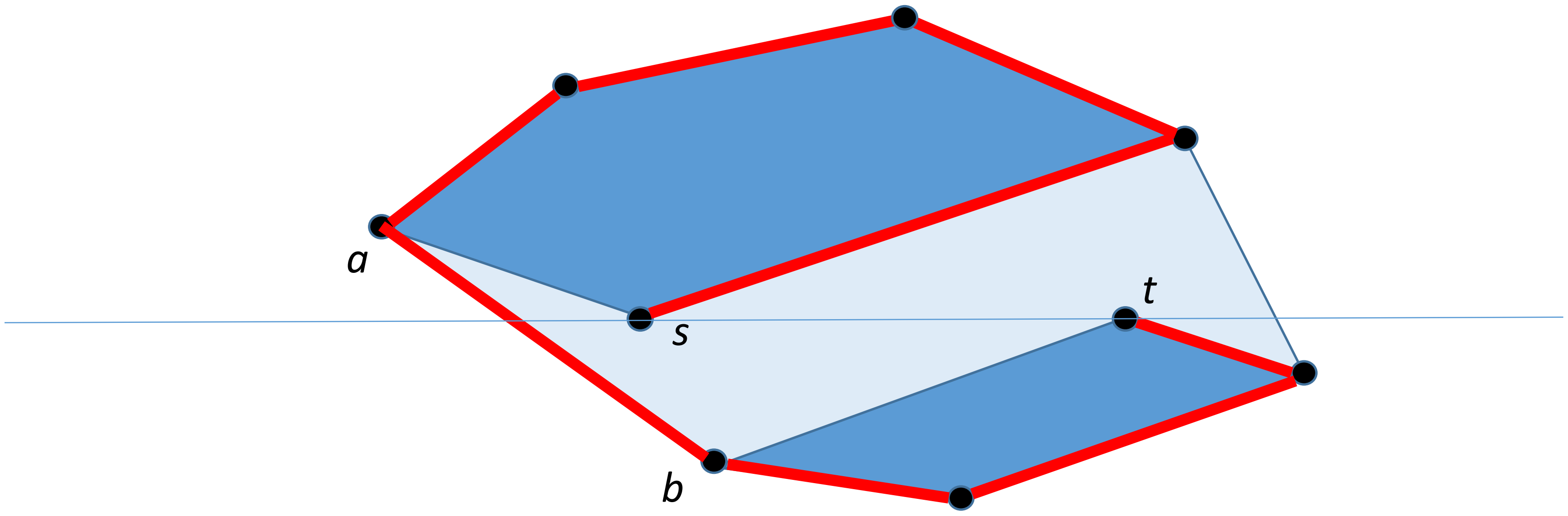

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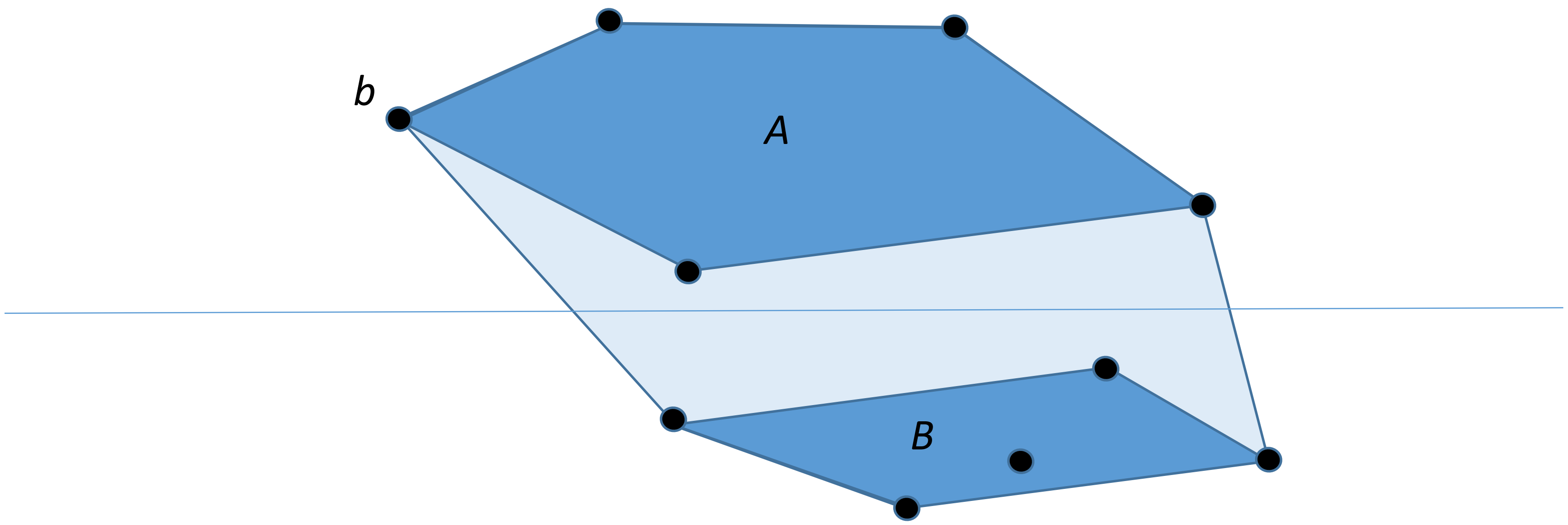

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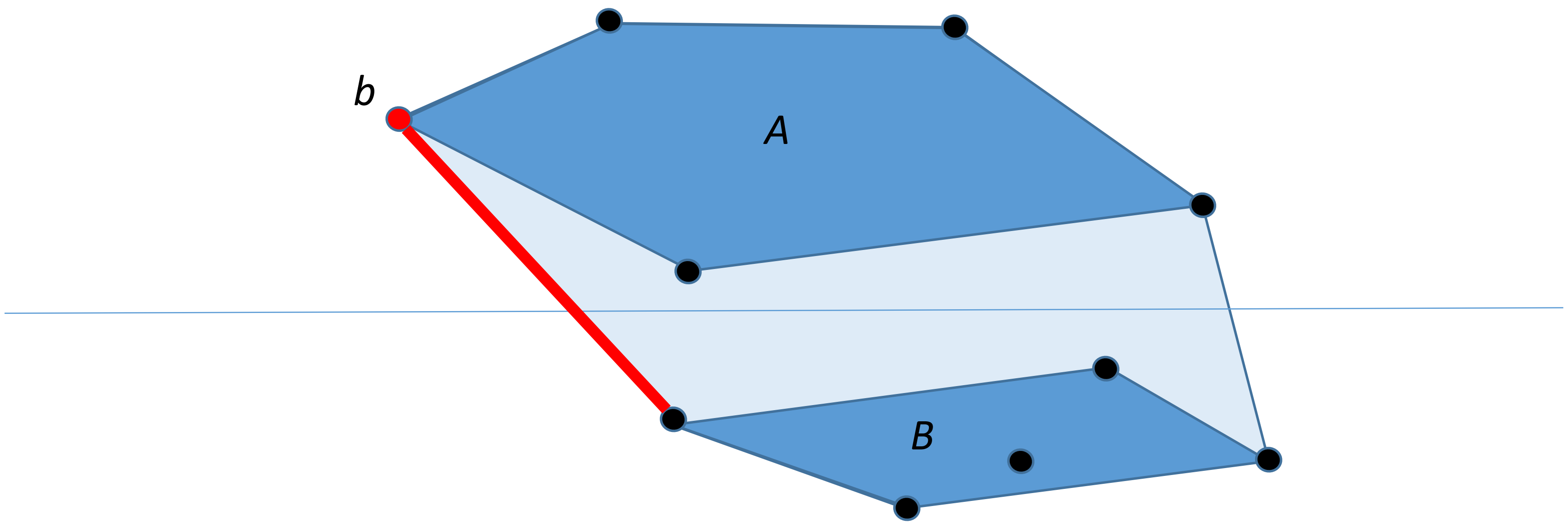

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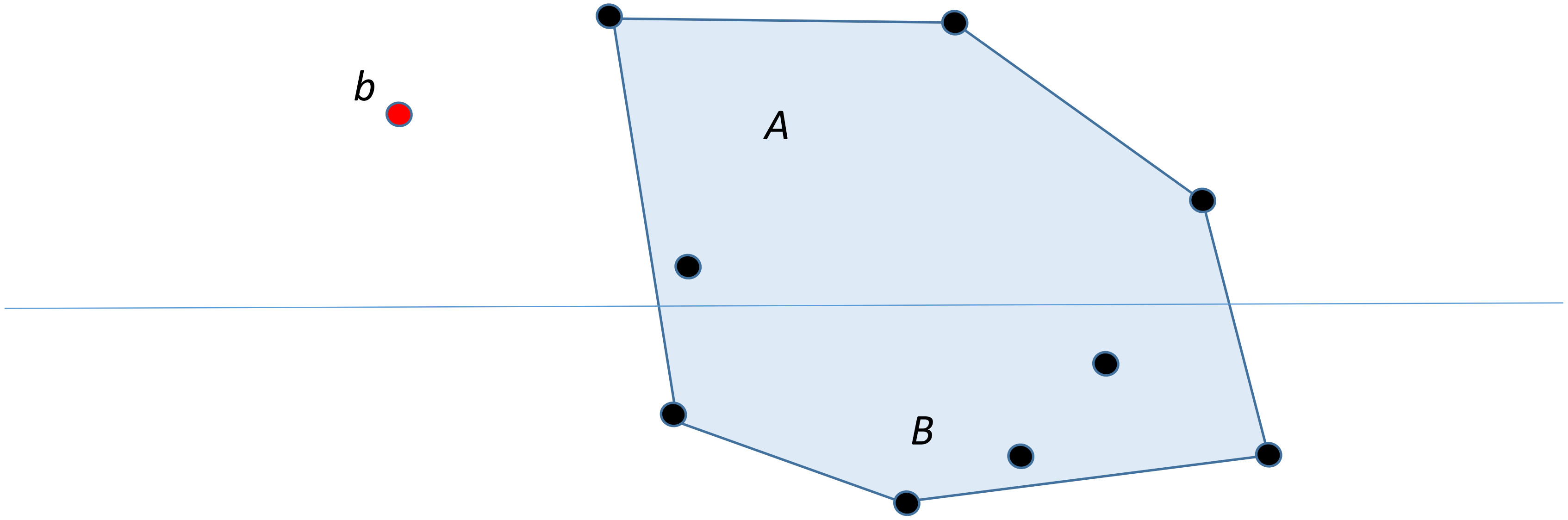

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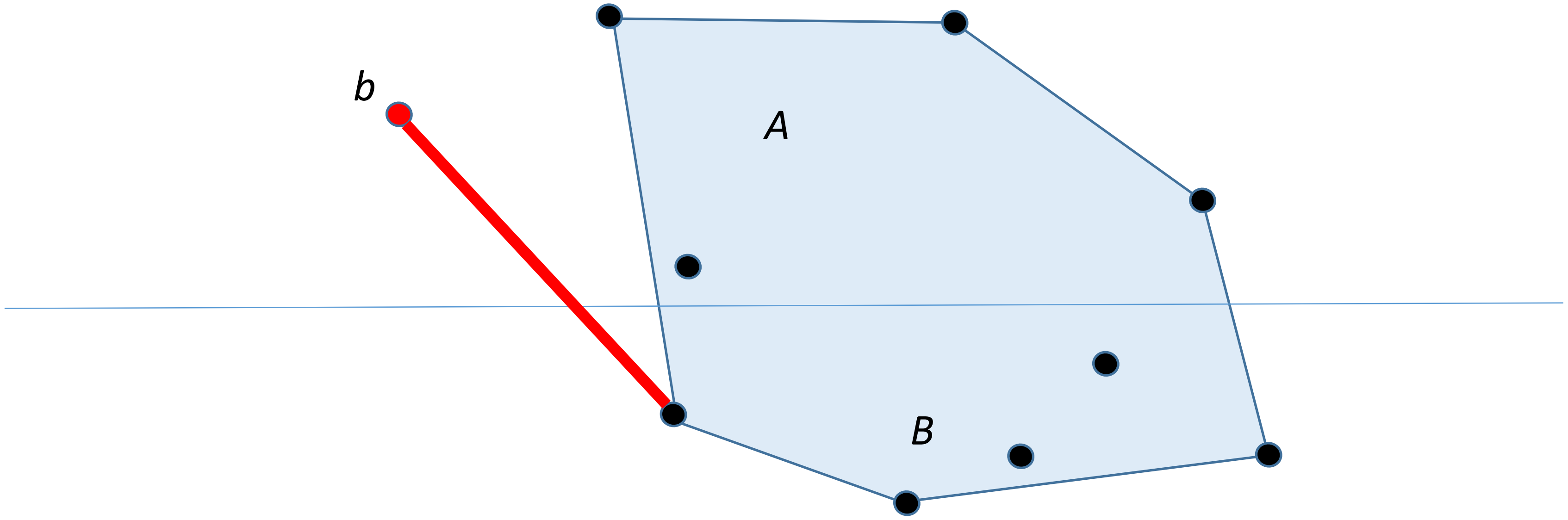

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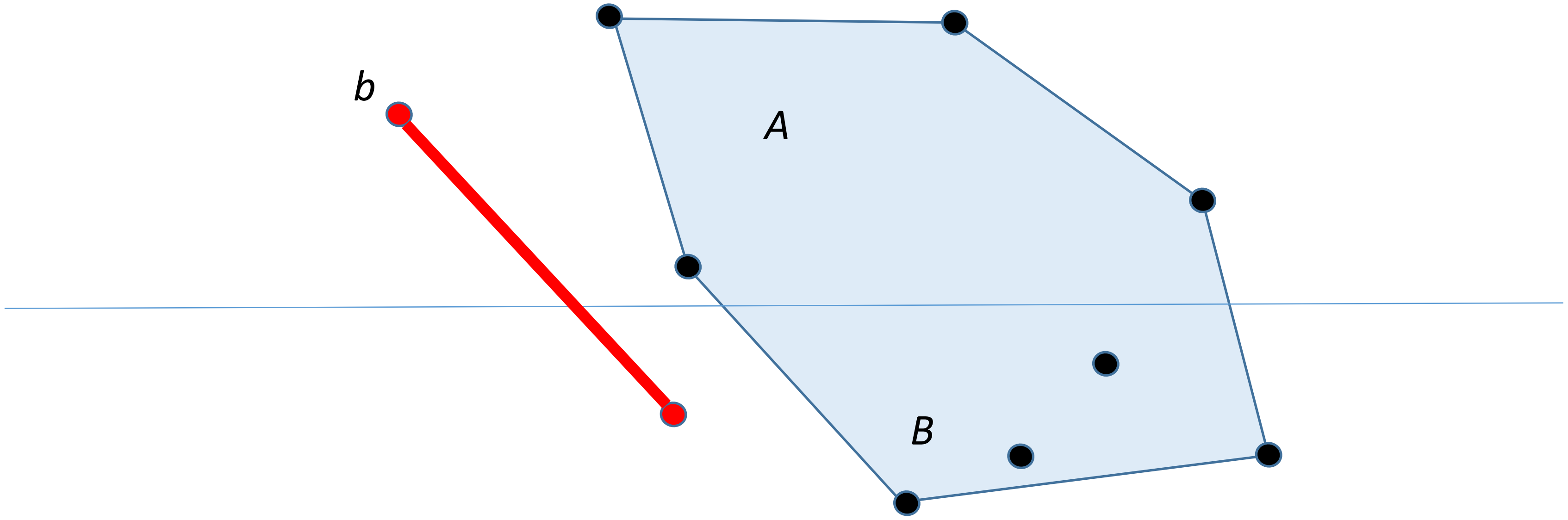

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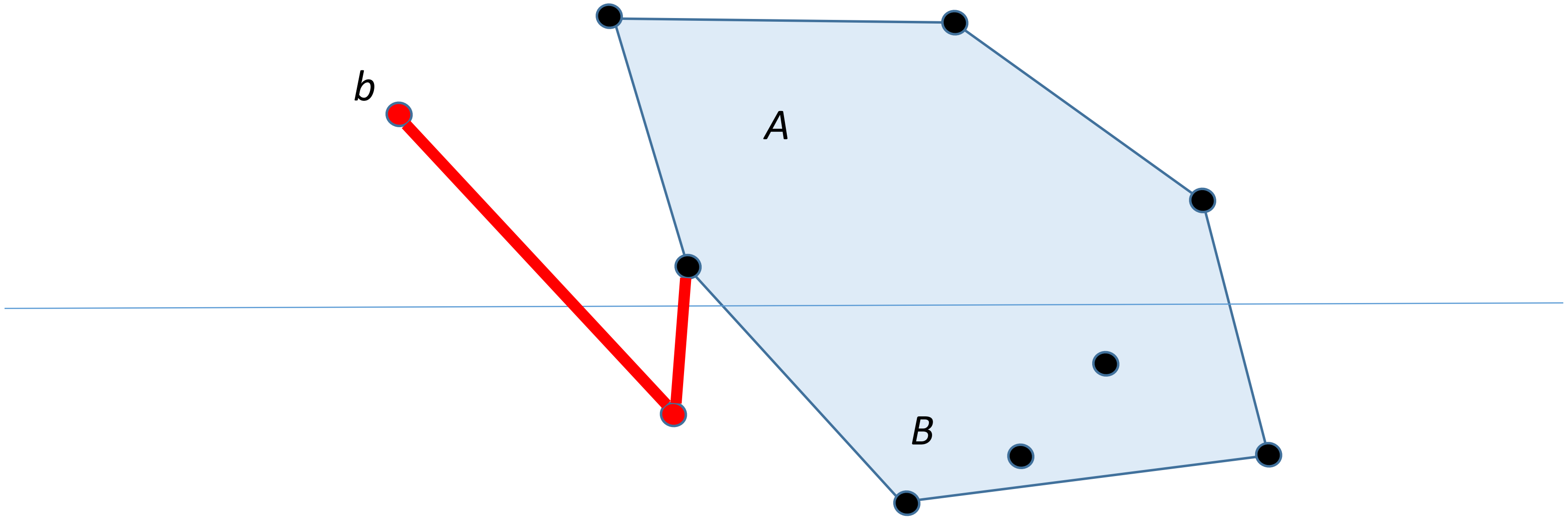

