CHARLES UNIVERSITY

FACULTY OF MATHEMATICS AND PHYSICS

STRUCTURE AND COMPLEXITY OF LOCALLY CONSTRAINED GRAPH HOMOMORPHISMS

Habilitation Thesis

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To my first mathematics teachers,
who showed me its beauty:

Hana Línová,
Eva Josífková,
and Karel Popp.

I would like to express my deep thanks to my colleagues and friends at the Department of Applied Mathematics and Institute for Theoretical Computer Science at Charles University in Prague for friendly an inspiring atmosphere that allowed me to concentrate on various interesting subjects in graph theorey and related fields. Even though I am aware that I cannot name all of them, I mention at least a dozen of names: Jan Kratochvíl, Jaroslav Nešetřil, Jiří Matoušek, Aleš Pultr, Martin Klazar, Pavel Valtr, Petr Kolman, Daniel Král', Tomáš Kaiser, Ondřej Pangrác, Jana Maxová, Martin Mareš and Robert Šámal.

Besides my closest colleagues at the department, I am bound to several foreign collaborators for fruitful discussions and joint work on interesting questions on graph homomorphisms — among others Andrzej Proskurowski from Eugene (Oregon), Jan Arne Telle from Bergen, Daniël Paulusma from Durham and Riste Škrekovski from Ljubljana.

Last but not least I thank my family for long lasting patience and everyday support. Without them my whole mathematical career would be impossible.

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Contents

Introduction

The subject of this thesis — locally constrained homomorphisms — indicates already by its name that it belongs to the field of discrete mathematics, in particular graph theory. Even though it might look relatively narrow focused at the first sight, it has various tight links to several other fields of mathematics stemming from algebra and mathematical structures via topology to more applied disciplines like the field of computational complexity, and in particular discrete optimization.

The thesis itself is aggregate of the following recently published articles:

Chapters 2 and 4 — Fiala, J., Paulusma, D., and Telle, J. A. Matrix and graph orders derived from locally constrained graph homomorphisms. In *MFCS* (2005), J. Jedrzejowicz and A. Szepietowski, Eds., vol. 3618 of *Lecture Notes in Computer Science*, Springer, pp. 340–351.

Chapter 3 — Fiala, J., and Maxová, J. Cantor-Bernstein type theorem for locally constrained graph homomorphisms. *European Journal of Combinatorics 7*, 27 (2006), 1111–1116.

Chapter 5 — Fiala, J., and Kratochvíl, J. Partial covers of graphs. *Discussiones Mathematicae Graph Theory 22* (2002), 89–99.

Chapter 6 — Fiala, J., and Kratochvíl, J. Complexity of partial covers of graphs. In *ISAAC* (2001), P. Eades and T. Takaoka, Eds., vol. 2223 of *Lecture Notes in Computer Science*, Springer, pp. 537–549.

Fiala, J., Kratochvíl, J., and Pór, A. On the computational complexity of partial covers of theta graphs. *Electronic Notes in Discrete Mathematics 19* (2005), 79–85.

Chapter 7 — Fiala, J., and Paulusma, D. A complete complexity classification of the role assignment problem. *Theoretical Computer Science 1*, 349 (2005), 67–81.

Chapter 8 — Fiala, J., Paulusma, D., and Telle, J. A. Algorithms for comparability of matrices in partial orders imposed by graph homomorphisms. In *WG* (2005), D. Kratsch, Ed., vol. 3787 of *Lecture Notes in Computer Science*, Springer, pp. 115–126.

Some of the results presented in Chapter 6 appeared in my PhD. thesis — FIALA, J. *Locally injective homomorphisms.* Charles University, Prague, 2000.

During the thesis compilation several notions that appear in the above articles and related literature under different names were unified together with symbols that are used here. In addition, an explanatory introduction was added as well as several examples and other supplementary material. The author confess that the text of the original manuscripts was literally transcribed into the thesis at several places.

History and motivation

Given two graphs G and H, a graph homomorphism is an edge preserving mapping between the underlying vertex sets $V_G \to V_H$.

A locally bijective homomorphism is required to be bijective between the neighborhood of every vertex of the source graph G and the neighborhood of its image in H.

Analogously we can define homomorphisms that are locally injective or locally surjective.

Let us briefly review history of the notion of locally constrained homomorphisms and their applications.

According to Boldi and Vigna [7] the first definition of a locally bijective homomorphism — called in that context *graph fibration* — can be contributed to Grothendieck [27]. He in late 50's translated the notion of fibration in homotopy theory into categorical terms. Independently, Sachs [57] in 1964 developed an equivalent notion of *graph divisors*, which was then intensively studied in the connection with the characteristic polynomial of a graph. Indeed, the fact that the chromatic polynomial of the quotient graph divides the characteristic polynomial of the original graph already became a folclore, see monographs of Cvetković, Doob and Sachs [12] or Godsil and Royle [23].

We traced one of the early occurrence of the notion of locally bijective homomorphism to Conway [4] who used it in early sixties to construct highly symmetric graphs, namely finite cubic 5 arc-transitive graphs. This approach was extended by Djoković [13] to a construction of a infinite class of finite fourregular 7-arc-transitive graph and by Gardiner [21] to the antipodal distance-regular graphs.

When G allows a locally bijective homomorphisms to a connected graph H, the cardinality of V_G is a multiple of $|V_H|$. If we denote the ratio $\frac{|V_G|}{|V_H|} = k$, we can say that G is a k-fold cover of H, or simply a *cover*.

The structure of the set of all k-fold covers of a given graph H was characterized in 1977 by Gross and Tucker [25] in terms of permutation voltage assignments in a symmetric group of k elements. A simpler characterization was given by Bodlaender [6] in 1989.

In 1984, Biggs [5] showed that covering graphs admit groups of automorphisms related to the group of the base graph. Hofmeister [32] in 1991 counted isomorphism classes of k-fold covers onto a fixed graph G.

In 1988, Negami [53] conjectured that the class of projective planar graphs is equal to the class of graphs that have a finite planar cover. The inclusion

$$\{H \text{ is projective planar}\} \subseteq \{H \text{ has a finite planar cover}\}$$

is trivial, but the opposite is difficult. So far the attempts to prove this conjecture involved Robetson-Seymour theorem of forbidden minors, which are known for the class of projective planar graphs. Only one of these 35 forbidden minors — namely the graph $K_{1,2,2,2}$ — resists to be shown that it allows no planar cover. However the conjecture is not proven yet, Hliněný and Thomas [31] in 1999 showed that the conjecture can allow at most 16 possible counterexamples (upto obvious constructions).

The locally bijective homomorphisms and graph covers became a standard construction in topological and algebraic graph theory, see monographs by Biggs and others [4, 26, 46].

Locally bijective homomorphisms have several applications in computer science.

Angluin [2] and also Angluin and Gardiner [3] showed in early 80's that classes of graphs closed under taking covers can not be recognized by a distributed computing environment with a finite fixed set of processor types. To prove the complete characterization, they conjectured that two graphs have a finite common cover if and only if they have the same degree refinement matrix, which was proved by Leighton [42] in 1982. In 1986 Mohar [51] adjusted this construction to classify the surface where the common cover can be embedded (depending on the surfaces hosting the underlying graphs).

Litovsky, Métivier and Zielonka [44] showed in 1993 that the families of series parallel graphs and planar graphs cannot be recognized by means of local computations. Courcelle and Métivier [11] exposed in 1994 other nontrivial minor-closed graph classes that cannot be recognized by local computations. Further models of local computations involving also locally injective and locally surjective homomorphisms were considered by Chalopin and Paulusma [8] in 2006.