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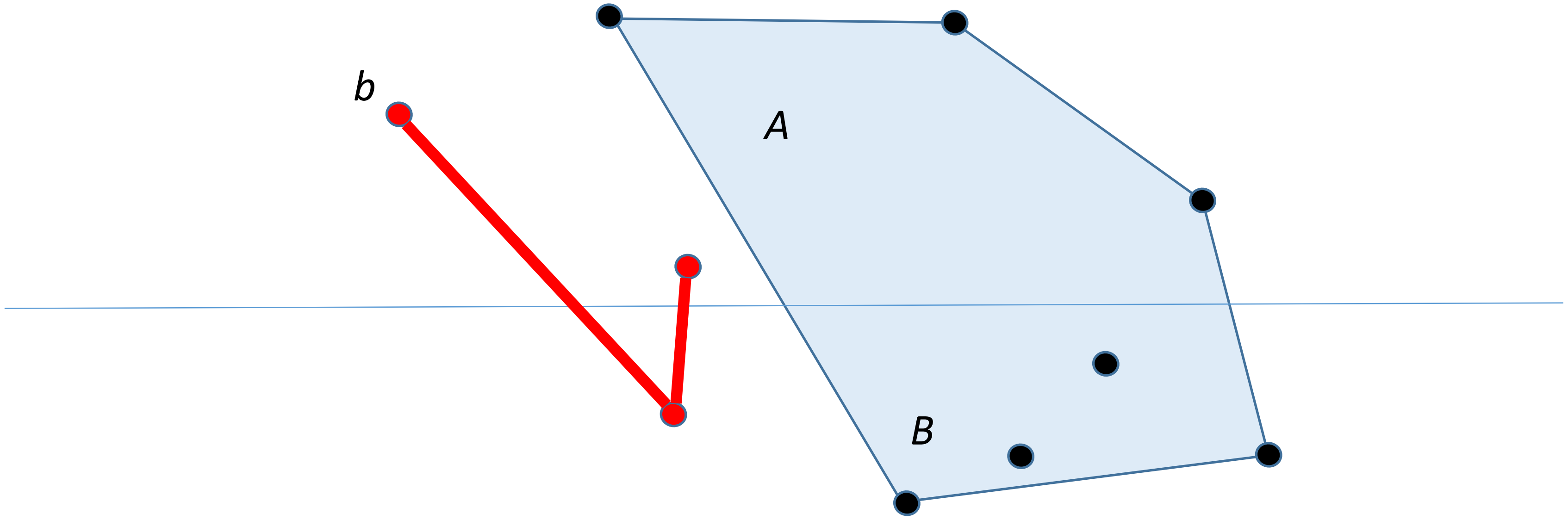

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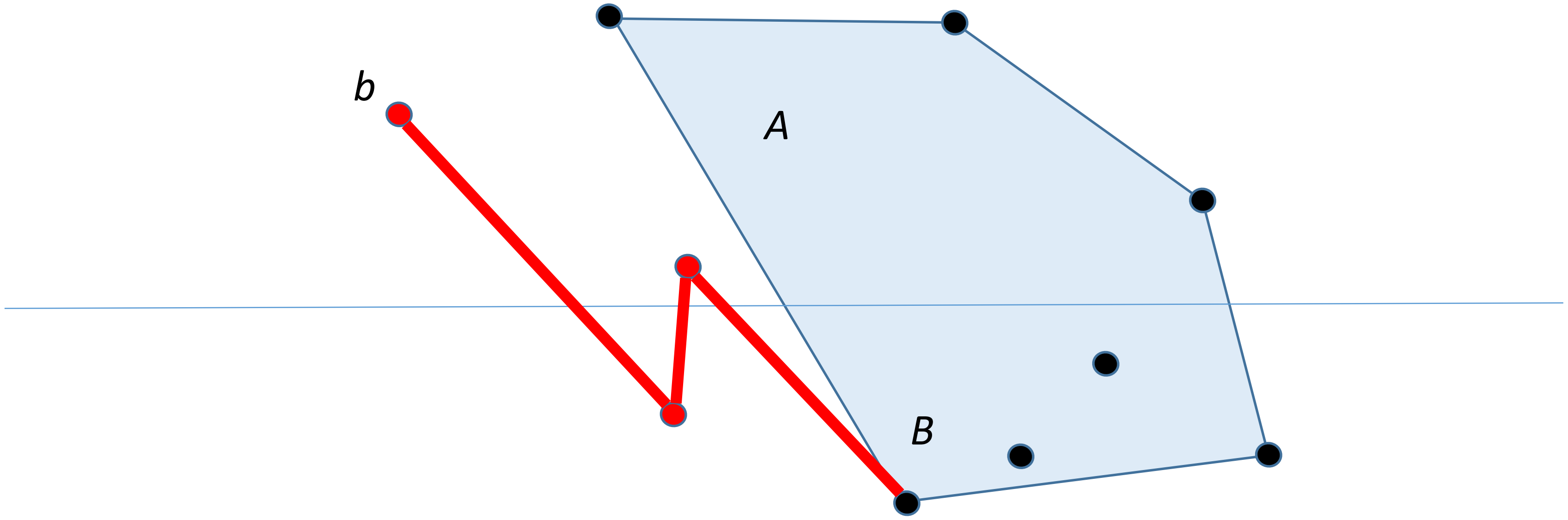

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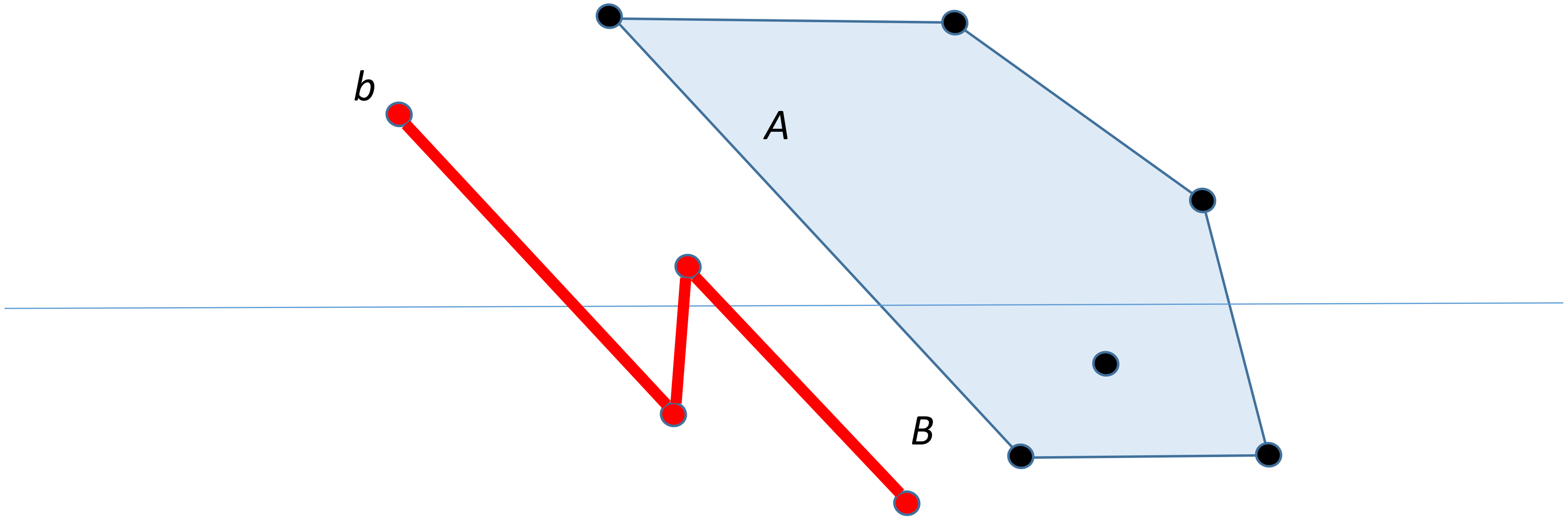

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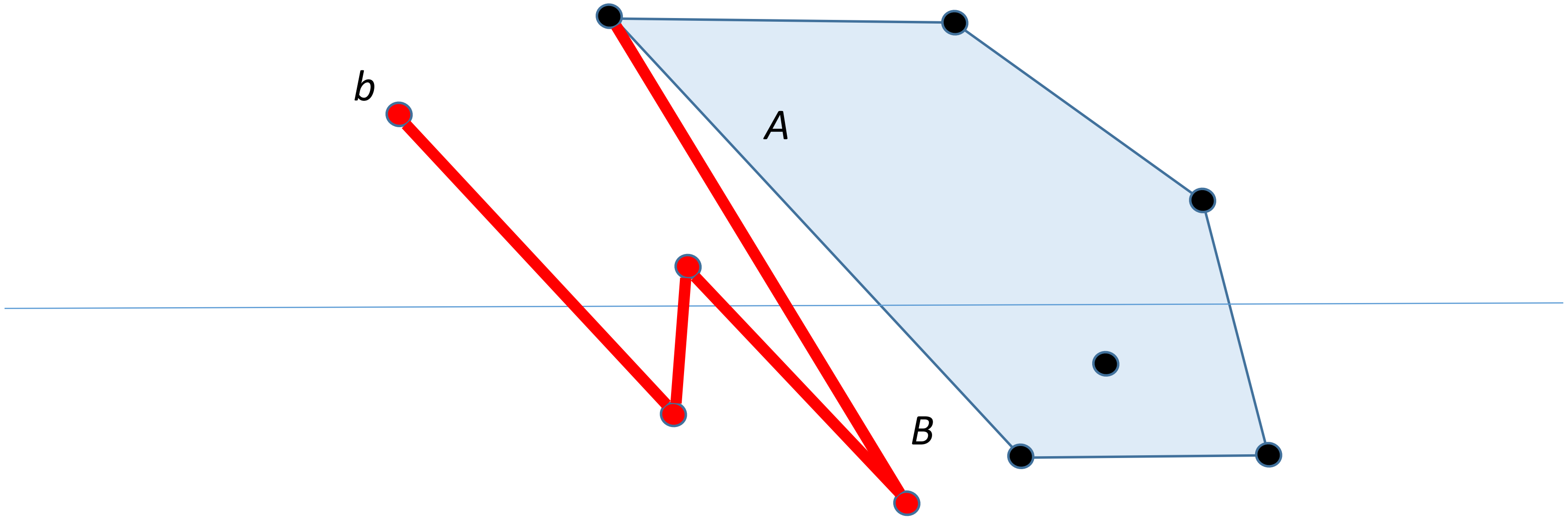

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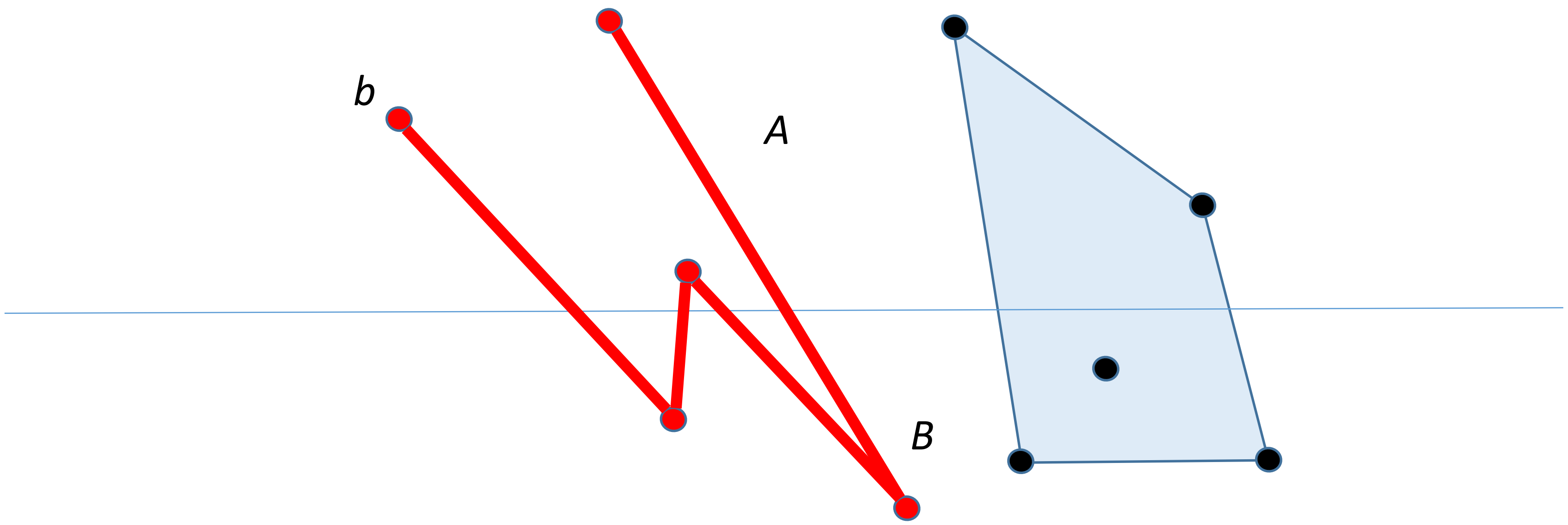

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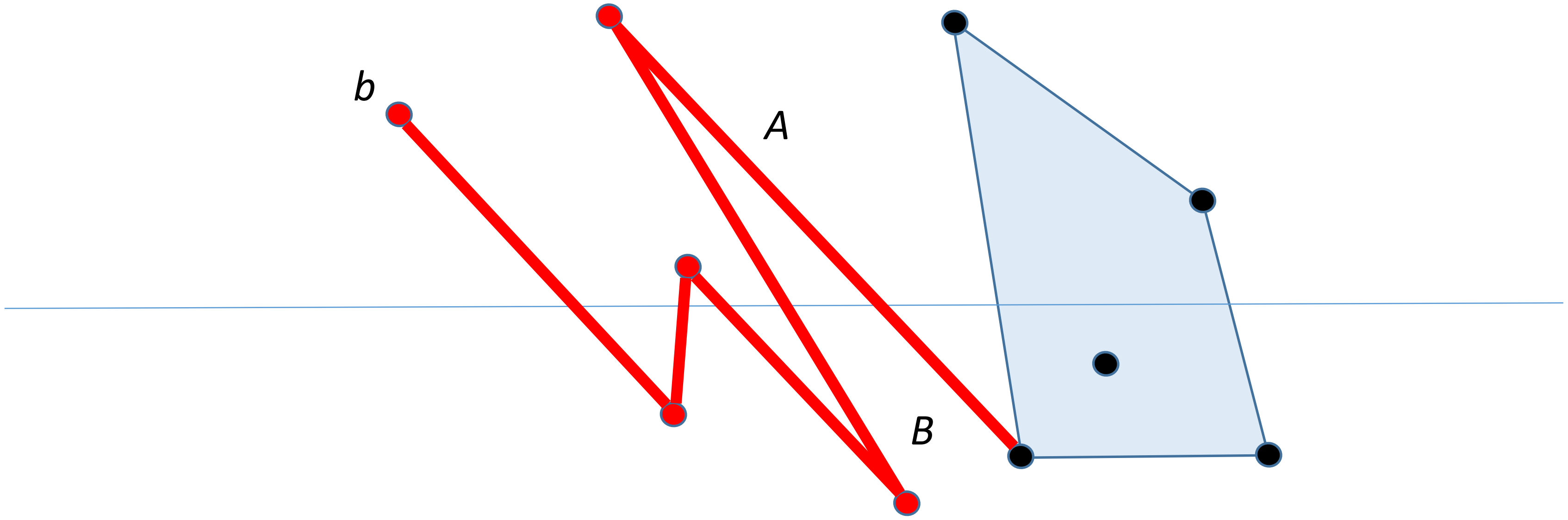

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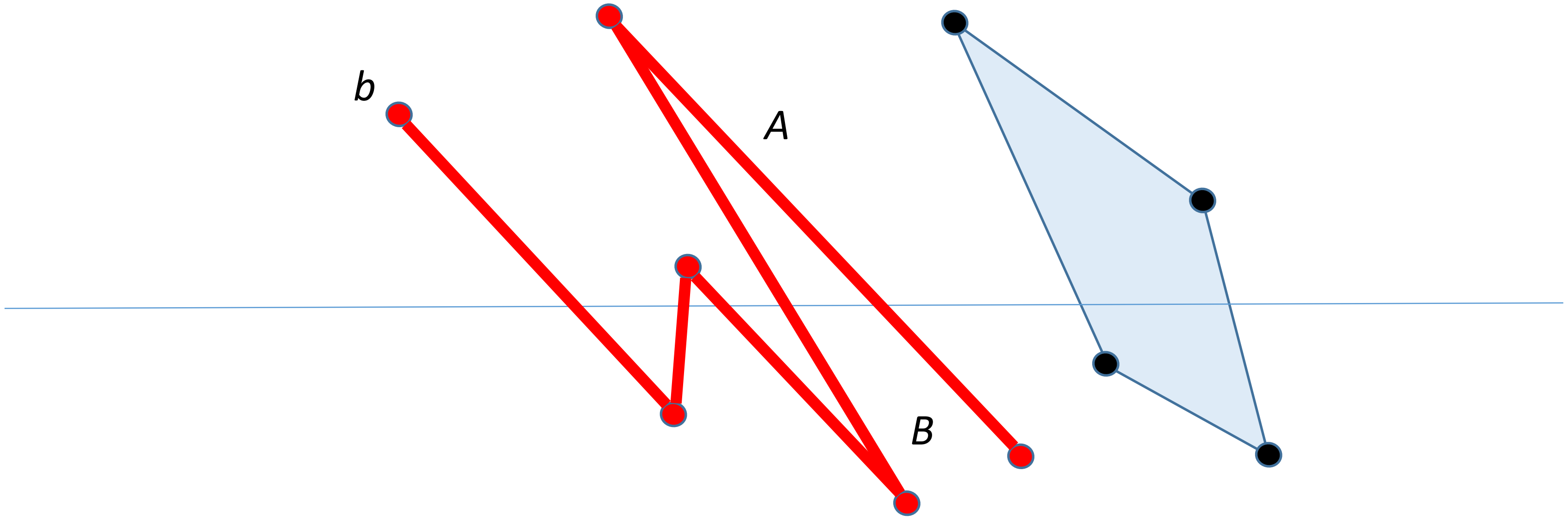