Bodlaender [6] proved in 1989 that every cover G of a connected graph H is a uniform emulation, that means that a parallel algorithm designed for the processor network G can be emulated on H where each node of H corresponds to a constant number of nodes of G. The same paper provided the complete characterization of covers of the ring, the grid, the cube, the cube connected cycles, the tree and the complete graphs. Moreover it is shown there that the decision problem whether a graph G covers a graph H is at least as hard as the graph isomorphism problem, even if the ratio $\frac{|V_G|}{|V_H|}$ is fixed.

In the concluding remarks Bodlaender asked the computational complexity of the decision problem H-LBiHom, where the question is whether a given graph G (the instance) allows a locally bijective homomorphisms to a fixed graph H (the parameter of the problem). Abello, Fellows and Stillwell [1] showed in 1991 that there are both polynomially solvable and NP-complete cases. The series of paper by Kratochvíl, Proskurowski and Telle [38, 37, 39, 36] from late 90's exhibits several approaches to establish the most accurate boundary between the graphs for which the H-LBiHomproblem is polynomially solvable and the NP-complete instances of H-LBiHom. Several nontrivial infinite classes of both polynomial and NP-complete instances were recognized, however, currently there is no plausible conjecture concerning a good characterization of graphs H, for which the H-LBiHom problem is polynomially solvable, is at hand (assuming, of course, $P\neq$NP).

Kratochvíl, Proskurowski and Telle showed that sufficiently connected regular graphs belong to NP-complete instances for the H-LBiHom problem. Their proof requires the existence of a graph G which satisfies the following property: For all its vertices u, the graph G allows an extension of a local isomorphism on the neighborhood of u into a locally bijective homomorphism $G \to H$. The construction of this *multicover* G involves an algebraic method that generalizes the building of common covers used by Angluin and Gardiner [3] and Leighton [42].

The other two kinds of local constraints have also interesting history and several applications.

Nešetřil [54] showed already in 1971 that every locally injective mapping $G \to G$ of a connected graph G is an isomorphism of G.

In his tutorial from 1983, Stallings [60] mentioned that every locally injective homomorphism $G \to H$ can be extended to a locally bijective homomorphism $G' \to H$ for G' being a supergraph of G.

Locally injective homomorphisms were applied in a hardness proof for the existence of distance constrained labelings of graphs [19], a notion stemming from a highly practical problem of interference-free frequency assignment for wireless networks.

Locally surjective homomorphisms were introduced by Everett and Borgatti [15], who called them role colorings. They originated in the theory of social behavior. The graph H, so called the role graph, models roles and their relationships, and we ask whether roles (i.e. the vertices of the role graph) can be assigned to individuals of a given society such that the relationships are preserved: Each person playing a particular role has among its neighbors exactly all necessary roles as are prescribed by the model.

The essential part of this thesis considers so called *equitable partition* of a graph. This notion introduced Corneil in his PhD. thesis in 1968 [10, 9] as a heuristic for the graph isomorphism problem. It is worth to mention that it was independently discovered by McKay [49] in 1976 in his master's thesis but with giving credits to Hopcroft's paper from 1971 [33] for the routine of minimizing states of a finite automaton. Also note that Boldi and Vigna [7] give in their survey credits for the notion of equitable partition to Schwenk [58] and Mowshowitz [52].

The wealth of the idea of equitable partition can be documented in several ways. It soon became a folklore notion, so it appears without previous reference in many works [2, 42, 36, 23]. Secondly, and later implemented as a subroutine of a graph isomorphism software called Nauty by McKay [50, 48]. For further derived methods for the graph isomorphism problem and its application in chemistry, see e.g. a recent tutorial by Tinhofer a Klin [61].

A wider class of partitions was considered in 1994 by Everett and Borgatti for various models of social network theory. They prove also several structural results, e.g. that the relation *being finer* imposes a lattice on the set of all equitable partitions of a fixed graph (mentioned already by McKay [50] in 1981). A recent survey on results in this direction was given by Lerner [43] in 2005.

The outline of the thesis

In this thesis we will focus our attention on the relationship between the existence of a locally constrained homomorphisms — in particular the computational complexity of the related problems on one side and the structure of the degree partitions, captured mainly in terms of degree matrices.

The thesis is organized as follows:

In Chapter 2 the notion of an equitable partition and its degree matrix is defined and several basic properties, including recognition/computation of such matrices are shown. We also present equivalent characterizations of degree matrices, e.g., by conditions on the dimension of the cycle space of some matrix-related graph.

The Chapter 3 is motivated by the celebrated Cantor-Bernstein theorem showing that the simultaneous existence of a surjective and injective mapping between two set provides a sufficient condition for the existence of a bijection between these sets. We show that an analogous statement to Cantor-Bernstein theorem holds also for locally constrained homomorphisms, in particular that a simultaneous existence of a locally injective and a locally surjective homomorphisms assures that both are indeed locally bijective.

In Chapter 4 we focus on various partial orders imposed by the existence of a locally constrained homomorphisms. These orders naturally arose on different ground sets like graphs, degree matrices, etc. These results generalize the use of degree refinement matrices to locally injective and locally surjective homomorphisms. We emphasize that such a relationship was not originally expected, since such degree conditions are not obvious for the non-bijective local constraints.

We then turn our attention on the computational complexity of the related problems. First, we show some recent progress on the H-LInHom problem in Chapter 6. In the consequent chapter we provide full characterization of the H-LSurHom problem.

The concluding Chapter 8 explores the computational complexity of matrix comparison problems related to orders presented in Chapter 4. We fully settle the computational complexity of these decision problems.

We recall here that the existence of a locally bijective homomorphism between two graphs is can be conditioned ty the existence of an isomorphism between their universal covers [42]. For the other two kinds of locally constrained homomorphisms this naturally raises the question, and conjecture, of a similar tight relationship between matrix comparison in the partial order and inclusion of universal covers. As a corollary of the structural results of Chapter 8, we apply our characterization theorem to disprove this enticing conjecture.

The thesis in accompanied by an appendix which summarizes used decision problems together with references to related algorithms or NP-hardness proofs.

Chapter 1

Preliminaries

In this chapter, we present the used notation used in this thesis. For the field of discrete mathematics we follow the standards established in monographs by Harary [28] or Matoušek and Nešetřil [47]. The notation from the field of computational complexity complies with the classical book by Garey and Johnson [22].

1.1 Relations

Relation A R on a set X is a subset of the Cartesian product $R \subseteq X \times X$. For a relation R on X we write xRy if and only if $(x, y) \in R$. A relation R is said to be *reflexive* if for all $x \in X : xRx$. It is *symmetric* if $xRy \Rightarrow yRx$. A relation is *antisymmetric* if $xRy \wedge yRx \Rightarrow x = y$. Finally a *transitive* relation has the property $xRy \wedge yRz \Rightarrow xRz$.

A reflexive, symmetric and transitive relation is called an *equivalence*. If R is an equivalence on X, then the set $[x] = \{y : xRx\}$ is called the *equivalence class* of $x \in X$. Observe that the collection of all equivalence classes is a partition of X.

A relation R which is reflexive and transitive is called a *quasiorder*. If it is in addition antisymmetric, then it is a *partial order*. From any quasiorder R a partial order R' can be obtained by the following construction: Take the inclusion-wise maximal equivalence relation $S \subseteq R$ and define X' the set of all equivalence classes of X. Define R' on X' such that $[x]R'[y]$ iff xRy. As R is transitive the construction of the partial order R' is independent on the choice of elements x and y.

We say that $x \in X$ is a *maximal element* of a partial order R if $\forall y \in X : \neg(xRy)$. The *maximum element* of an partial order is $x \in X$ such that $\forall y \in X : yRx$. The maximum element may not exist, but if it exists it is unique. We may speak about maximal and maximum elements of a set $Y \subseteq X$ — they are the maximal and maximum element of the relation R restricted onto the set Y. The *supremum* of a set $Y \subseteq X$ in the partial order

R is the minimum element of the set $Z = \{x : y \in Y \Rightarrow yRx\}$.

Analogously we define an *minimum* and the *minimal* elements and the *infimum* of a set.

A *lattice* is a partially ordered set (X, R) where every 2-element subset $Y = \{x, y\} \subseteq X$ has the infimum, denoted by $x \wedge y$, and also the supremum, denoted by $x \vee y$.

A typical example of a lattice can be constructed as follows: for a set X take the set Y of all equivalence relations on X, ordered by inclusion. In other words, if $R \subseteq R'$ we say that R is finer than R' and it is equivalent with the condition $xRy \Rightarrow xR'y$ for all $x, y \in X$.

Observe that the above defined structure $(Y, \subseteq)$ is a lattice. The infimum is obtained straightforwardly $R \wedge R' = R \cap R'$. For the supremum $R \vee R'$ we take the minimum equivalence on X that contains $R \cup R'$, i.e. the *transitive closure* of $R \cup R'$.

Note that the equivalences, partial orders and other relations can be build also on classes instead of sets.

1.2 Graphs

For a set V we denote by $\binom{V}{2}$ the set of all unordered pairs from V, i.e. all it's 2-element subsets.

A *graph* G is a pair (V_G, E_G), where V_G is a set of so called *vertices* (or nodes, equivalently) and $E_G \subseteq \binom{V_G}{2}$ is a set of its *edges*.

We say that G is finite if it's vertex set V_G is finite and vice-versa. If not stated otherwise we denote a finite vertex set by $V_G = \{v_1, v_2, ..., v_n\}$. Here n is the *order* of the graph G. Similarly, m usually denotes the cardinality of the edge set, called *size* of G and we write $E_G = \{e_1, ..., e_m\}$.

For vertices and edges we often use small letters $u, v, \ldots$ and $e, f, \ldots$. An edge e containing two vertices u and v will be written as (u, v) to emphasize that u and v are distinct vertices (in contrary to the set-like notation $\{u, v\}$). In the above case we also say that u and v are *adjacent* and that u and v are *incident* with e.

1.2.1 Graph generalizations

The notion of graph can be extended in several ways: The edges may be formed out of ordered pairs and we get so called *directed graph* or shortly *digraph* $\vec{G} = (v, \vec{E})$. We call directed edges *arcs* and indicate them as $\vec{e} = [u, v]$ meaning that the edge is oriented from the vertex u towards v. Here u is the *tail* of $\vec{e}$ and v is its *head*.

Only distinct vertices can be connected by an edge in the definition of the graph. If we allow presence of edges that connect some vertices with themselves (called *loops*) and also occurrence of more edges connecting the same pair of vertices, we speak about a *multigraph*. A multigraph can be also directed in the above way, it is often plausible to have mixed both directed and undirected edges in the same structure. The *multiplicity* of an (un)directed edge or loop is the number of its occurrences. Formally we denote a multigraph as a 5-tuple $G = (V, E, \vec{E}, L, \vec{L})$ where E and $\vec{E}$ are multisets of undirected edges and directed edges, and similarly, L and $\vec{L}$ are the analogous multisets of loops.

If we intend to emphasize that a graph has no directed or multiple edges and also no loops, we say that it is a *simple graph.* The notion of *bidirected graph* will be reserved only for directed multigraphs where two adjacent vertices are connected by edges in both directions and all present edges and loops have multiplicity one.

Graphs and multigraphs are often depicted by drawings, where vertices are represented by points in the plane, and edges are drawn as arcs that connect the adjacent pairs of vertices. Edge orientations can be indicated by an arrows, see Fig. 1.1.

We can further allow edges to contain more than two vertices. In such a case we obtain *hypergraph* $H = (V_H, E_H)$ where $E \subseteq \mathcal{P}(V_H)$ is the set of its *hyperedges.* It is often assumed that hyperedges are distinct unordered r-tuples. These hypergraphs are called r*-uniform.* In the above setting, simple graphs are just 2-uniform hypergraphs.

1.2.2 Local properties

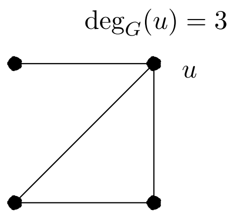
For a vertex u in a graph G we call the set of all vertices adjacent to u the *neighborhood* of u and denote it by $N_G(u)$. In this sense a vertex v is a *neighbor* of u if u and v are adjacent. The set $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u. To avoid confusion when both types of neighborhood are used the former one may be also called *open neighborhood.*

We mention here that even though in the thesis are considered infinite graphs (multigraphs, hypergraphs) we restrict ourselves only to those with finite degrees. The *degree* of a vertex $u \in V_G$ is the cardinality of the set of edges incident with it, and is denoted by $\deg_G(u)$. Note that in a simple graph this is exactly the cardinality of its neighborhood, while in a multigraph we have to encounter multiplicity of each edge incident with u.

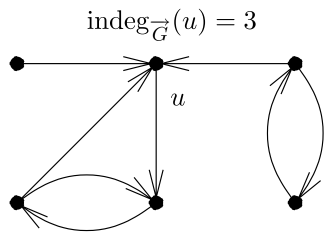
For directed (multi)graphs we similarly establish the *indegree* $\mathrm{indeg}_{\vec{G}}(u)$ as the cardinality of the set of arcs oriented towards the given vertex, and analogously the *outdegree* $\mathrm{outdeg}_{\vec{G}}(u)$ for the number of outgoing arcs. In a mixed hypergraph the degree of a vertex is then often defined via undirected edges only.

Each undirected loop is encountered twice in the degree of the incident vertex, while a directed loop increments both indegree and outdegree by one.

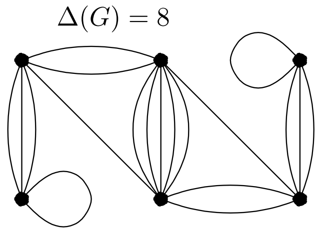
A graph is k*-regular* if it consists only of vertices of degree k. A 3-regular graph is also called a *cubic graph.*



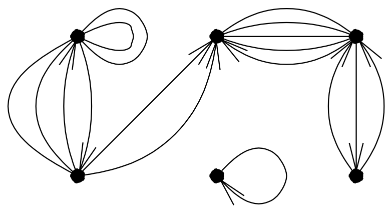
a) a simple graph of order $n = 6$
and size $m = 5$



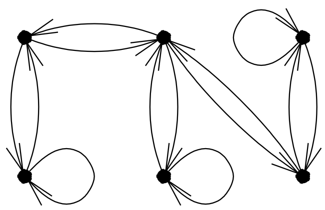
b) a directed graph



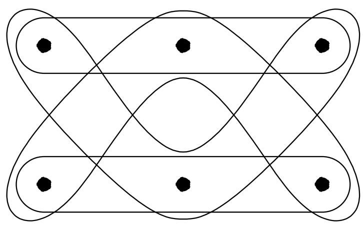
c) an undirected multigraph



d) a multigraph with both edge types



e) a bidirected graph



f) a 3-uniform 2-regular hypergraph

Figure 1.1: Graph and multigraph examples

If all degrees in an undirected multigraph G are bounded, we may define its *maximum degree* $\Delta(G)$. Similarly $\delta(G)$ stands for the *minimum degree* of a graph G..