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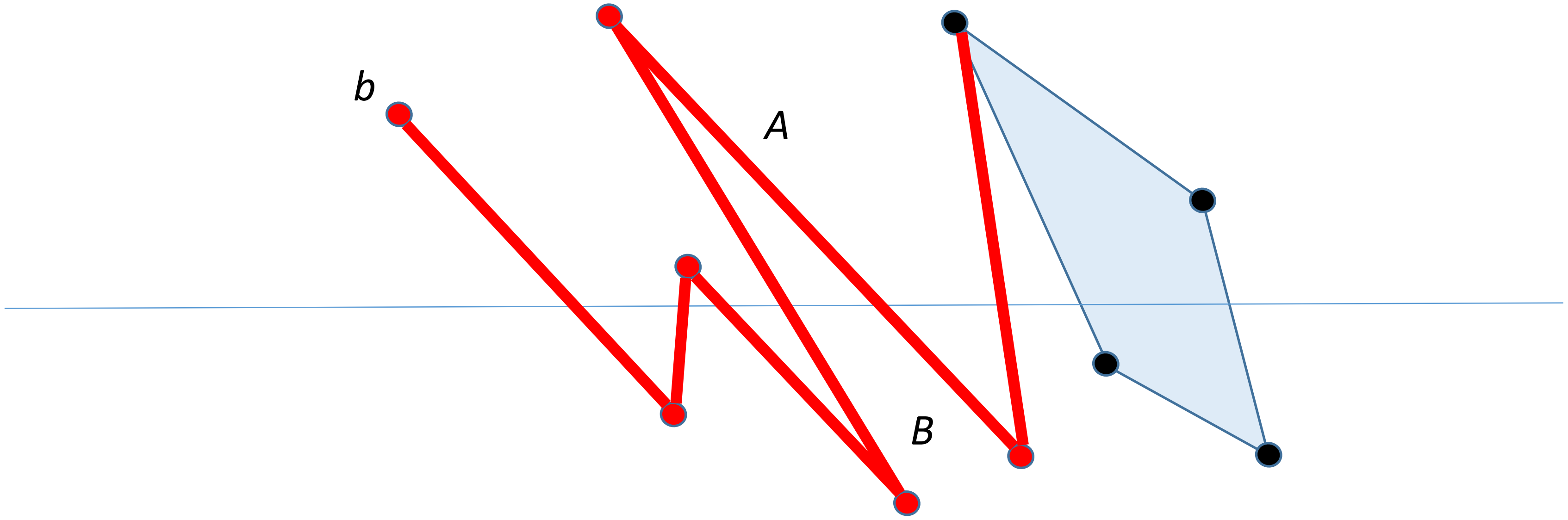

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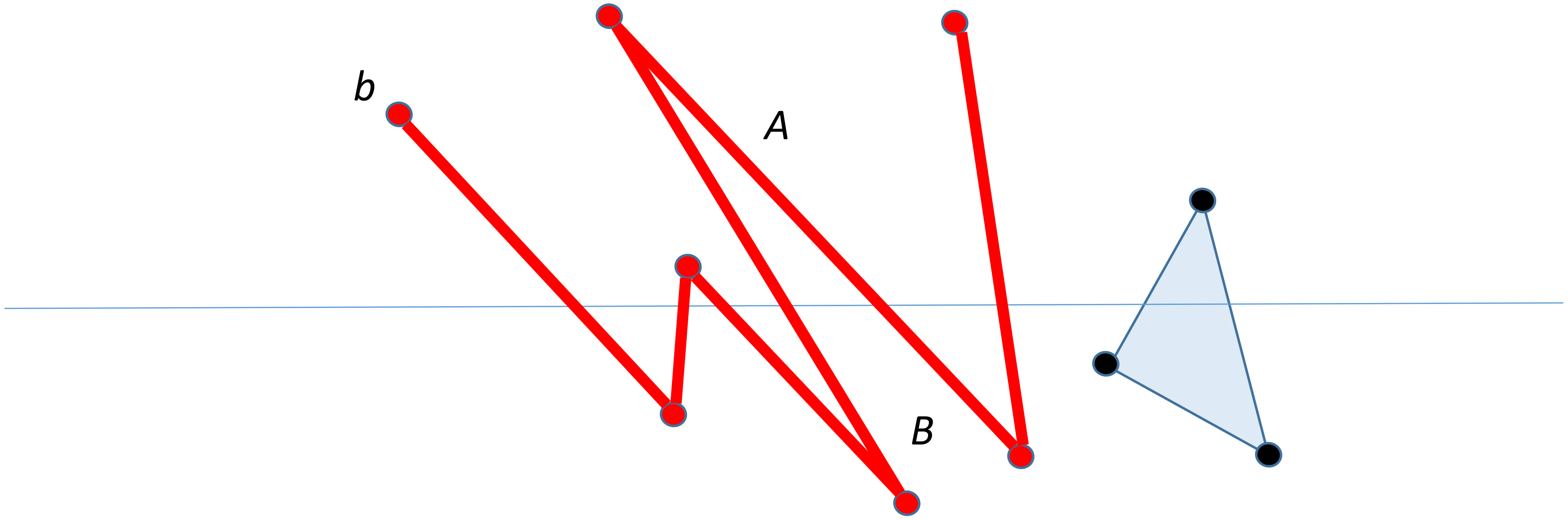

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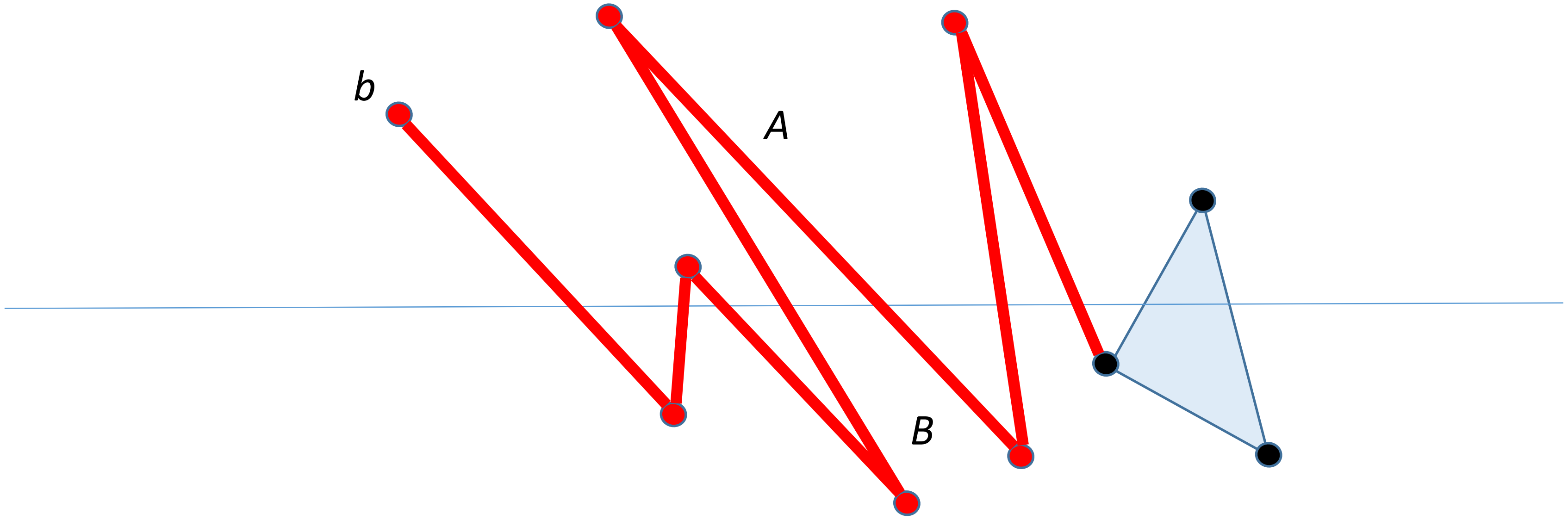

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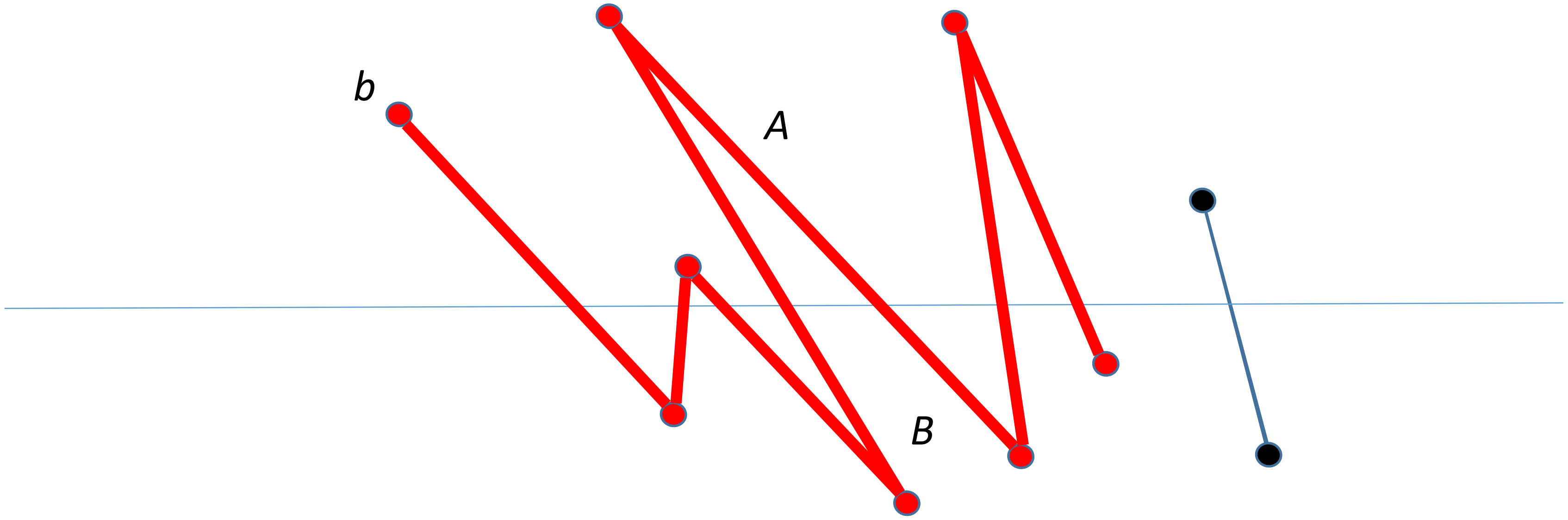

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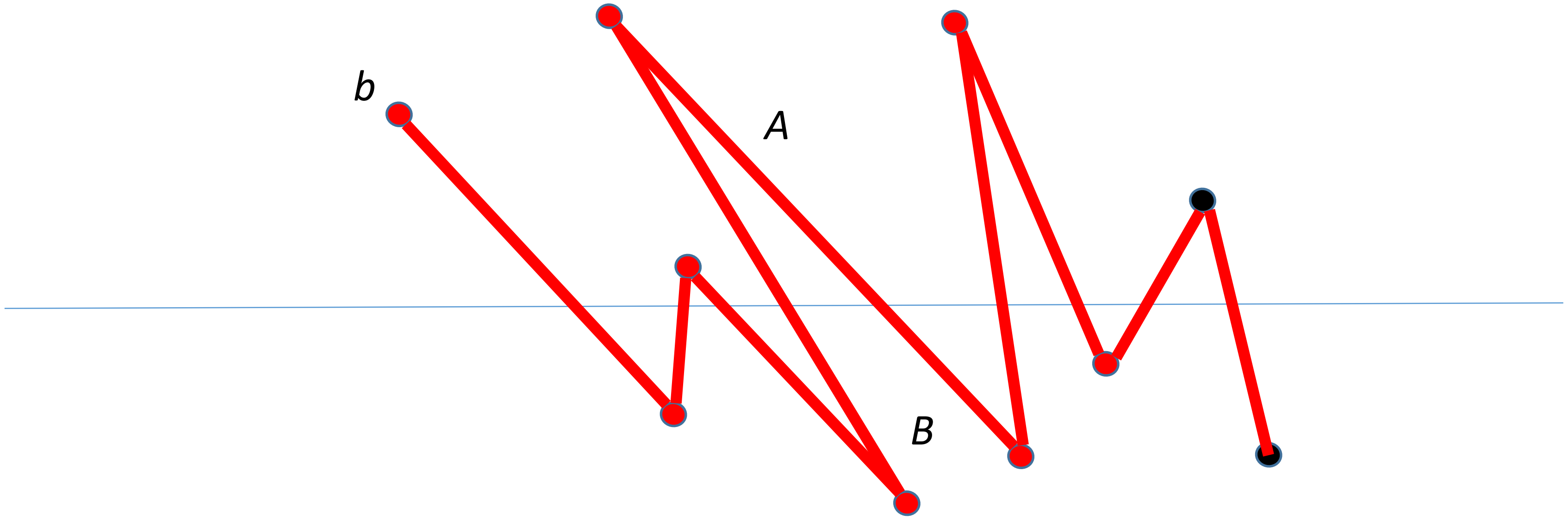
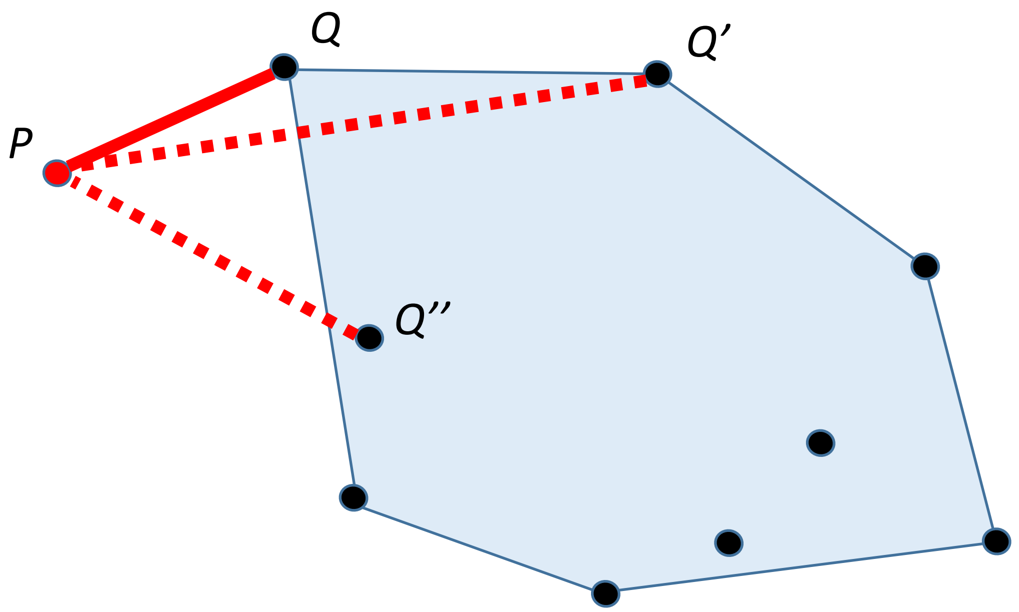


A few technical lemmas

Definition: A point P outside of conv(S) sees a point $Q \in S$ if $PQ \cap$ conv(S) = Q.



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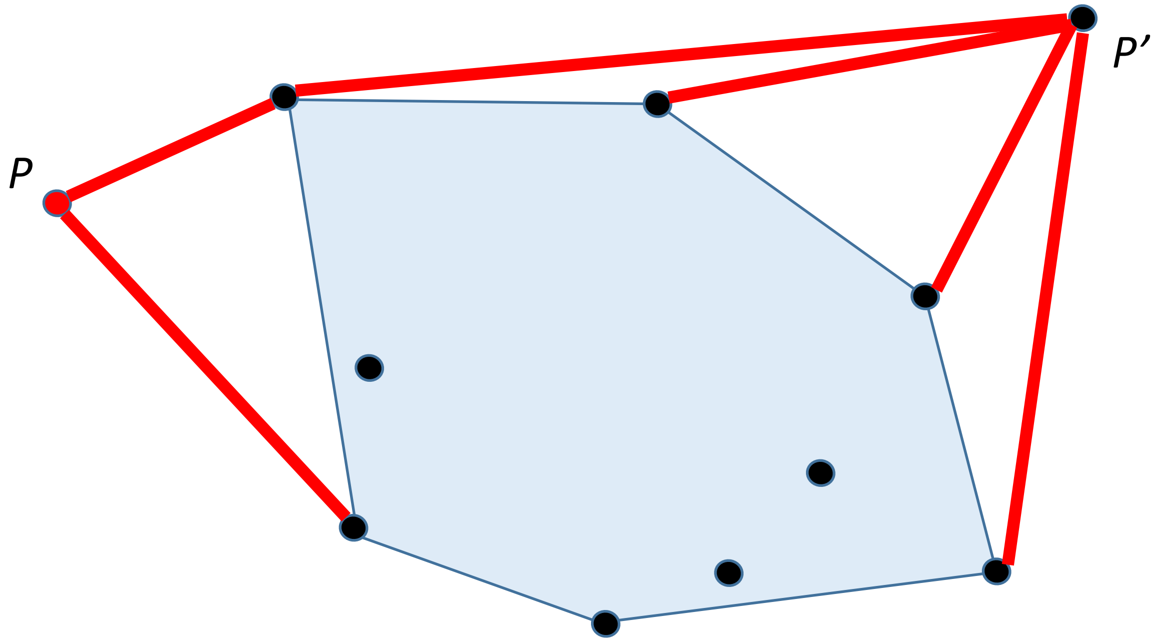

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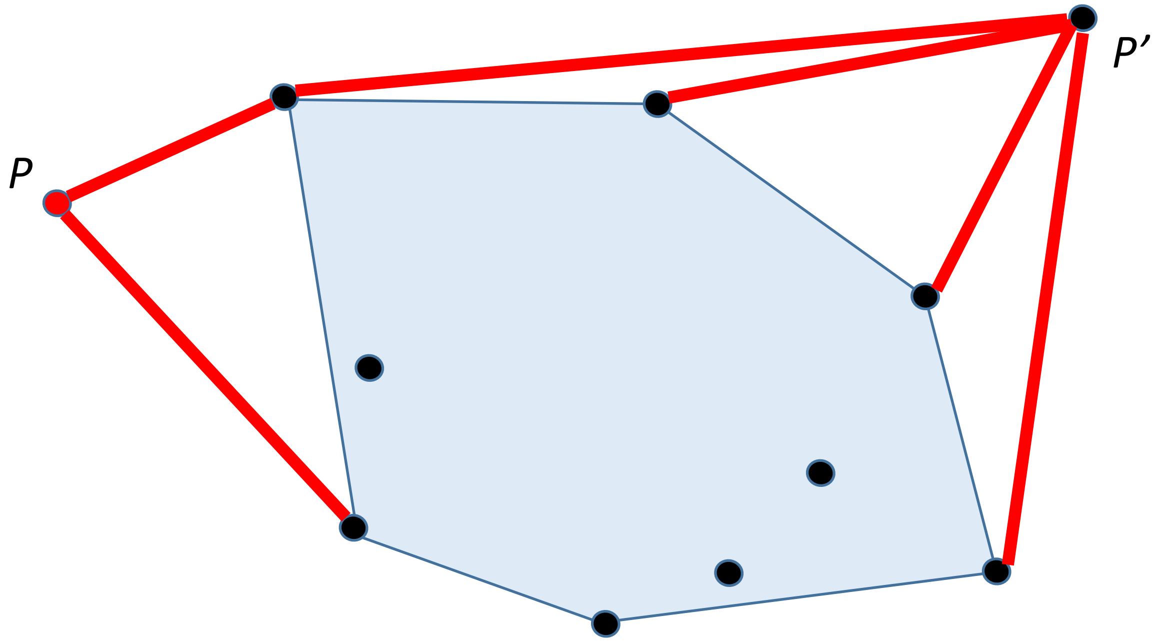

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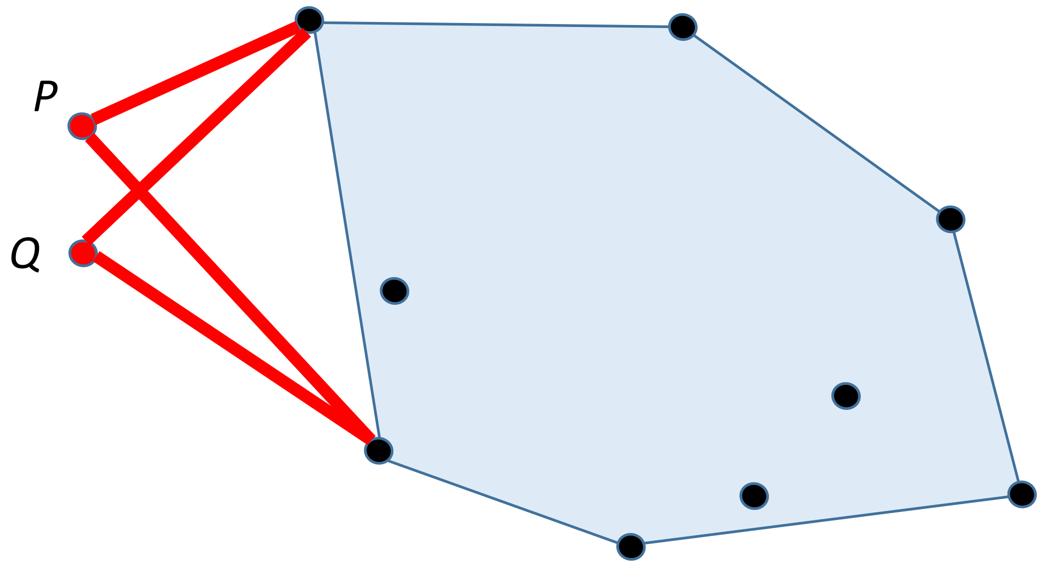

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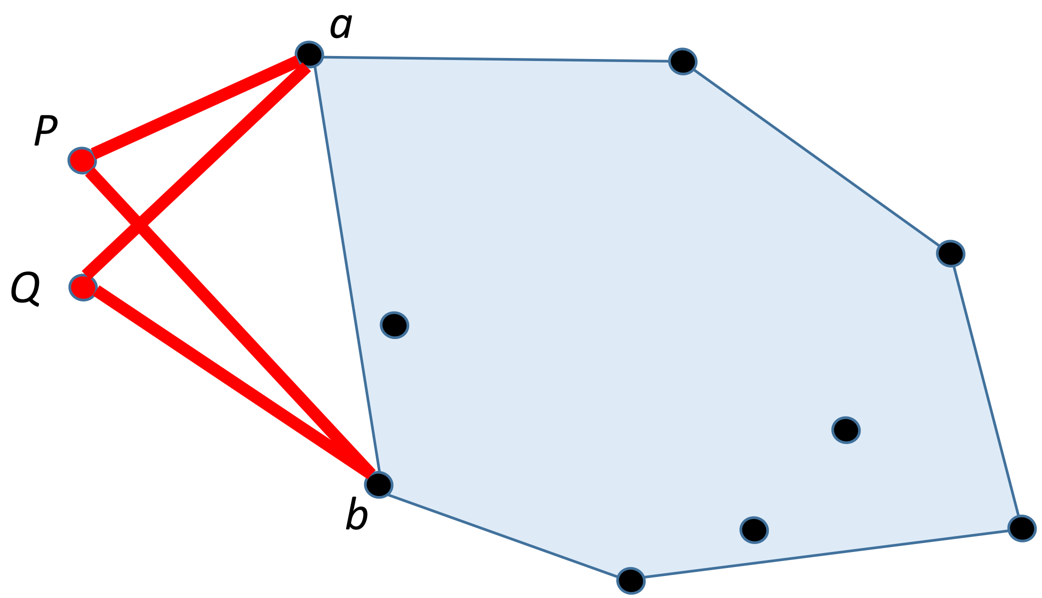

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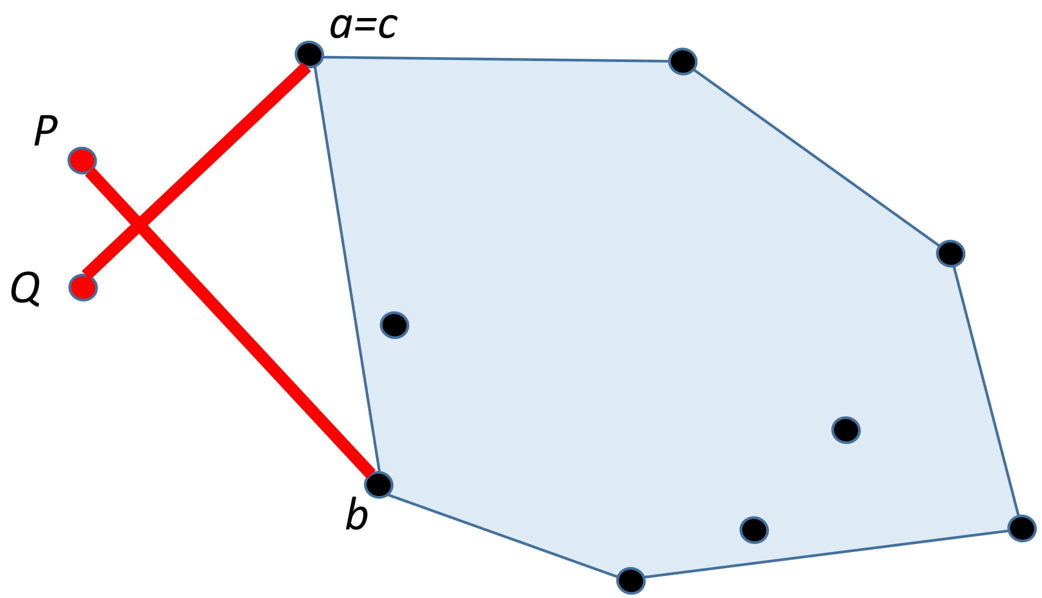

2 paths

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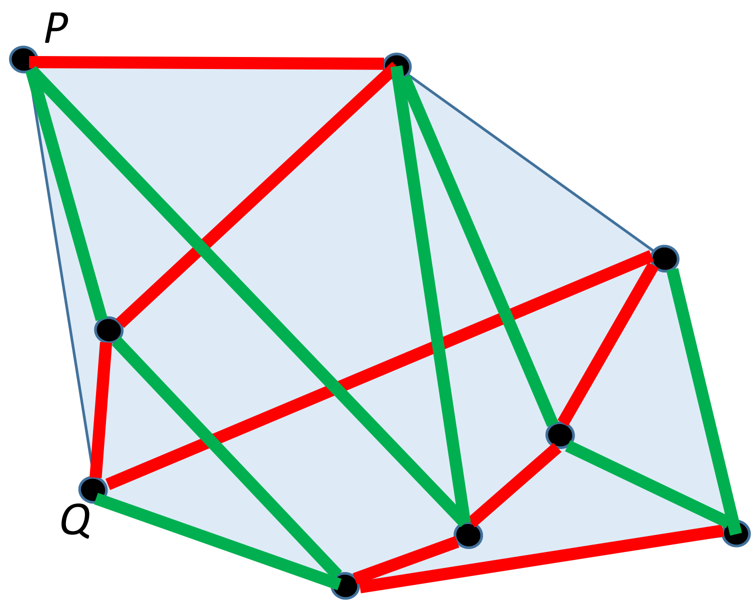

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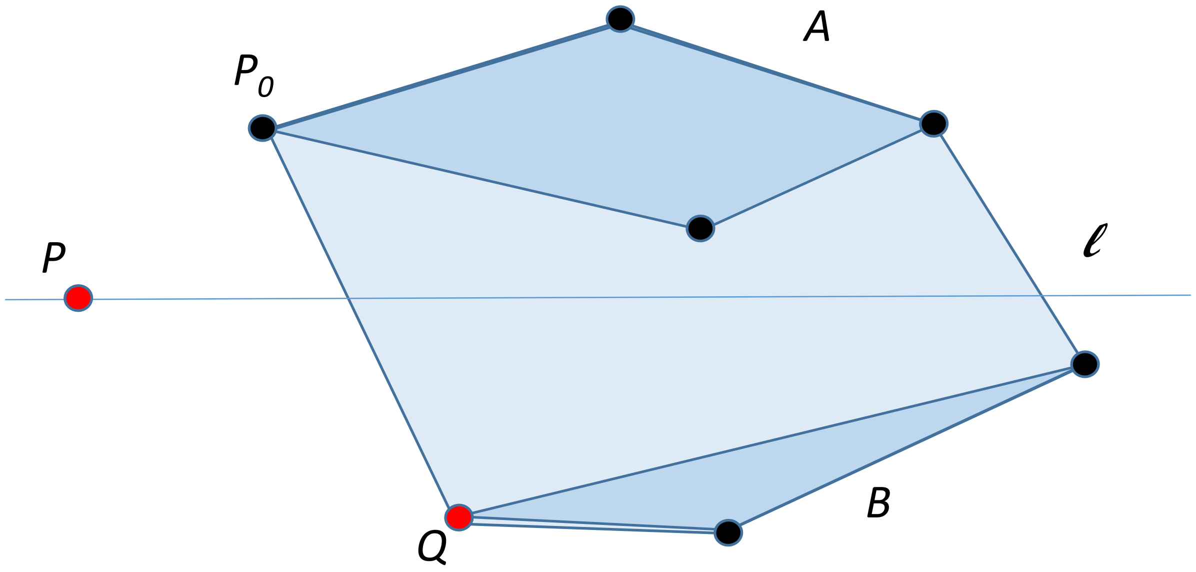

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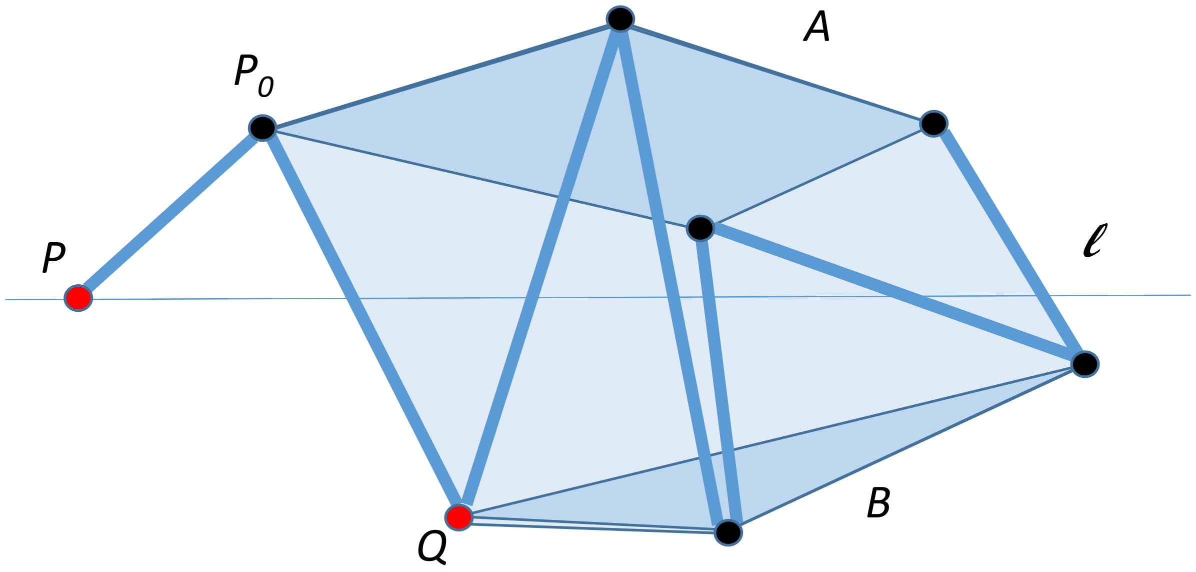

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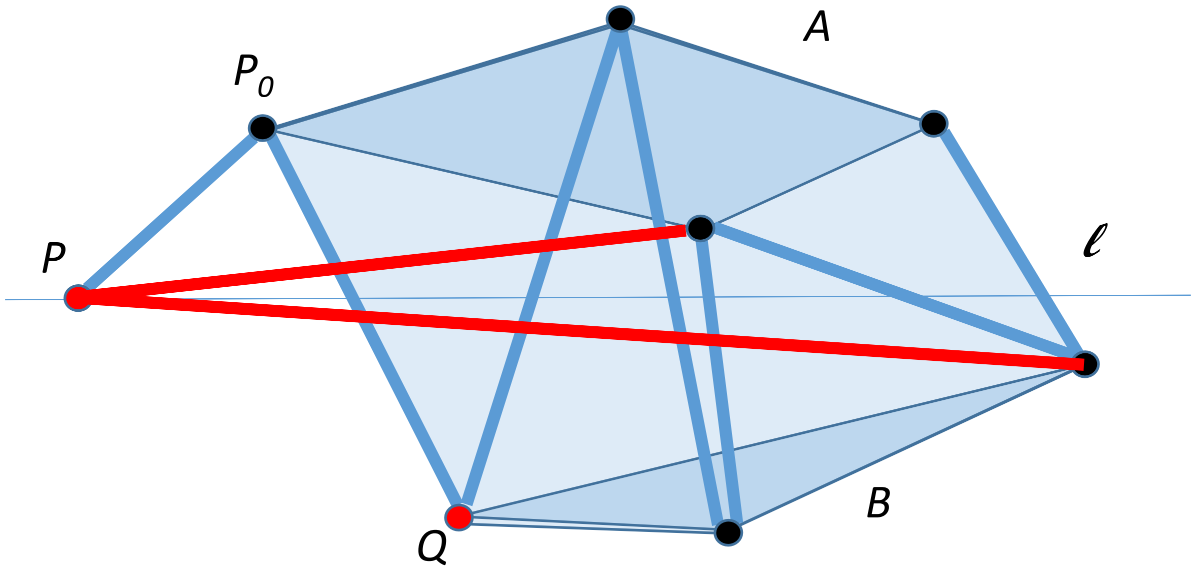

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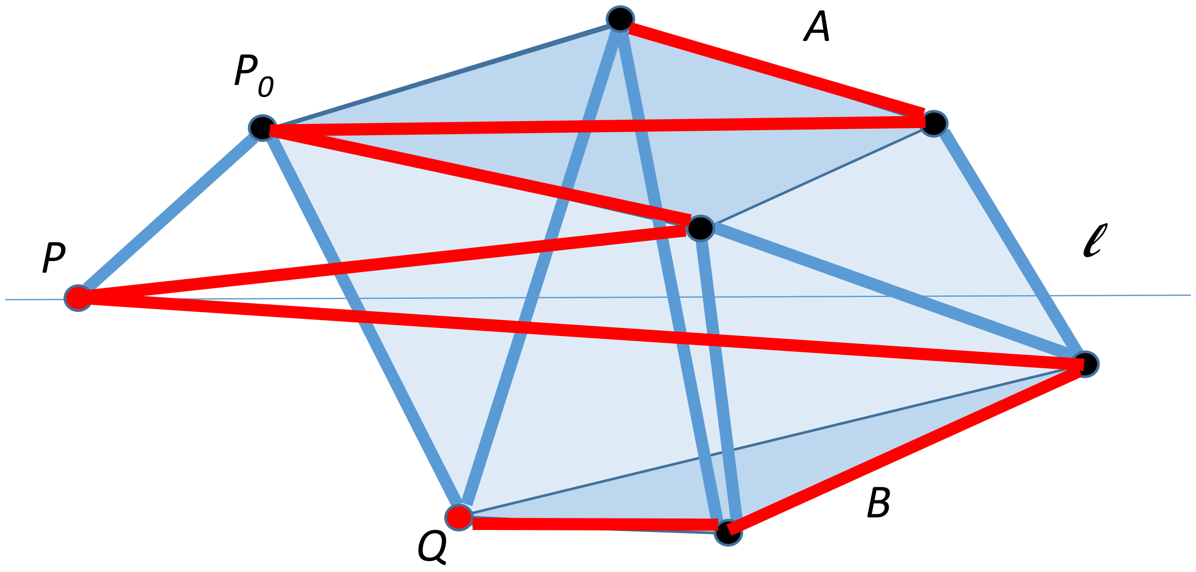

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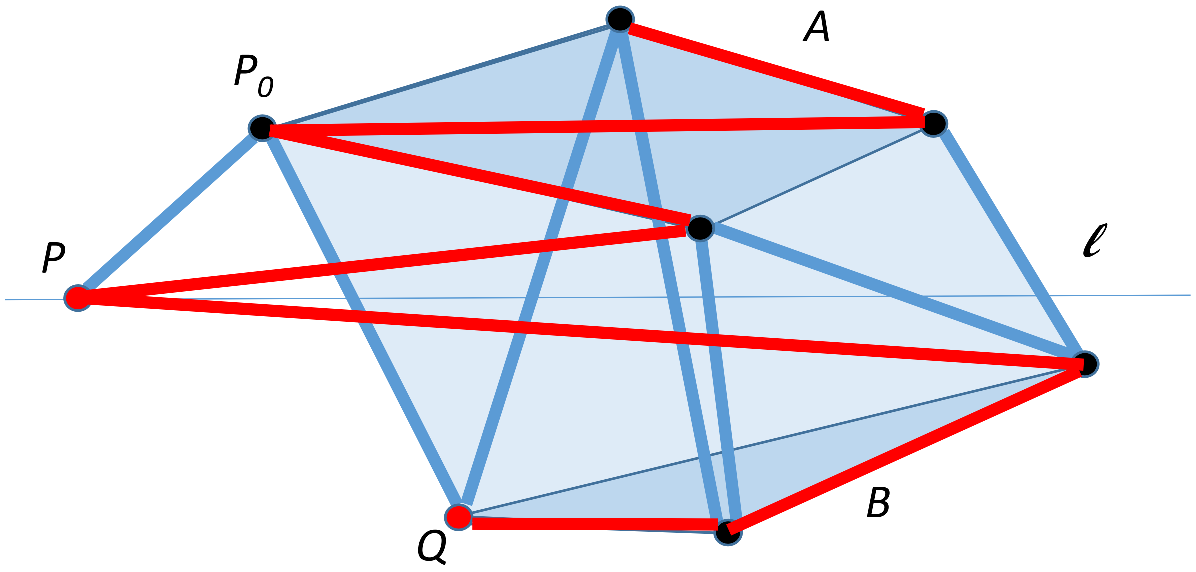

b) Subcase $|S|$ even and PQ is a halving line and $|A(P) \cup A(Q)| \geq 3$.