The *adjacency matrix* A_G of a finite undirected multigraph G with vertex set $\{v_1, \ldots, v_n\}$ is an $n \times n$ matrix, where the entry $(A_G)_{i,j} = a_{i,j}$ is equal to the multiplicity of the edge (v_i, v_j). The matrix is symmetric for undirected graphs and it is 0,1 valued for simple graphs. If the graph does not contain a loop, then all entries on the diagonal are zeros. (This notion can be extended to directed mutigraphs, but since we will not use it in this thesis, we omit the specification here.)

1.2.3 Global properties

We say that a graph $H = (V_H, E_H)$ is a *subgraph* of a graph $G = (V_G, E_G)$ if it consists of some vertices and some edges of G, i.e. $V_H \subseteq V_G$ and $E_H \subseteq E_G$. If H is a subgraph of G, we write $H \subseteq G$. When E_H consists of all edges of G that connect vertices from H, we say that H is an *induced subgraph* of G, formally $E_H = E_G \cap \binom{V_H}{2}$.

If a subgraph H contains all vertices of G, it is called a *factor*. A k-regular factor is called a k-factor. A 1-regular subgraph is called a *matching*, while 1-factor is *perfect matching*.

The graph consisting from the edges not presenting in a given graph G is called the *complement* $\overline{G}$ of G, formally $\overline{G} = (V_G, \binom{V_G}{2} \setminus E_G)$.

The *empty graph* is a graph with no edges $(\{v_1, \ldots, v_n\}, \emptyset)$ and the *complete graph* is it's complement. In other words every pair of vertices in the complete graph is adjacent. The complete graph on n vertices is denoted by K_n.

The vertex set of a *bipartite graph* is formed by the disjoint union of two sets (blocks of bipatition), where edges connect only vertices from different blocks. If all such edges are present and the two blocks are of sizes n and n' then the graph is called *complete bipartite graph* $K_{n,n'}$. It can be viewed also as the complement of the disjoint union of graphs K_n and $K_{n'}$. The *star* S_n is equivalent to the graph $K_{1,n}$, i.e., it contains n vertices of degree one connected to a single central vertex. If vertices of a bipartite graph are of degree k in one block of the bipatition and are of degree l in the other block, we call the graph a *(k, l)-semiregular graph*.

A sequence of vertices of a graph G, such that the consecutive pairs are adjacent, is called a *walk* in G. If all vertices in a walk are distinct, then it is called a *path*.

The length of a finite walk (path) is measured in the number of edges traversed, i.e., a walk of length k consists of $k + 1$ vertices.

A finite path of length at least three, where only the first and the last vertex coincide, is called a *cycle*. The length of the shortest cycle in a graph G is called *girth* and is denoted by $\operatorname{girth}(G)$.

We also use the name path of length $n-1$ for the graph P_n consisting of n vertices $v_1, \ldots, v_n$ and edges $E_{P_n} = \{(v_i, v_{i+1}) \mid i = 1, \ldots, n-1\}$. Analogously, for $n \geq 3$ the cycle of length n, denoted by C_n, is the graph formed from P_n by adding the edge (v_1, v_n).

A graph G is said to be *connected* if for every pair of vertices u, v there exists a finite path starting in u and ending in v. The length of the shortest path connecting u and v is the *distance* $\mathrm{dist}_G(u, v)$ between vertices u and v. Since the distance satisfies the triangle inequality in every connected graph G, it also imposes a natural metric on V_G.

When G is disconnected, the inclusion-wise maximal subgraphs of G are called *components*. In many cases we will assume that vertices of G are ordered such that vertices of the same component come in one block. This yields that the adjacency matrix of a disconnected graph can be split into main submatrices which are adjacency matrices of the components.

The greatest distance between a pair of vertices of a finite connected graph G is called the *diameter* of G, and is indicated by $\mathrm{diam}(G)$.

We say that a simple graph G is *vertex k-connected*, if for every pair of vertices u, v there exist at least k paths connecting u and v, and these paths are pairwise disjoint on their inner vertices.

Analogously, a simple graph G is *edge k-connected*, if for every pair of vertices u, v, at least k edge disjoint paths join u to v.

The maximal vertex 2-connected induced subgraphs of a graph G are called *blocks* of G.

A set of vertices $V' \subset V_G$ is called the *cutset* of G, if the subgraph spanned on $V_G \setminus V_{G'}$ has more components than G.

A set of edges $E' \subset E_G$ is called the *edge cutset* of G, if (V_G, E') has more components than G.

An one-vertex cutset is called an *articulation* or a *cutvertex*. An edge-cutset of size one is called a *bridge*.

A graph which contains no cycle is called a *forest*. A connected forest is a *tree*. It follows from the above definitions that in a tree for any pair of given two vertices the path that connects them always exists and is unique. A tree, which is also factor of a given graph G, is called *spanning tree* of G.

A *unicyclic graph* contains only one cycle as a subgraph. A *cactus* is a graph where each block is a cycle.

1.2.4 Graph morphisms

A *homomorphism* from a graph $G = (V_G, E_G)$ to $H = (V_H, E_H)$ is a vertex mapping $g : V_G \to V_H$ satisfying the property that for any edge (u, v) in E_G, we have $(g(u), g(v))$ in E_H as well. In other words, $g(N_G(u)) \subseteq N_H(g(u))$ holds for all vertices $u \in V_G$. To indicate homomorphisms we use the arrow notation $f : G \to H$, or also shortly $G \to H$, when only the existence of such a mapping is essential. Frequently, we use the fact that the composition of

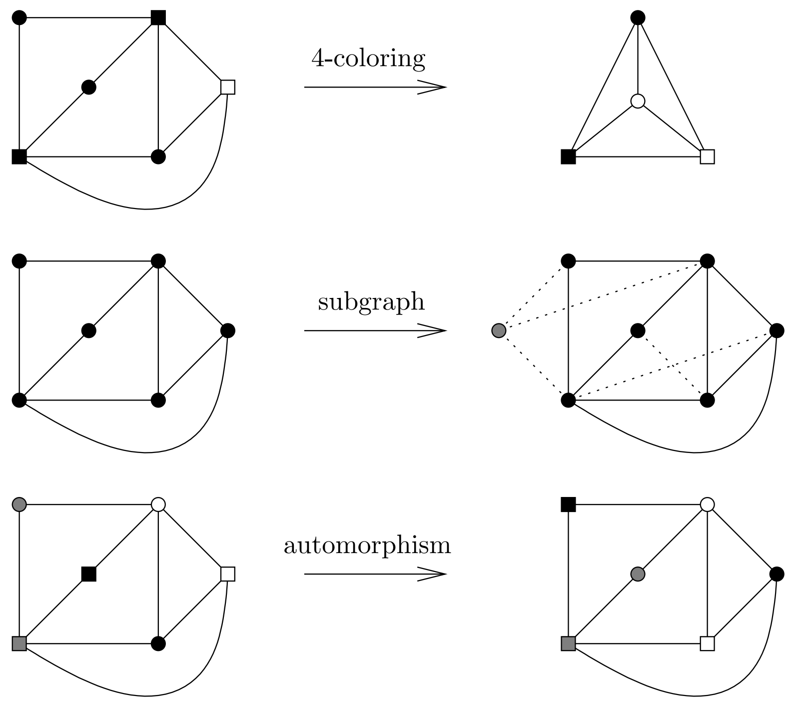


Figure 1.2: Examples of special graph homomorphisms
In the first and in the last case, the mappings are indicated by vertex shapes.

two homomorphisms $f : F \to G$ and $g : G \to H$ provides a homomorphism $g \circ f : F \to H$.