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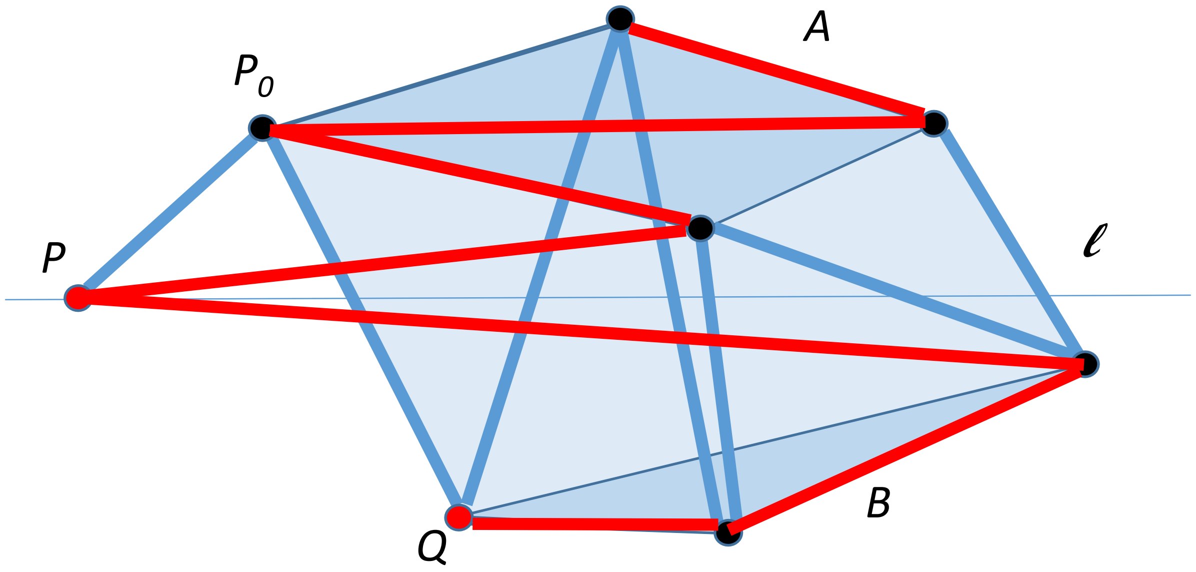

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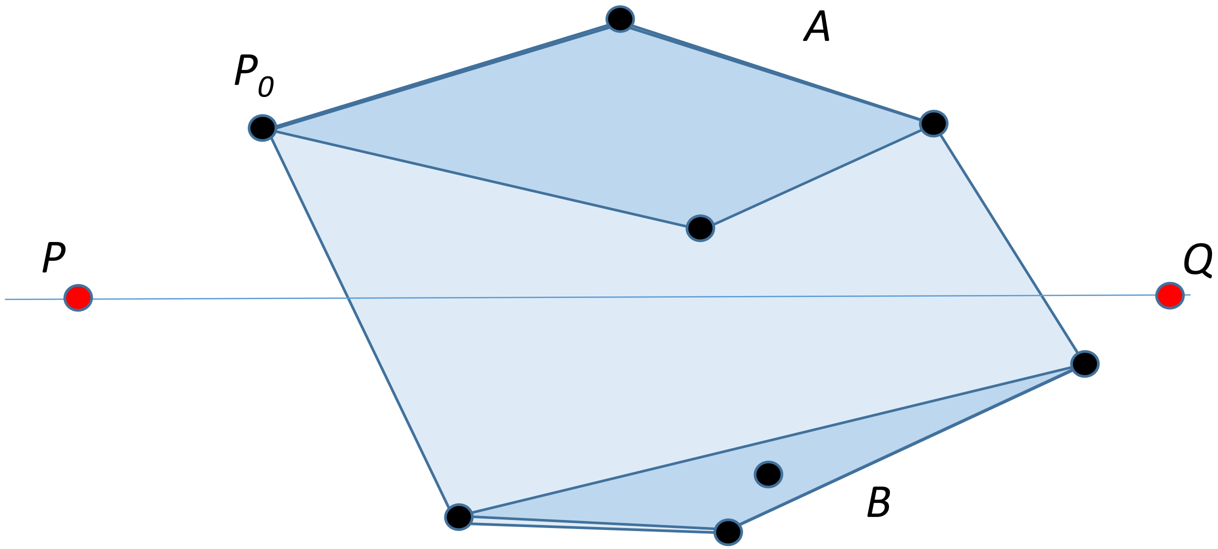

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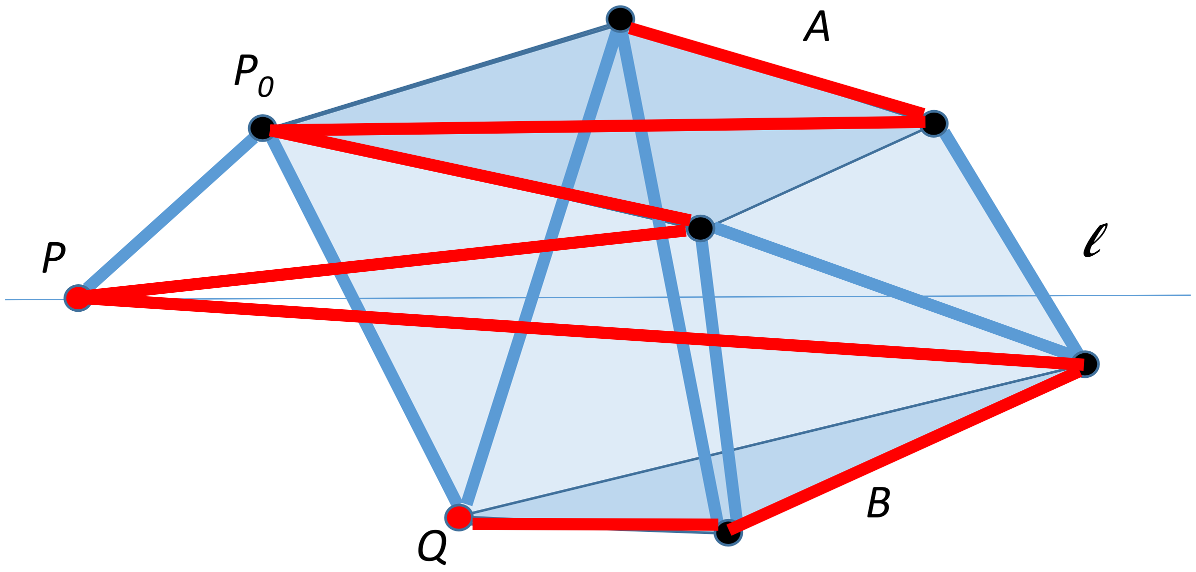

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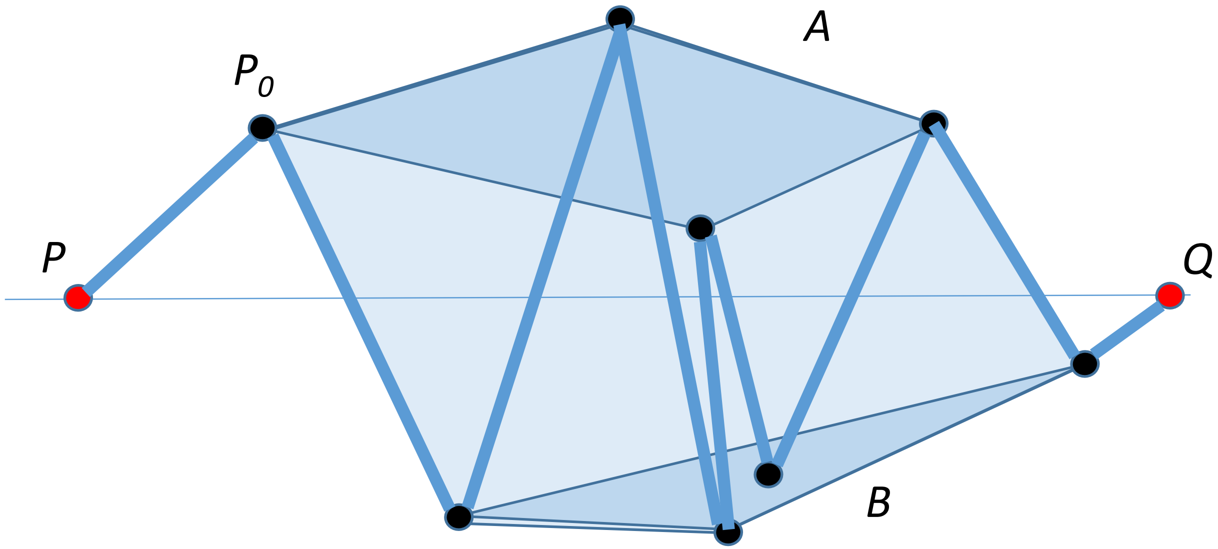

2 paths

Theorem 2: Let P and Q be two (not necessarily distinct) points of S, lying on the boundary of conv(S), and let $|S| \geq 5$. Then S admits 2 edge-disjoint plane spanning paths, one starting in P, the other one starting in Q, and none of them using the edge PQ (in case P and Q are distinct).

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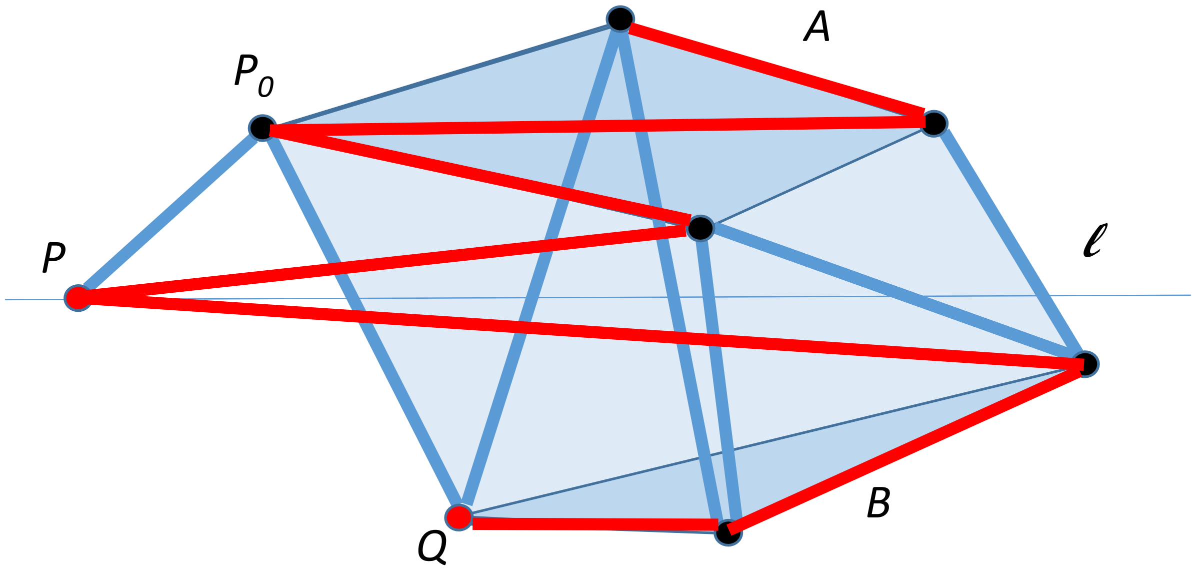

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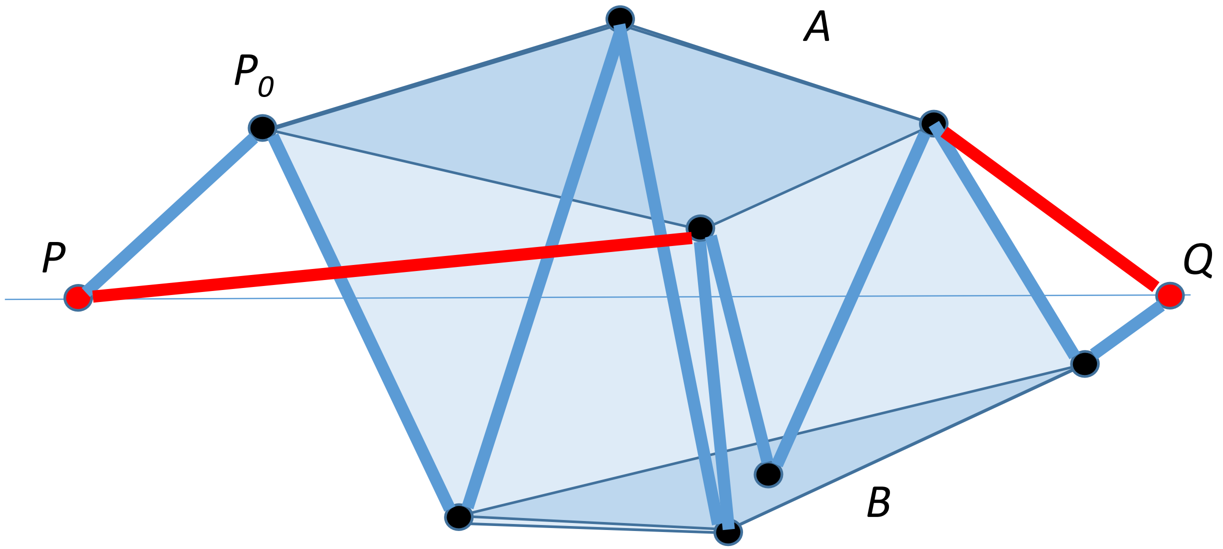

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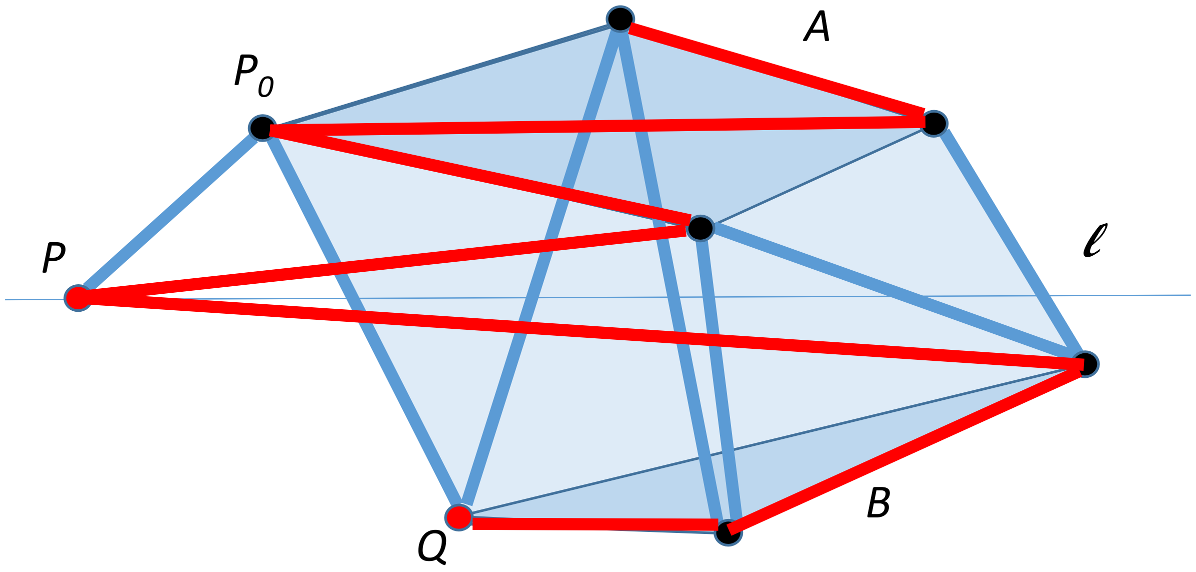

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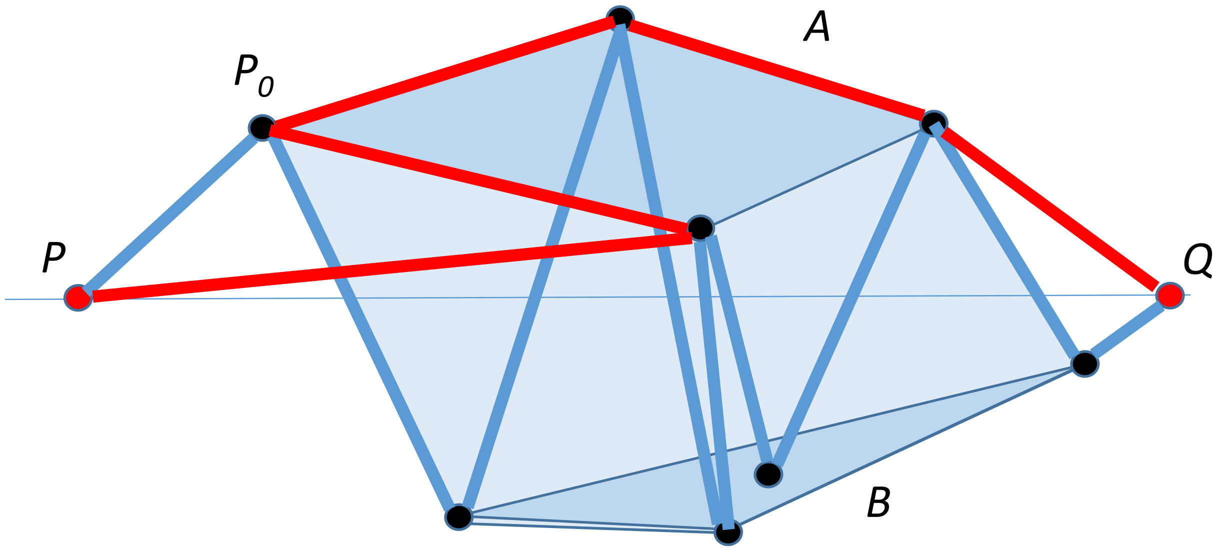