Examples of graph homomorphism are depicted in Fig. 1.2. Note that homomorphisms the complete graphs corresponds to proper graph colorings. Any such $f : G \to K_k$ assigns each vertex of G one of the k vertices of K_k (a color), such that adjacent vertices have distinct images. Therefore homomorphisms $f : G \to K_k$ are often called k-colorings, while with a general target graph H they are sometimes named H-colorings. Note that graphs that allow a homomorphism to K_2 are exactly bipartite graphs.

We further distinguish two special cases: When a homomorphisms $f : G \to H$ is an one-to-one mapping where f^{-1} is a homomorphism as well, we call f an *isomorphism.* In such case we say that G and H are isomorphic and denote it by $G \simeq H$.

Isomorphic graphs have the same structure since they differ only in the elements used to distinguish vertices. The existence of an isomorphism provide an equivalence relation on the class of all graphs. When dealing with structural properties of graphs it is sufficient to examine only one graph from each equivalence class. We call this collection of distinct graphs the class of nonisomorphic graphs. Its subclass consisting only of finite and connected nonisomorphic graphs is denoted by $\mathcal{G}^c$.

An isomorphism $f : G \to G$ is called an *automorphism*. It follows directly from the definition that the set of automorphisms of a graph G together with the operation of composition provide so called *automorphism group*. We denote this group by $\mathrm{Aut}(G)$ and its neutral element — the identity — by *id*.

For two graphs G and H the *product graph* $G \times H$ is the graph with the vertex set $V_{G\times H} = V_G \times V_H$. The set of edges $E_{G\times H}$ is defined such that $((u, v), (x, y)) \in E_{G\times H}$ if and only if $(u, x) \in E_G$ and $(v, y) \in E_H$.

Observe that the *projections* $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$, defined by $\pi_1(u, v) = u$ and $\pi_2(u, v) = v$, are both graph homomorphisms.

1.3 Computational complexity

A problem L in complexity theory is defined as a set of binary words. The standard *decision problem* for L means the question whether a binary word w belongs to L or not.

The computational time complexity of a decision problem L is defined as the minimal function $f(n)$, where $f(n)$ is the upper bound on the number of steps such that a hypothetical Turing machine needs to correctly answer the question whether $w \in L$ for every binary word of length at most n.

Frequently, it is sufficient to know only the asymptotic growth rate of the function $f(n)$. Hence, the time complexity is expressed in O-notation with the following sense:

$$\begin{aligned}
f = O(g) &\iff \exists c, n_0, \forall n \geq n_0 : f(n) \leq c \cdot g(n) \\
f = o(g) &\iff \lim_{n\to\infty} f(n)/g(n) = 0 \\
f = \Omega(g) &\iff \exists c, n_0, \forall n \geq n_0 : f(n) \geq c \cdot g(n) \\
f = \omega(g) &\iff \lim_{n\to\infty} g(n)/f(n) = 0 \\
f = \Theta(g) &\iff \exists c, c', n_0, \forall n \geq n_0 : c \cdot g(n) \leq f(n) \leq c' \cdot g(n)
\end{aligned}$$

The meaning is that O and o provide asymptotic upper bounds on the function f, while Ω, ω are lower bounds. Moreover, the small letters indicate that the growth is much faster or much slower. The last symbol Θ indicates the case when the bound is asymptotically tight.

For the problems related to discrete objects, like graphs and integer matrices, it is natural to obey the above setting in the following manner.

Instead of designing a Turing machine we describe an algorithm in a high level procedural language involving conditions and loops. We assume frequently that elementary arithmetic operations and tests can be executed in constant time (therefore omitting a logarithmic factor in the running time). We also allow elementary set operations and tests (membership, deletion, union, etc.). As these may affect significantly the running time, we will discuss them specifically with each algorithm.

The size of an instance will be denoted by $\langle\cdot\rangle$ meaning the length of the shortest word that can encode it.

According to the above needs we assume that a graph is encoded as a binary word whose length is proportional to the number of vertices and edges of the given graph. (Again, omitting a logarithmic factor that is needed to encode vertices' numbers). Therefore for the *size* of a graph with n vertices and m edges viewed *as an instance* of a decision problem is $\langle G\rangle = O(n+m)$ (we hope that no confusion arises with the notion of the size of a graph that express the number of edges). As connected graphs have at least $n-1$ edges we can express the running time only in m as because in this case $\langle G\rangle = O(m) = O(n^2)$.

When the matrix elements are only subject to arithmetic operations then the size of a matrix of order $k \times l$ is equal to $\langle M\rangle = O(kl)$. If an algorithm uses integers stored in the matrix to control the flow of the computation, we have to measure matrix size more precisely. Hence, if $m_{i,j}$ is its element with the maximum absolute value, we assume $m_{i,j} \neq 0$ and define $\langle M\rangle = O(kl(1+\log|m_{i,j}|))$.

1.3.1 Traditional complexity classes

The traditional division on tractable and hard problems is based on the classification whether they allow an algorithm running in a polynomial time or not. Formally, we define the class P containing all problems whose running time can be bounded by a polynomial function of the size of the input. A proof of the fact $L \in \mathsf{P}$ is usually done via construction of an algorithm together with a proof of its time complexity. These algorithns often use problems from P as subroutines. A catalogue of some polynomially solvable problems uses in this thesis is in the Appendix.

A further step in time complexity hierarchy is the class NP which contains all problems L with the following property: there exists a polynomial p such that a word w of length n belongs to L if and only if there exists a word w' of length $p(n)$ such that $(w, w') \in L'$ for some problem $L' \in \mathsf{P}$. The existence of the word w' is often expressed as a witness of the membership $w \in L$. The abbreviation NP stands for *nondeterministically polynomial* since to design an algorithm for L it suffices first nondeterministically guess w' and then verify $(w, w') \in L'$ in polynomial time.

The following inclusion is straightforward from the definition $\mathsf{P} \subseteq \mathsf{NP}$ It is widely expected that $\mathsf{P} \neq \mathsf{NP}$ and this open problem is the prominent challenge in contemporary theoretical computer science.

To convince that the time complexity of some problem cannot be asymptotically bounded by a slowly growing function (like polynomials) a concept of polynomial time reductions and hard problems has been established.

We say that some problem L' can be *polynomially reduced* to a problem L (and write $L' \propto L$) if there exists an algorithm (formally a Turing machine) that transforms every word w' in time polynomial in the size of w' into a word w such that $w \in L \iff w' \in L'$. Then the time complexity of L is not smaller than the time complexity of L' upto a multiplicative polynomial factor. The question whether $w' \in L'$ can be decided if one performs the reduction and then asks whether $w \in L$, since the answer is correct also for the original question whether $w' \in L'$.

We define that a problem L is hard for some class of problems $\mathcal{L}$ (and write that L is $\mathcal{L}$-hard) if every problem $L' \in \mathcal{L}$ can be polynomially reduced to L. Moreover, if L itself belongs to $\mathcal{L}$ we say that it is $\mathcal{L}$-complete.