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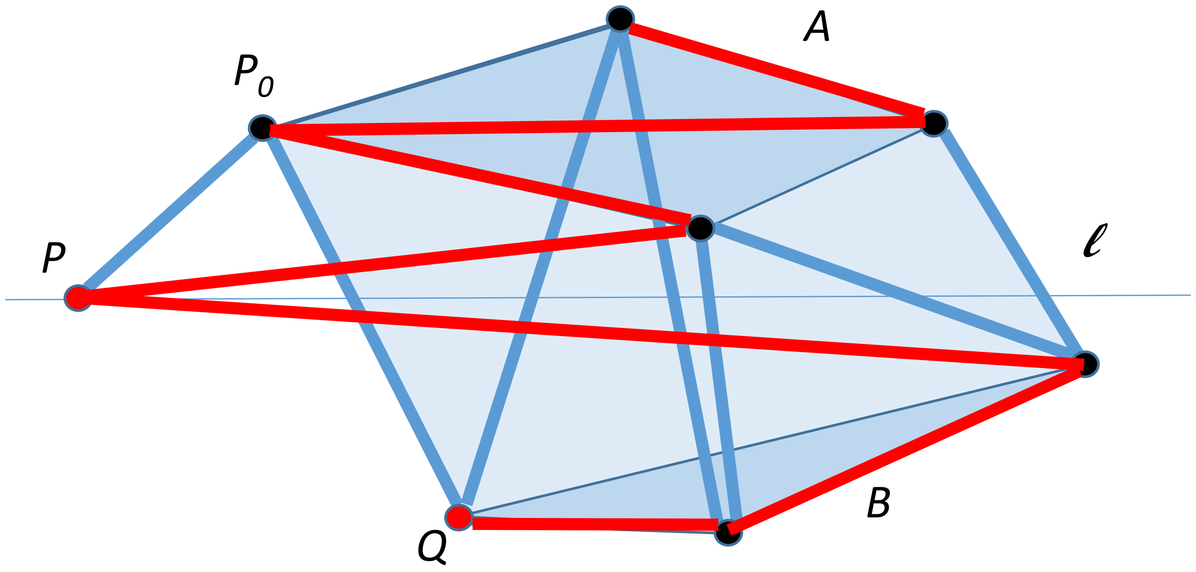

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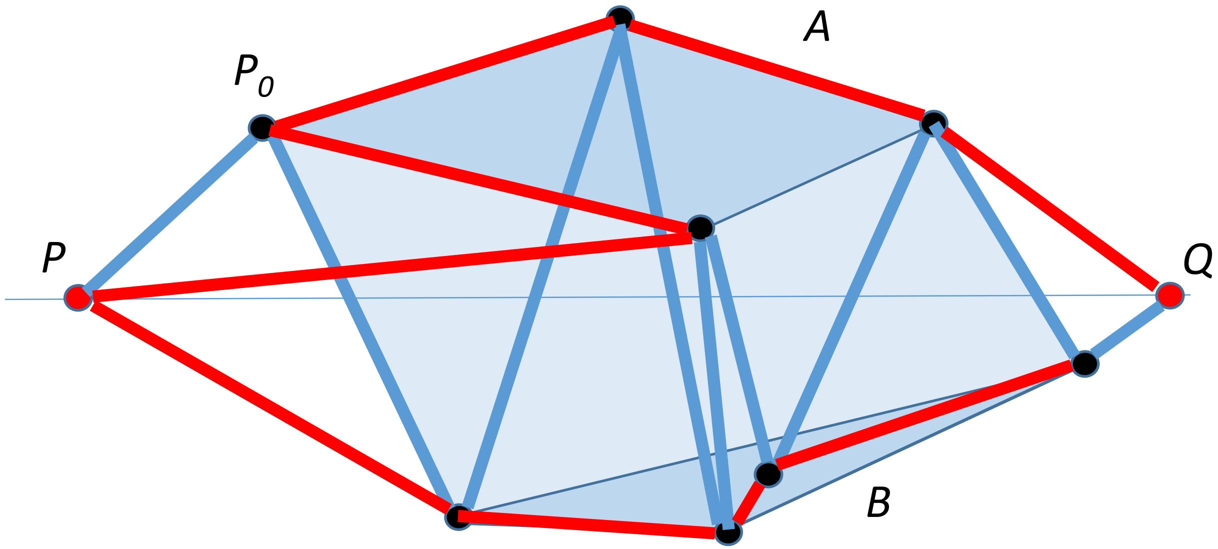

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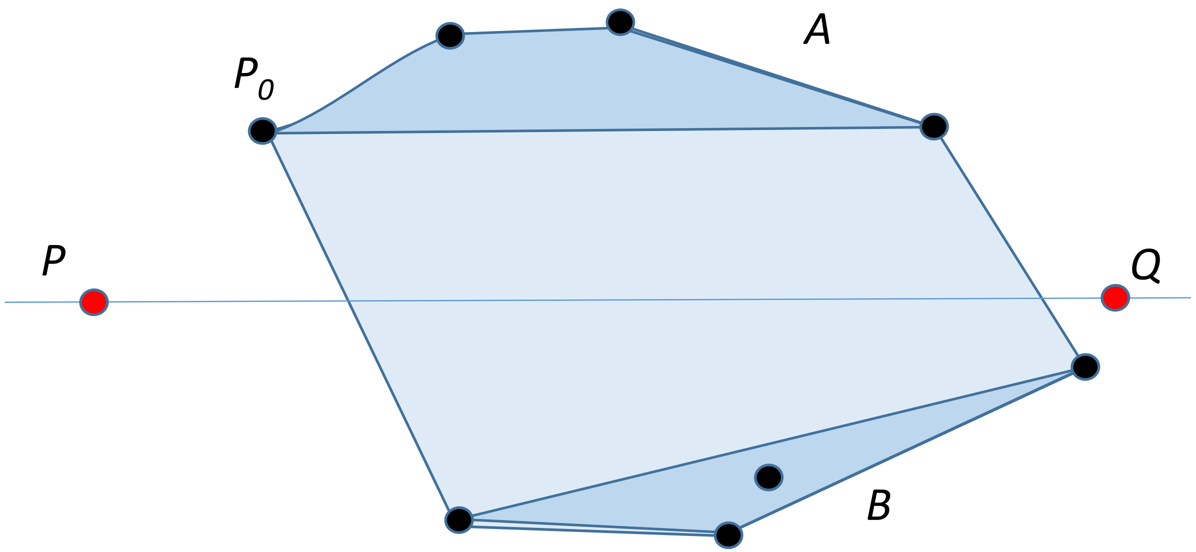

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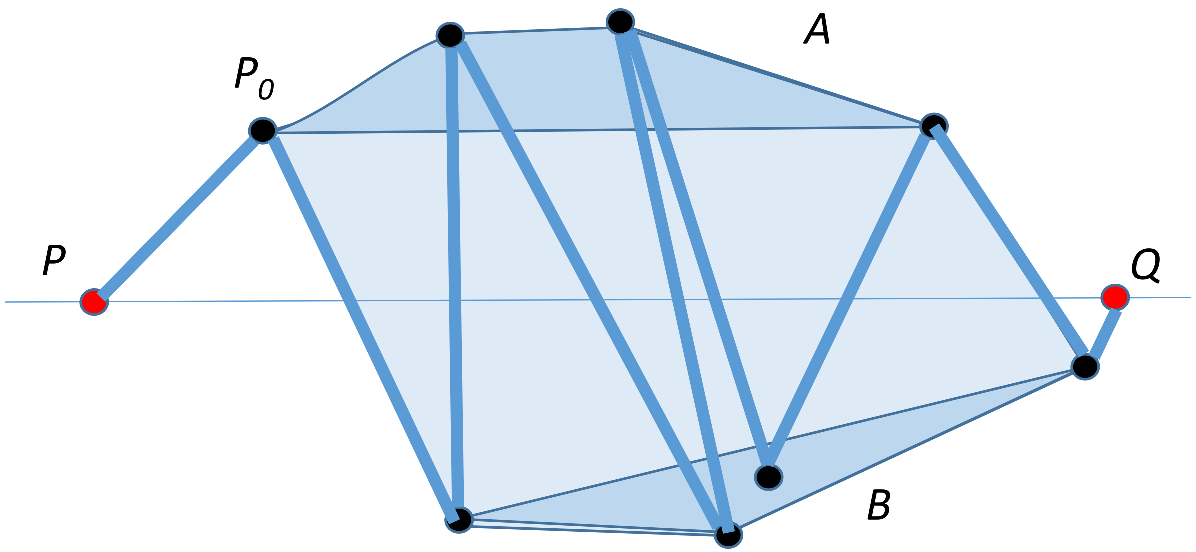

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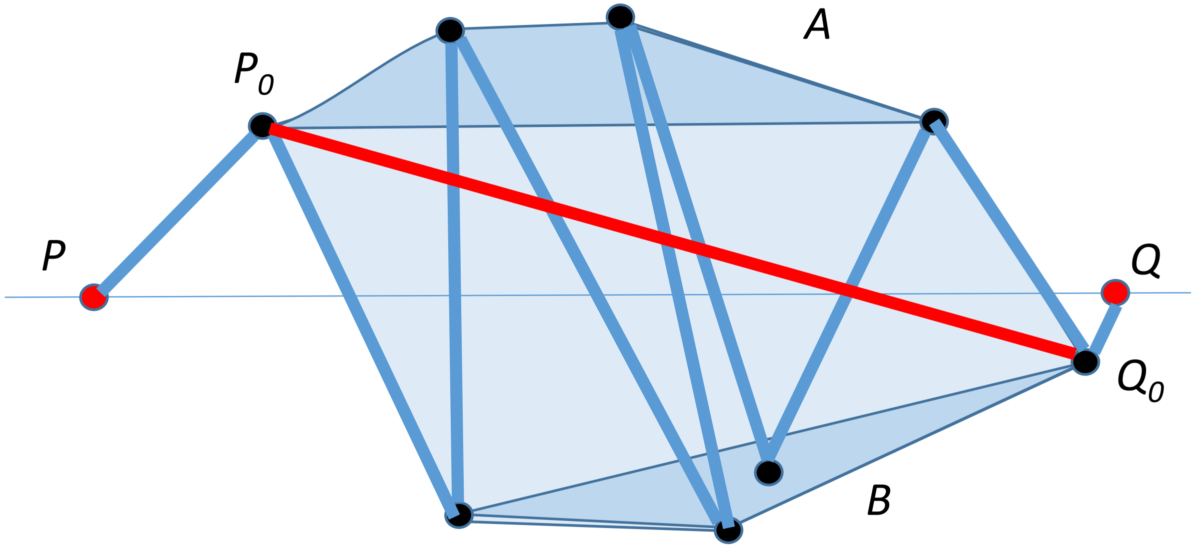

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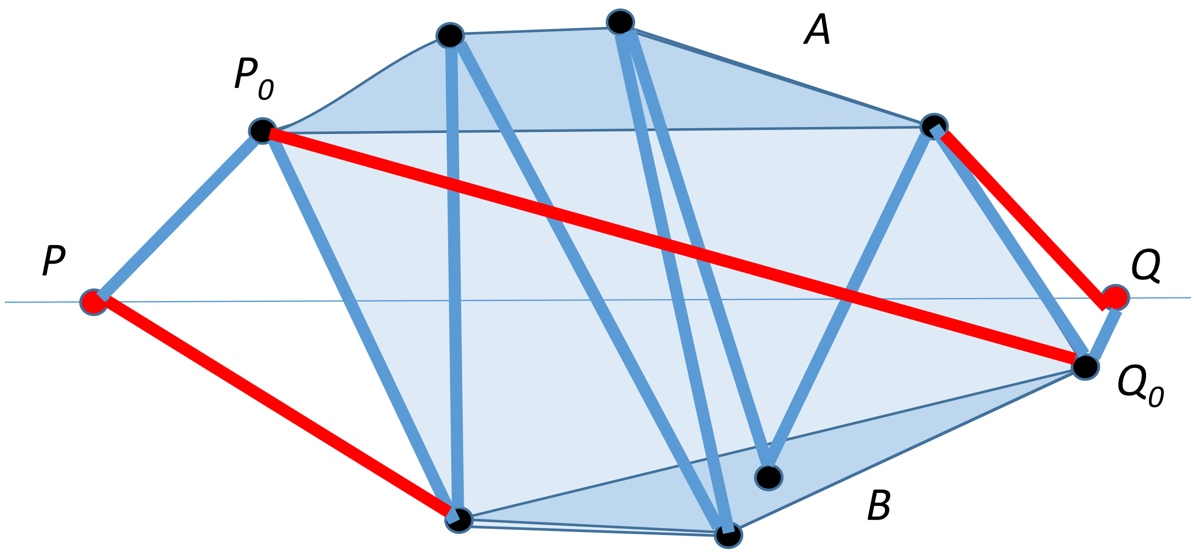

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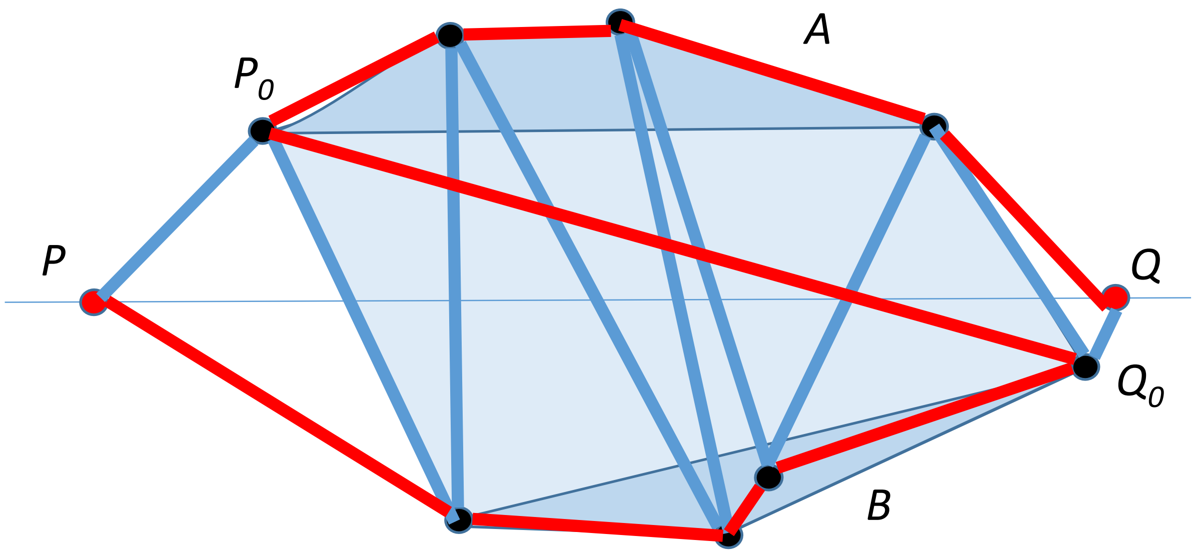

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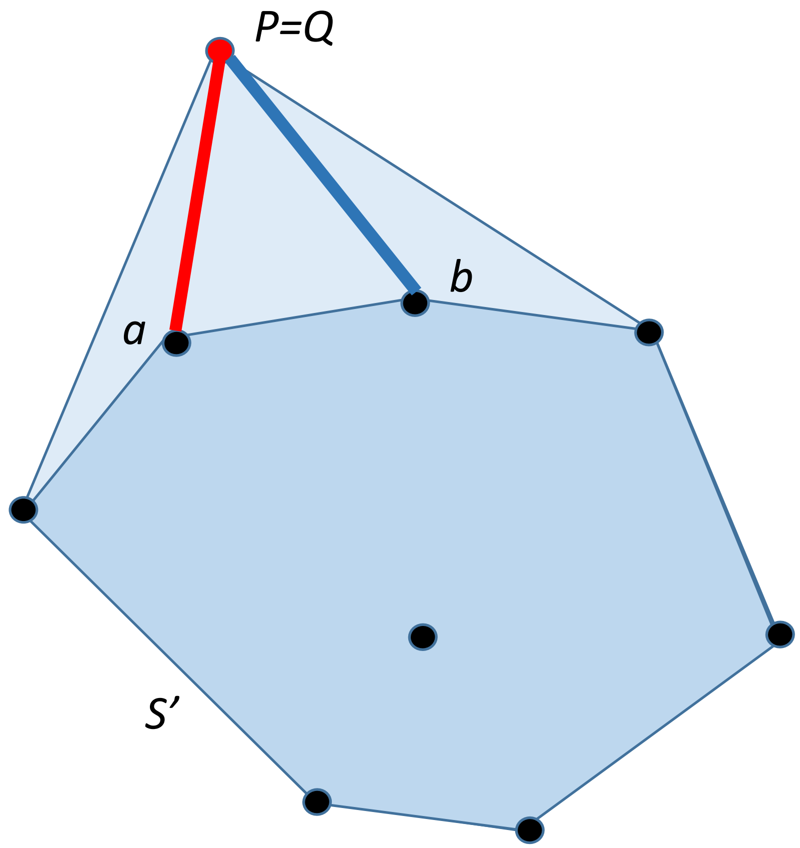

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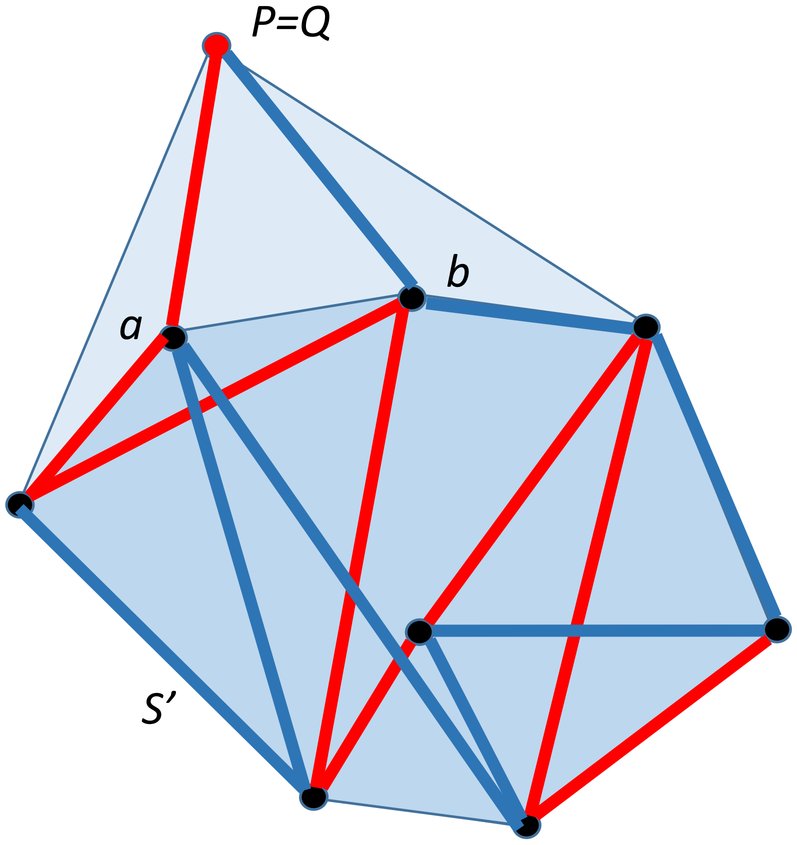
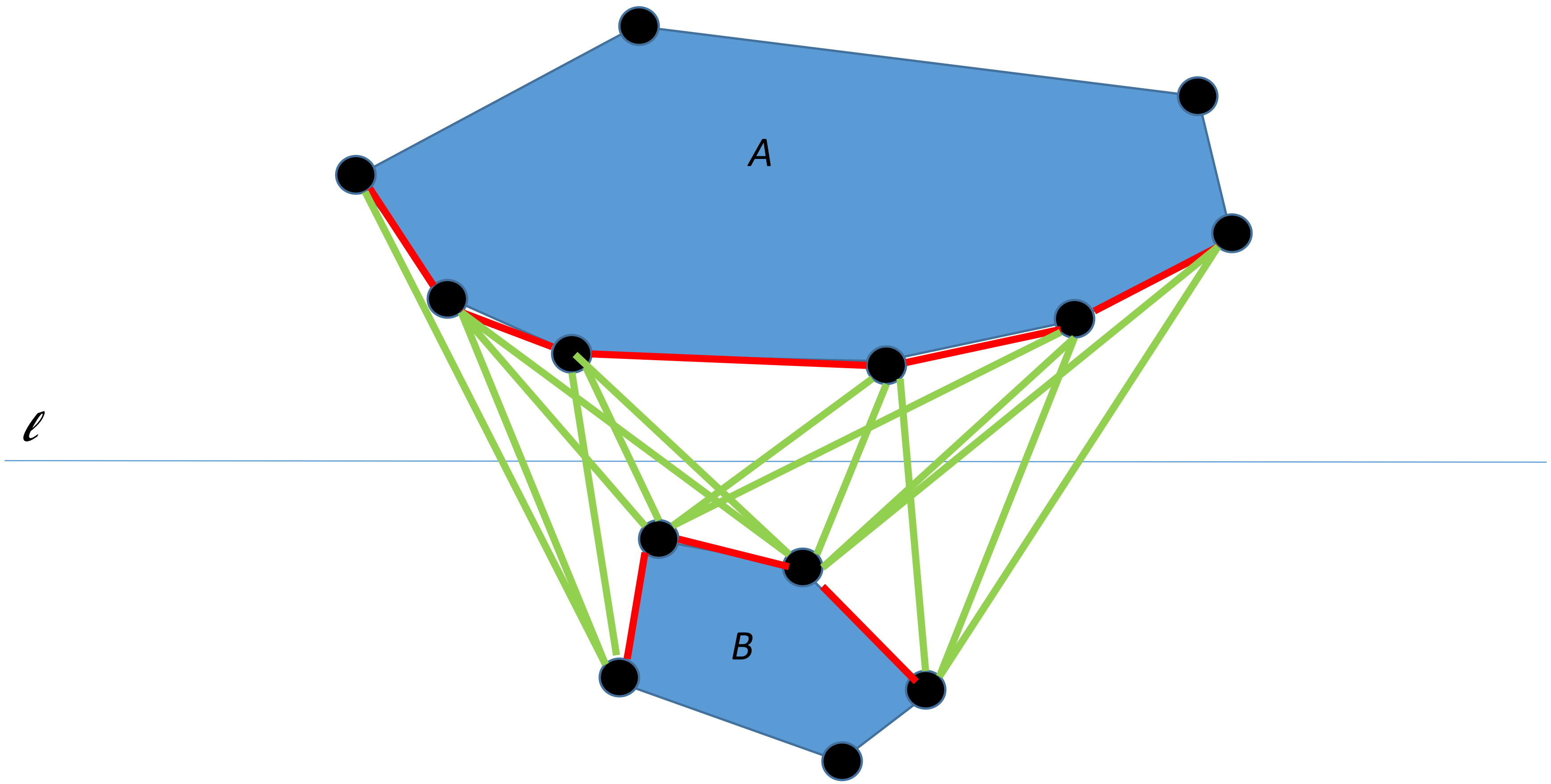


Two more technical notions

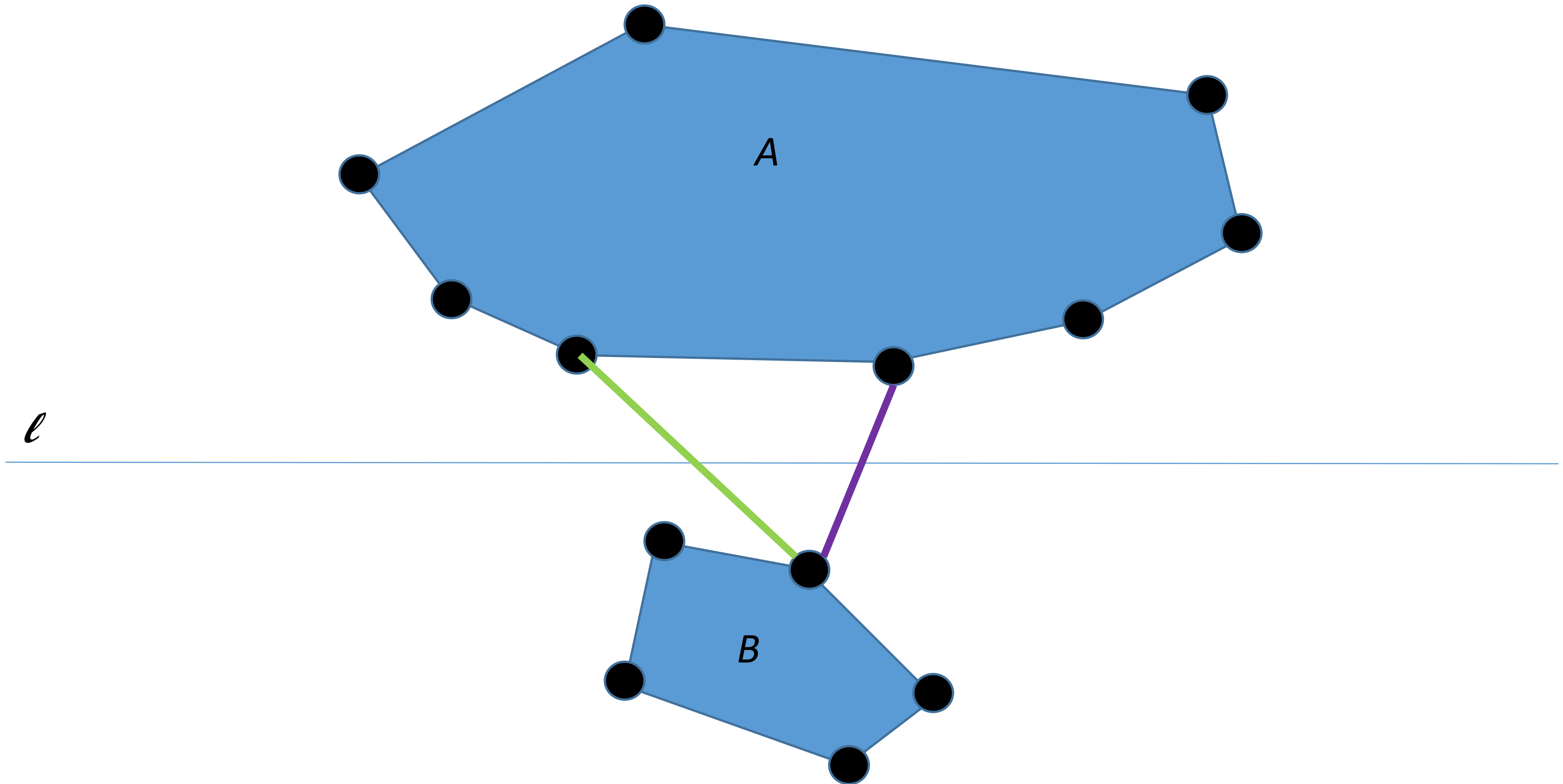
Definition: Let (A,B) be a separation of S. The visibility graph $\mathrm{Vis}(A,B)$ of the separation is the graph with vertex set S and edges PQ s.t. $P \in A(Q)$ and $Q \in B(P)$.



Two more technical notions

Definition: Let (A,B) be a balanced separation of S and let Z be a zig-zag path w.r.t. (A,B). An edge $e \in E(\text{Vis}(A,B))$ is called free if e does not belong to Z.

Lemma 2: Let (A,B) be a balanced separation of S of $|S| \geq 10$ points and let Z be a zig-zag path w.r.t. (A,B). If Z leaves at least 2 free edges, then S admits 3 edge-disjoint plane spanning paths.



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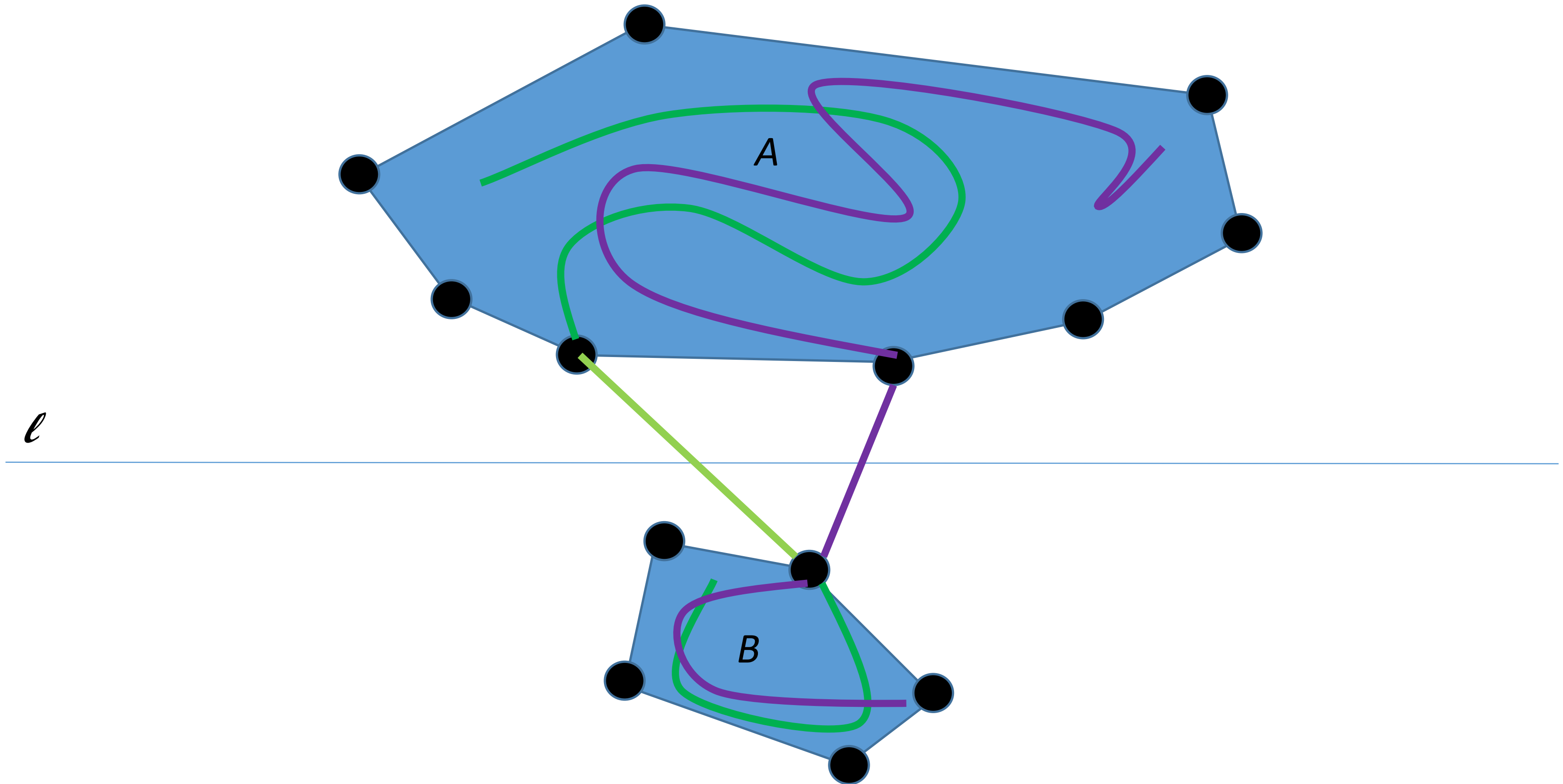
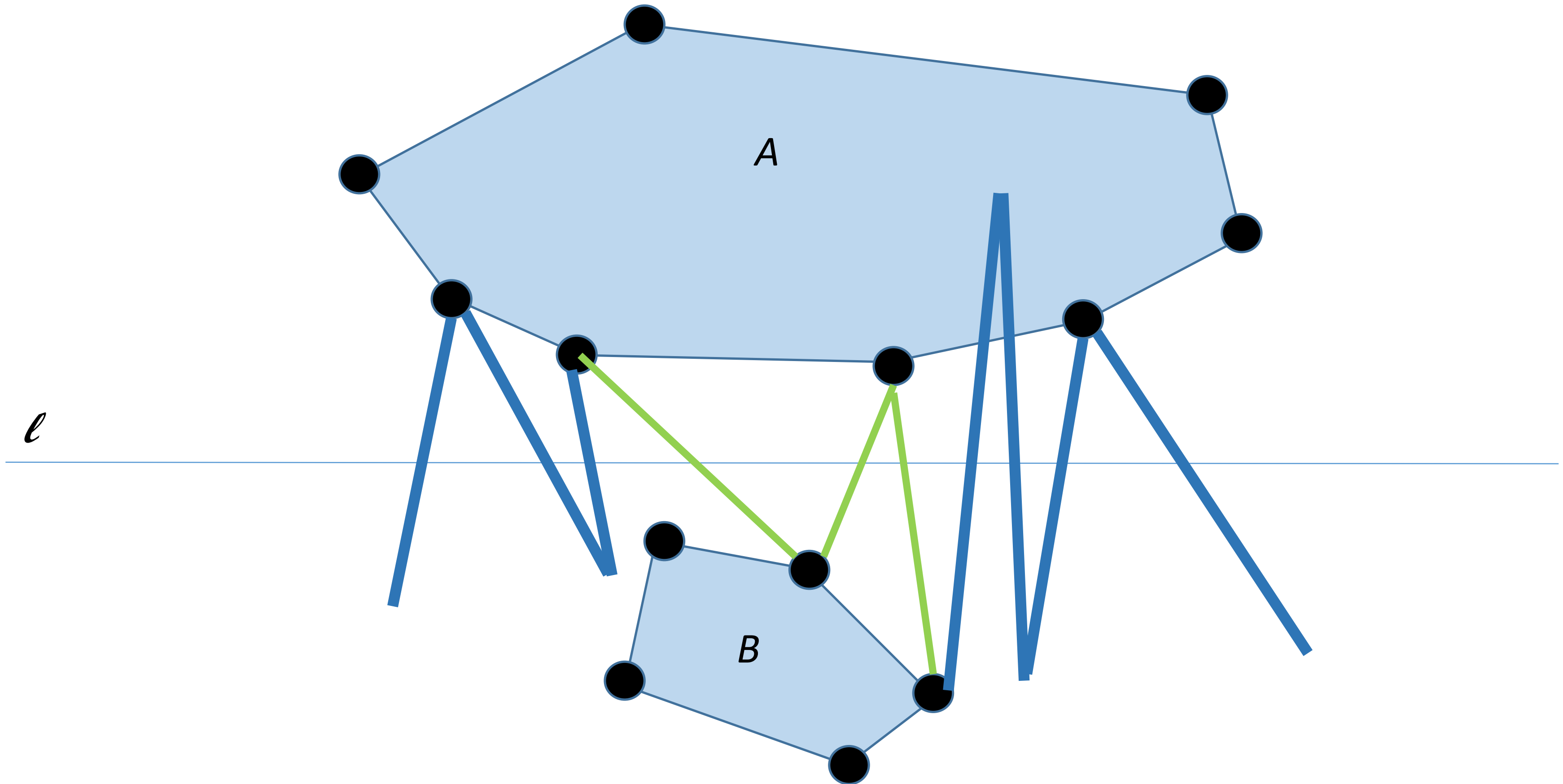


3 Paths

Lemma 3: Let (A,B) be a balanced separation of S of $|S| \geq 10$ points and let Z be a zig-zag path w.r.t. (A,B). If Z uses 3 consecutive edges of Vis(A,B), then S admits 3 edge-disjoint plane spanning paths.