In this thesis we will consider the class of NP-complete problems as the class of difficult problems (even though there are problems known with higher complexity). The assumption $\mathsf{NP} \neq \mathsf{P}$ is then equivalent that none of the NP-complete problems allows a polynomial time algorithm which complies with our original classification of hard and tractable problems. If we follow the above guidelines showing that some problem L is NP-complete it's enough to provide a polynomial time reduction *from* an NP-complete problem L' (i.e. $L' \propto L$) and also show existence of a witness to get $L \in \mathsf{NP}$. As a classical problem we mention here satisfiability of Boolean formulae (SAT). A collection of this and other NP-complete problems is presented in the Appendix.

1.4 Few concepts from linear algebra

We use $\mathbb{N}$ for the set of natural numbers, and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ resp. for the sets of integers, real and complex numbers, respectively. If $n \in \mathbb{N}$, then $[n]$ denotes the set $\{1, 2, ..., n\}$.

For p being a prime, the symbol $\mathbb{Z}_p$ stands for the ring of residues modulo p.

The kernel and the rank of a matrix M over given ring are denoted by $\ker(M)$ and $\mathrm{rank}(M)$. The transpose of a matrix M is denoted by M^T.

For a graph G with m edges we represent each edge $e \in E_G$ by a unit vector in the vector space $\mathbb{Z}_2^m$, called the *edge space* $\mathcal{E}_G$ of the graph G. Each vector in this space is the characteristic vector for a set of edges of a subgraph of G.

The *cycle space* $\mathcal{S}_G$ of G is the linear subspace of $\mathcal{E}_G$ generated by vectors corresponding to all cycles in G. Vectors in the cycle space corresponds to subgraphs, where every vertex has an even degree. We denote the dimension of a linear subspace $\mathcal{S}$ by $\dim(\mathcal{S})$. Let T be a spanning tree of a graph G. For every edge e not in T there is a unique cycle in the graph $T + e$. Since there are $|E_G| - |V_G| + 1$ of these edges in G, it follows that $\dim(\mathcal{S}_G) = |E_G| - |V_G| + 1$.

For an integer valued $k \times l$ matrix M define $m^* := 2 + \max\{|M_{i,j}| \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq l\}$.

We conclude this section with a proof of a technical lemma on solutions of systems of linear equations:

Lemma 1.1. *Let M be an integer valued $k \times l$ matrix with $l > k$. If $M\mathbf{x} = \mathbf{0}$ allows a nontrivial nonnegative solution, then it allows a nontrivial nonnegative integer solution $\mathbf{x}$ with at most $k+1$ nonzero entries and with $\langle x_i \rangle = O(k \log(km^*))$ for each entry x_i.*

Proof. If a solution $\mathbf{x}$ with more than $k+1$ positive coefficients exists, then the columns corresponding to $k+1$ of these variables are linearly dependent. Let the coefficients of such a linear combination form a vector $\mathbf{x}'$. Obviously $M\mathbf{x}' = \mathbf{0}$, but the entries of $\mathbf{x}'$ may not be necessarily nonnegative.

Without loss of generality we assume that at least one of the entries in $\mathbf{x}'$ is positive. Then, for a suitable value $\alpha = -\min\{\frac{x_i}{x'_i} \mid x'_i > 0\}$ the vector $\mathbf{x} + \alpha\mathbf{x}'$ is also a nontrivial nonnegative solution with more zero entries than $\mathbf{x}$.

By repeating this trimming iteratively we obtain a nontrivial nonnegative solution with at most $k+1$ nonzero entries. As the other entries are zero, we may restrict the matrix M to columns corresponding to nonzero entries of the solution. It may happen that the rank of the modified matrix decreases. Then we reduce the number of rows until the remaining ones become linearly independent. By repeating the above process we finally get an $k' \times (k'+1)$ matrix N of rank $k' \leq k$, such that $N\mathbf{y} = \mathbf{0}$ allows a nontrivial nonnegative solution $\mathbf{y}$. Such $\mathbf{y}$ can be extended to a solution $\mathbf{x}$ of the original system by inserting zero entries.

Without loss of generality we assume that the first k' columns of N are linearly independent, and we arrange them in a regular matrix R. Then its inverse can be expressed as $R^{-1} = \frac{\mathrm{adj}(R)}{\det(R)}$, where $\mathrm{adj}(R)$ is the adjoint matrix of R. By the determinant expansion we have that $\det(R) \leq k'!(m^*)^{k'} \leq k!(m^*)^k \leq k^k(m^*)^k$. Then we find that $\langle \det(R) \rangle = O(k \log(km^*))$. Each element of $\mathrm{adj}(R)$ is a determinant of a minor of R and hence is smaller than $(k-1)^{k-1}(m^*)^{k-1}$.

Now consider the integral valued matrix $N' = \det(R) \cdot R^{-1}B$. Then

- $\mathbf{y}$ is a solution of $N'\mathbf{y} = \mathbf{0}$ if and only if $N\mathbf{y} = \mathbf{0}$.
- The first k' columns of N' form the matrix $\det(R) \cdot I_{k'}$.
- In the last column the entries $z_1, \ldots, z_l$, are all negative (if $\det(R) > 0$) or all positive (otherwise).

If $\det(R) > 0$ then $\mathbf{y} = (-z_1, \ldots, -z_{k'}, \det(R))$ is a nonnegative nontrivial integral solution to $N\mathbf{y} = \mathbf{0}$. In the other case we swap the sign and choose $\mathbf{y} = (z_1, \ldots, z_{k'}, -\det(R))$. As each $z_i \leq ka^* \max_{i,j}(\mathrm{adj}(R)_{i,j}) \leq k^k(m^*)^k$, we obtain $\langle z_i \rangle = O(k \log(km^*))$, which concludes the proof. □

Chapter 2

Degree matrices

2.1 Equitable partition and degree matrices

Any locally bijective graph homomorphism, with graph isomorphism as a special case, preserves not only vertex degrees but also degrees of neighbors and degrees of neighbors of these neighbors and so on. To capture this property the following notions have been defined [9, 24].

Definition 2.1. A *equitable partition* of a graph G is a partition of the vertex set V_G into blocks $\mathcal{B} = \{B_1, \dots, B_k\}$ such that whenever two vertices u and v belong to the same block B_i, then for any $j \in \{1, \dots, k\}$ we have $|N_G(u) \cap B_j| = |N_G(v) \cap B_j| = m_{i,j}$. The $k \times k$ matrix M such that $(M)_{i,j} = m_{i,j}$ is a *degree matrix*.

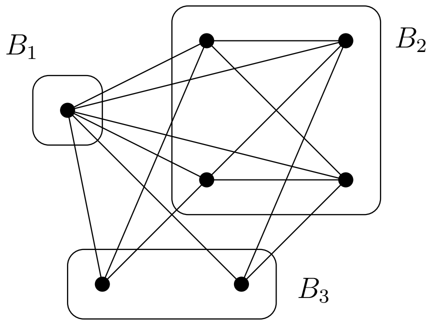
Examples of equitable partitions of some graphs and the matrices are depicted in Fig 2.1.

Observe that any permutation of rows and columns of a degree matrix is again a degree matrix corresponding to the same partition of G, only the blocks are ordered in a different manner. A finite graph G can allow several distinct equitable partitions, with an adjacency matrix itself being associated with the largest one. *Degree refinement matrices*, which will be considered in the next section, are on theother extreme.

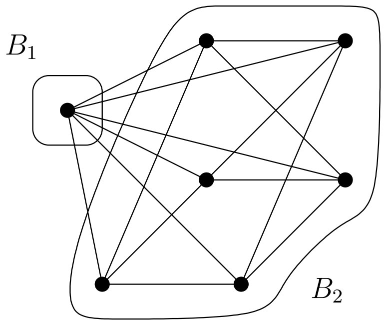
We denote the set of all degree matrices of a graph G by $\mathcal{M}_G$. The set of all degree matrices of finite connected graphs is denoted by $\mathcal{M}^c$.

The number of degree matrices of a single graph can be surprisingly large. For example, any partition of vertices of a complete graph K_n is an equitable partition and the number of non isomorphic partitions is equal to the number of possible partition on n into summands whose order of magnitude is $\exp(O(\sqrt{n}))$ [62].

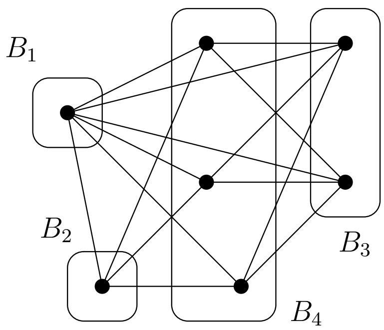
It depends essentially on the equitable partition, how tightly the degree matrix describes the graph. For a k-regular graphs the comparison is most striking: the adjacency matrix describes all edges of the graph. On the other



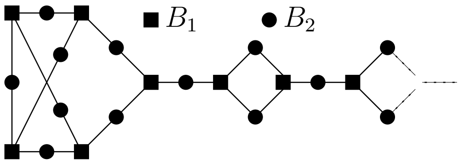
$$M = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$



$$M = \begin{pmatrix} 0 & 6 \\ 1 & 3 \end{pmatrix}$$



$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$



$$M = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

Figure 2.1: Examples of equitable partitions and degree matrices

hand the matrix coming from the equitable partition with a single block is an one-element matrix (k).

Note also that for an infinite graph an equitable partition may not exist. The example is the infinite path $(\mathbb{N}, \{(i, i+1) \mid i \in \mathbb{N}\})$. In this graph the vertex 1 is the only of degree one, hence must be in its own class, and any other vertex can be distinguished by its distance to the vertex 1. Such infinite graphs therefore have no degree matrix.

We continue with characterization of degree matrices. The first observation is that whenever in an equitable partition $N_G(u) \cap B_j = \emptyset$ for some $u \in B_i$, it must be also $N_G(v) \cap B_i = \emptyset$ for any $v \in B_j$

This immediately implies that for any degree matrix M of size k,

$$m_{i,j} = 0 \text{ if and only if } m_{j,i} = 0 \text{ for all } 1 \leq i < j \leq k.$$

We call integer matrices that have the above property *0-symmetric.* Note that there exist nonnegative 0-symmetric matrices that are not degree matrices of any *finite* graph. Consider the following matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Observe that in any graph the vertices of any pair of blocks B_i, B_j induce a $(m_{i,j}, m_{j,i})$-semiregular bipartite graph. When the graph is finite the block sizes are related by $m_{i,j}|B_i| = m_{j,i}|B_j|$. Hence, the entries first row and column justify that sizes of the three blocks are the same $|B_2| = |B_1 = |B_3|$ while the entries $m_{2,3}$ and $m_{3,2}$ yield $|B_2| = 2|B_3|$, which is impossible.

This makes the following decision problem interesting.

DEGREE MATRIX DETERMINATION (DMD)
Instance: A square matrix M.
Question: Is M a degree matrix of a finite graph G?

To determine the complexity of DMD we use some special weighted bidirected graphs. Let $w : E_{\vec{G}} \to \mathbb{N}$ be a positive weight function defined on the arc set of a bidirected graph $\vec{G}$. We say that a cycle $v_1, v_2, \ldots, v_c, v_1$ in such a graph $\vec{G}$ has the *cycle product identity* if

$$\left(\prod_{i=1}^{c-1} \frac{w[v_i, v_{i+1}]}{w[v_{i+1}, v_i]} \right) \frac{w[v_c, v_1]}{w[v_1, v_c]} = 1,$$

In other words, a cycle has the cycle product identity if the product of arc weights going clockwise around the cycle is the same as the product counterclockwise. We say that the bidirected graph $\vec{G}$ has the cycle product identity if every cycle of $\vec{G}$ has the cycle product identity. Using induction on the cycle length immediately yields:

Observation 2.2. *A weighted bidirected graph $\vec{G}$ has the cycle product identity if and only if every induced cycle of $\vec{G}$ has the cycle product identity.*

Proof. The proof goes by induction on the length c of a cycle C in $\vec{G}$. If $c = 3$ then the cycle is certainly induced. Assume that $c \geq 4$ and that the cycle $C = v_1, v_2, \ldots, v_c, v_1$ is not induced. Then we find a pair of arcs $[v_i, v_j]$ and $[v_j, v_i]$ for some indices $i, j : 2 \leq |i-j| \leq c-2$. These two arcs split the cycle C into two smaller cycles $C_1 = v_1, v_2, \ldots, v_i, v_j, v_{j+1}, \ldots, v_c, v_1$ and $C_2 = v_i, v_{i+1}, \ldots, v_j, v_i$. Note that the product of edge weights clockwise around the cycle C is equal to the the product of edge weights clockwise around the cycles C_1 and C_2 divided by $w[v_i, v_j]w[v_j, v_i]$. Likewise the product of edge weights counter-clockwise around C is equal to the product of counter-clockwise products around cycles C_1 and C_2 divided by $w[v_i, v_j]w[v_j, v_i]$. By induction we conclude that the cycle C has the cycle product identity. □

Weighted bidirected graphs are in one-to-one correspondence with 0-symmetric matrices. For such a matrix M we define the weighted bidirected graph $\vec{F}_M$ as follows: Its vertex set $V_{\vec{F}_M}$ consists of vertices $\{v_1, \ldots, v_k\}$. There is an arc $[v_i, v_j]$ with weight $m_{i,j}$ if and only if $m_{i,j} \neq 0$. In the opposite direction a from weighted bidirected graph $\vec{G}$ the corresponding 0-symmetric matrix can be transformed from the adjacency matrix by replacing ones by the arc's weights. See Fig. 2.2 for an example.

For the solution of the DMD problem we present here two auxiliary structures. First, let F_M be the underlying simple graph of $\vec{F}_M$, i.e., $V_{F_M} = V_{\vec{F}_M} = \{v_1, \ldots, v_k\}$ and (v_i, v_j) is an edge of F_M, whenever $i \neq j$ and $[v_i, v_j]$ and $[v_j, v_i]$ are arcs of $\vec{F}_M$ (note that all loops of $\vec{F}_M$ are automatically omitted).

Then we define the *weighted incidence matrix IM* to be the real $|E_{F_M}| \times k$ matrix whose rows are indexed by edges of F_M and its content is defined by:

$$\text{For } e = (v_i, v_j) \text{ where } i < j \text{ let } (IM)_{e,l} = \begin{cases} m_{i,j} & \text{when } l = i, \\ -m_{j,i} & \text{for } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

We now present our characterization of degree matrices of connected graphs.