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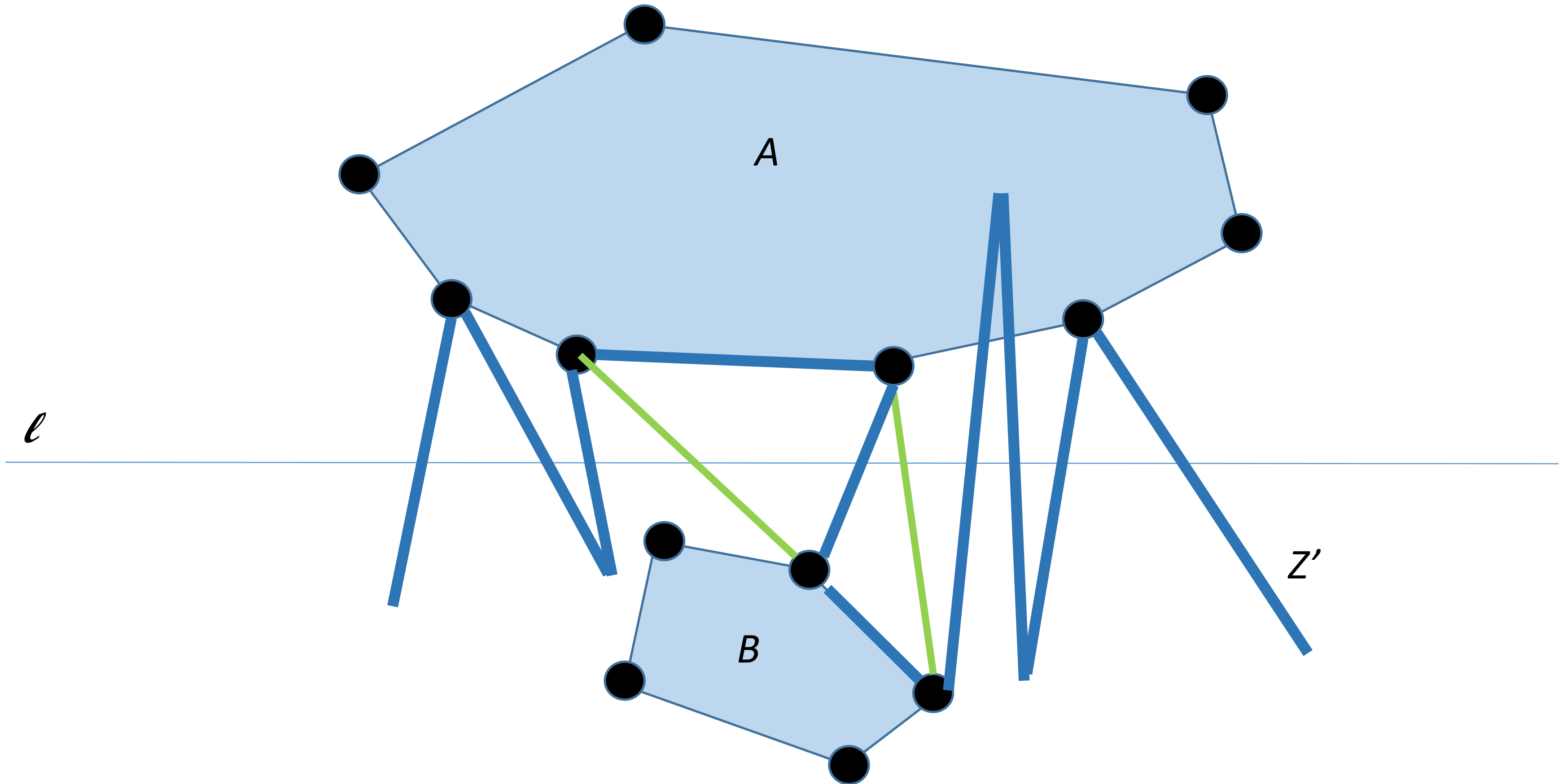
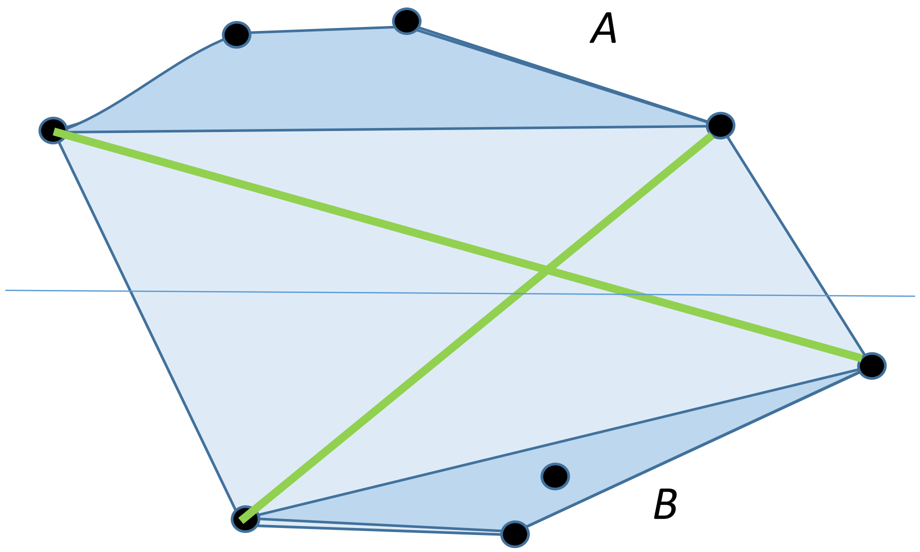


3 Paths

Theorem 3: Let *S* be a set of points in the plane. Then

- *S* allows a balanced separation (*A*,*B*) such that Vis(*A*,*B*) contains 2 crossing edges, or



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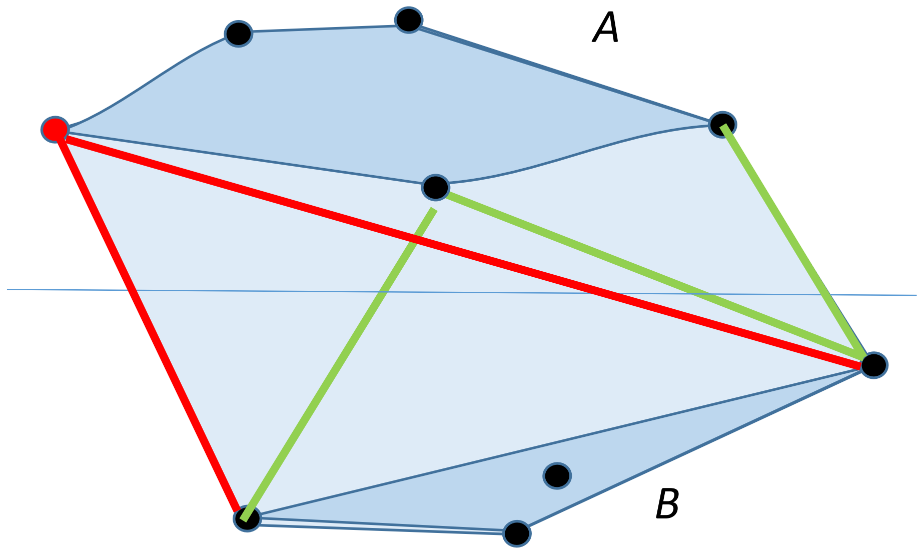

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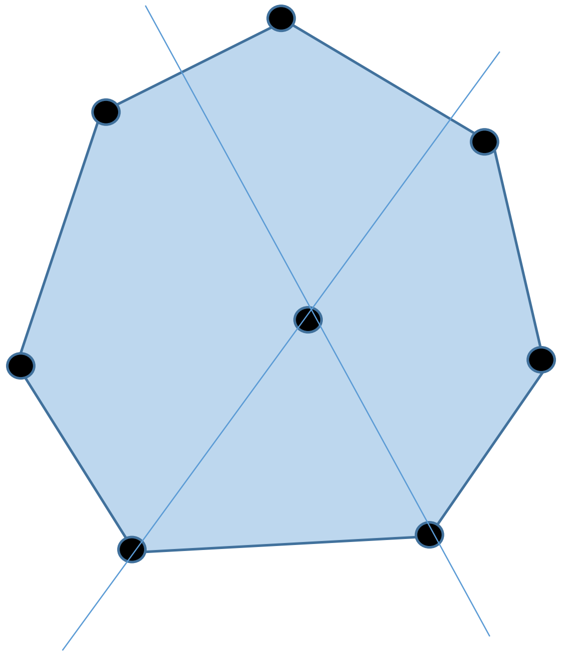
- $n=|S|$ is even and *S* is in the wheel configuration.

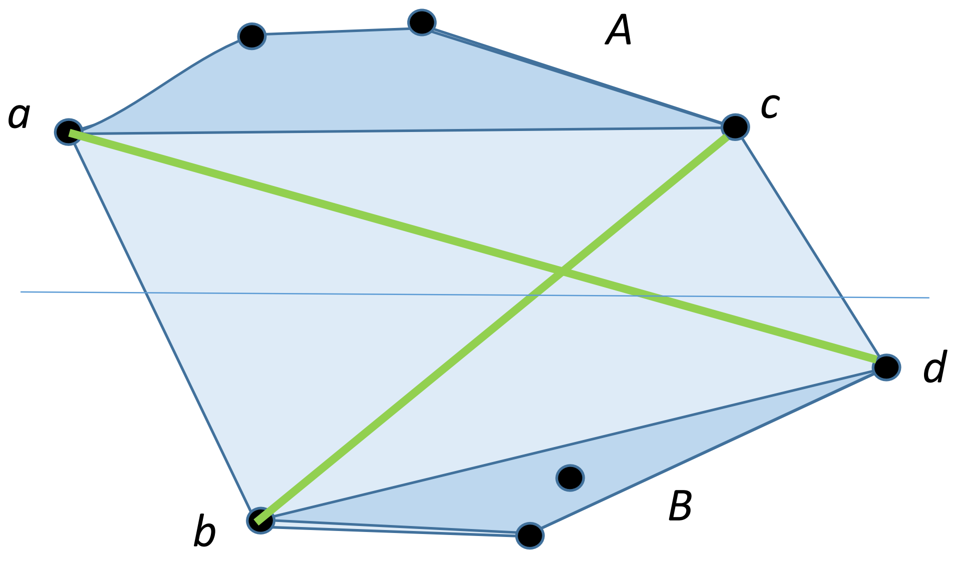


3 Paths

Theorem 4: Every set of $|S| \geq 10$ points admits 3 edge-disjoint plane spanning paths.

Proof:

Case A. S allows a balanced separation (*A*,*B*) such that Vis(*A*,*B*) contains 2 crossing edges.



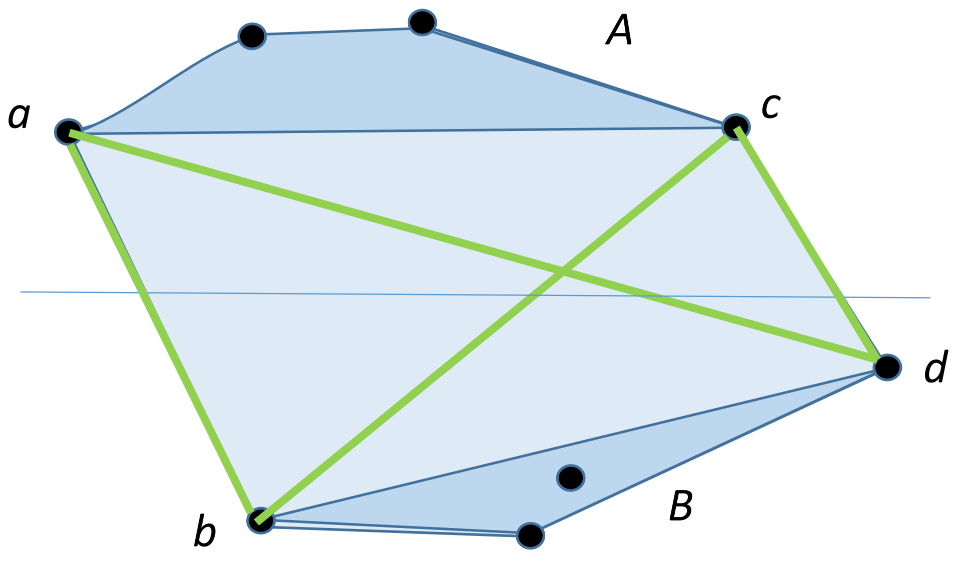
3 Paths

Theorem 4: Every set of $|S| \geq 10$ points admits 3 edge-disjoint plane spanning paths.

Proof:

Case A. S allows a balanced separation (*A*,*B*) such that Vis(*A*,*B*) contains 2 crossing edges.

Then Vis(*A*,*B*) contains consecutive vertices $a,c \in A$ and $b,d \in B$ and all 4 edges *ab, ad, bc, cd*. Consider the Abellanas zig-zag path. It cannot contain all 4 edges (mind the crossing). If it contains 3 of them, apply Lemma 3. If it uses at most 2 of them, it leaves at least 2 free edges, and apply Lemma 2.



3 Paths

Theorem 4: Every set of $|S| \geq 10$ points admits 3 edge-disjoint plane spanning paths.

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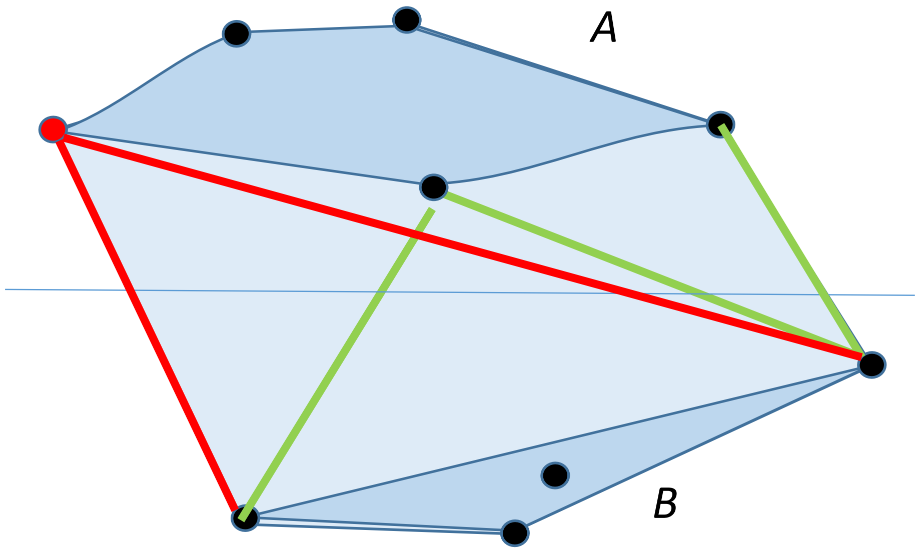
Case B. S allows a balanced separation (A,B) such that Vis(A,B) contains an empty path of length 3 and a bridged vertex incident with two edges of Vis(A,B).



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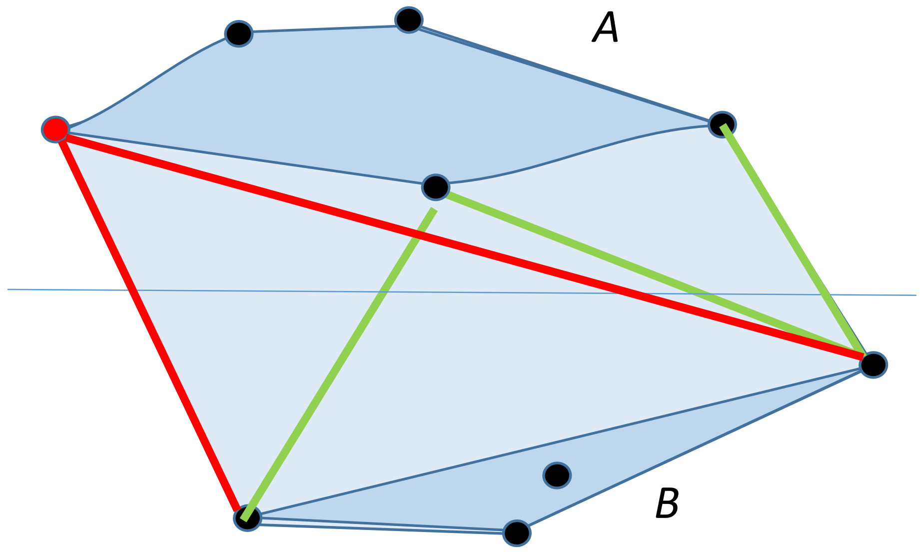
Case B. S allows a balanced separation (*A*,*B*) such that Vis(*A*,*B*) contains an empty path of length 3 and a bridged vertex incident with two edges of Vis(*A*,*B*).

Consider Abellanas zig-zag path *Z* starting in the bridged vertex. The visibility graph Vis(*A*,*B*) contains at least 2 edges incident to this vertex, and only one of them is in the path. So it leaves at least 1 free edge. If all 3 edges of the empty path belong to Z, use Lemma 3. Otherwise, one of these 3 edges is free, and apply Lemma 2.



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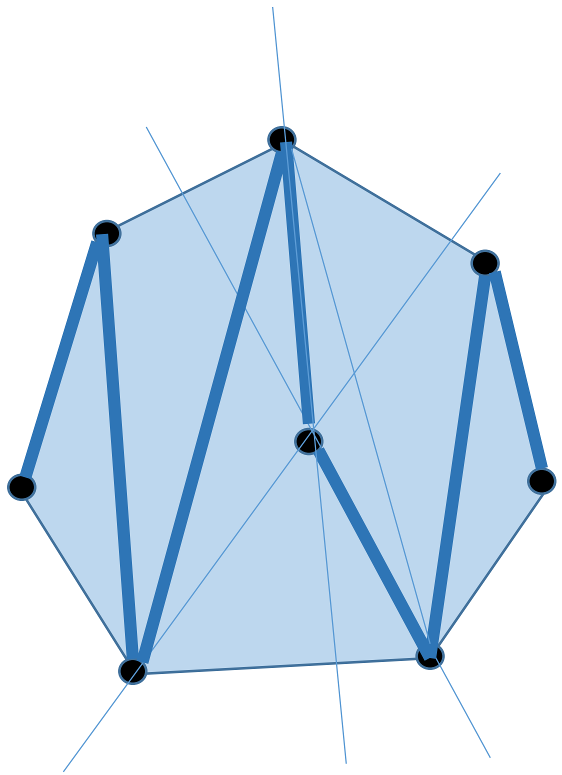
Proof:

Case C. S is in the wheel position.

An ad hoc construction shows that *S* has $(n\text{-}2)/2 \geq 3$ edge-disjoint plane spanning paths.



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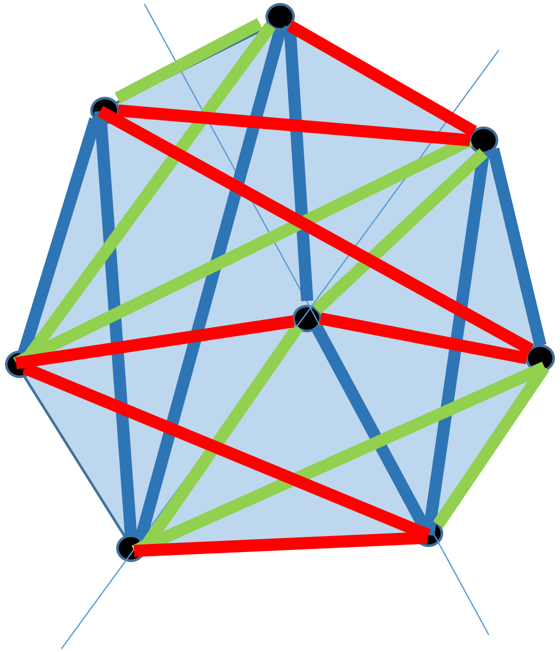

3 Paths

Final comment: All steps of the proof were constructive. Thus given a set *S* of at least 10 points, we can construct 3 edge-disjoint plane spanning paths for *S* in polynomial time.

HAPPY BIRTHDAY, ZSOLT!