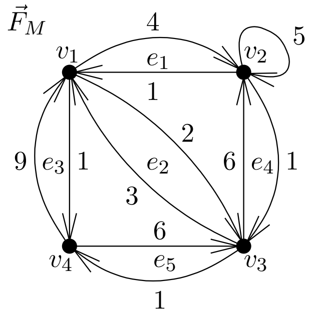
Theorem 2.3. *The following statements are equivalent:*

(i) *M is a degree matrix of a graph $G \in \mathcal{C}$.*

(ii) *$\vec{F}_M$ is a connected weighted bidirected graph satisfying the cycle product identity.*

(iii) *M is 0-symmetric and* $\dim(\ker(IM)) = 1$.

$$M = \begin{pmatrix} 0 & 4 & 2 & 1 \\ 1 & 5 & 1 & 0 \\ 3 & 6 & 0 & 1 \\ 9 & 0 & 6 & 0 \end{pmatrix}$$



$$IM = \begin{pmatrix} 4 & -1 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 1 & 0 & 0 & 9 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 1 & -6 \end{pmatrix}$$

Figure 2.2: Example of a 0-symmetric matrix M and the corresponding weighted bidirected graph $\vec{F}_M$ and also of its weighted incidence matrix IM

(iv) M is 0-symmetric and $\dim(\ker(IM^T)) = \dim(\mathcal{S}_{F_M})$.

Proof. *(i)* $\Rightarrow$ *(ii)* Since M is a degree matrix, it is 0-symmetric and $\vec{F}_M$ is a weighted bidirected graph. As the underlying graph G is connected, the graph $\vec{F}_M$ is connected as well.

Let $C = v_1, \ldots, v_c, v_1$ be a cycle in $\vec{F}_M$. where each vertex v_i corresponds to the block B_i of the equitable partition $\mathcal{B}$. As G is finite, we have $m_{i,j}|B_i| = m_{j,i}|B_j|$ for any pair of arcs $[v_i, v_j], [v_j, v_i]$ of C. Then we write:

$$\left(\prod_{i=1}^{c-1} \frac{w[v_i, v_{i+1}]}{w[v_{i+1}, v_i]}\right) \frac{w[v_c, v_1]}{w[v_1, v_c]} = \left(\prod_{i=1}^{c-1} \frac{m_{i,i+1}}{m_{i+1,i}}\right) \frac{m_{c,1}}{m_{1,c}} = \left(\prod_{i=1}^{c-1} \frac{|B_{i+1}|}{|B_i|}\right) \frac{|B_1|}{|B_c|} = 1$$

and see that C satisfies the cycle product identity.

(ii) $\Rightarrow$ *(iii)* Since $\vec{F}_M$ is an weighted bidirected graph, M is 0-symmetric. Take a real vector $\mathbf{b} \in \ker(IM)$ such that $b_1 = 1$. If $e = (v_i, v_j)$ is an edge of F_M then the e-th row of IM provides $b_j = \frac{m_{i,j}}{m_{j,i}} b_i$. Now consider a path P_{1l} in $\vec{F}_M$ from the vertex v_1 to any vertex v_l corresponding to the l-th row of M. Walking along the path and by multiplying the above fractions for consecutive vertices we see that the value of b_l is uniquely determined. Moreover, as $\vec{F}_M$ satisfies the cycle product identity, we get the same number also via any other path between v_1 and v_l. As the graph $\vec{F}_M$ is connected, we can uniquely determine the value of any other coordinate in the vector $\mathbf{b}$ from the value of b_1. Hence, any vector from $\ker(IM)$ is a multiple of the vector $\mathbf{b} = (b_1, \ldots, b_k)$ and $\ker(IM)$ has dimension one.

(iii) $\Rightarrow$ *(i)* We first determine the block sizes of a candidate graph G. We do this with respect to the following two facts.

Claim 2.4. *(1) For $p \geq 1$ there exists a p-regular graph on n vertices if and only if $n \geq p + 1$ and np is even.*

For the existence one can involve well known theorems about graph factors (see e.g. [28]). Either n is even and then K_n can be factorized into $n-1$ disjoint perfect matchings. Any union of p such matchings provides a p-regular graph. Or n is odd and then K_n can be fatorized into $\frac{n-1}{2}$ disjoint 2-factors. Analogously, any $\frac{p}{2}$ of them yield the desired graph.

An explicit construction can be done as well: In both cases take $V_H = \{u_1, \ldots, u_n\}$. For an even p define the edges by $E_H = \{(u_i, u_j) \mid 1 \leq i < j \leq n,\ j - i \in \{1, 2, \ldots, \frac{p}{2}, n - \frac{p}{2}, \ldots, n\}\}$. For and odd p the number of vertices n must be even, so we can take $E_H = \{(u_i, u_j) \mid 1 \leq i < j \leq n,\ j - i \in \{1, 2, \ldots, \frac{p-1}{2}, \frac{n}{2}, n - \frac{p-1}{2}, \ldots, n\}\}$

(2) *There exists a (p,q)-semiregular graph with the degree-p side having m vertices and the degree-q side having n vertices if and only if $m \geq q, n \geq p$ and $mp = nq$.*

We provide a construction analogous to the above. Define the vertex set as $V_H = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and take edges $E_H = \{(u_i, v_j) \mid 1 \leq i \leq m, 1 \leq j \leq n,\ jq - ip < pq \mod mn\}$.

The necessity is in both cases obvious.

As IM is an integer valued matrix, we can choose an positive integer vector $\mathbf{b}$ from $\ker(IM)$ such that

- $b_i \geq m_{i,i} + 1$ for all i.
- $b_i m_{i,i}$ is even for all i. $(*)$
- $b_i \geq m_{j,i}$ for all i and all $j \neq i$.

Then we construct a graph G with the vertex set being the disjoint union of k blocks $B_1, \ldots, B_k$ such that each B_i contains exactly b_i vertices. The edge set E_G is chosen such that:

- The subgraph induced by B_i is $m_{i,i}$-regular for $1 \leq i \leq k$.
- The subgraph consisting of edges between vertices of any pair of blocks B_i and B_j is $(m_{i,j}, m_{j,i})$-semiregular.

Clearly, the graph G has M as one of its degree matrices.

(iii) $\Leftrightarrow$ *(iv)* Note that $\dim(\ker(IM)) = 1$ if and only if $\mathrm{rank}(IM^T) = \mathrm{rank}(IM) = k - 1$ which is equivalent to

$$\dim(\ker(IM^T)) = |E_{F_M}| - \mathrm{rank}(IM^T) = |E_{F_M}| - k + 1 = \dim(\mathcal{S}_{F_M}).$$

Observe that the statement is valid even though $\ker(IM^T)$ contains real vectors while the cycle space $\mathcal{S}_{F_M}$ is a subspace of $\mathbb{Z}_2^{|E_{F_M}|}$. □

Corollary 2.5. *The* DMD *problem can be solved in polynomial time.*

Proof. First we check whether the matrix M is 0-symmetric. If it is, we construct the graph $\vec{F}_M$. Let $M_1, \ldots, M_p$ be the submatrices of M corresponding to the components of $\vec{F}_M$. For each M_i we compute $\ker(IM_i)$ and use Theorem 2.3.

The time complexity of such an algorithm is upperbounded by $O(k^2)$, since we can process edges of F_M sequentially for each edge in constant time decide whether the weights breaks the cycle product identity in so far explored graph or not. (Assuming constant time for arithmetic operations with weights.) □

In Theorem 2.3 were only considered matrices that are the degree matrix of some *finite* graph. In fact any 0-symmetric matrix is a degree matrix of an infinite graph. If we would like to characterize degree matrices of infinite connected graphs, then it is enough to test whether also the graph $\vec{F}_M$ is connected. This is since a construction of a k-regular graph on $\mathbb{Z}$ or of a (k,l)-semiregular on $\mathbb{Z} \cup \mathbb{Z}$ can be constructed similarly as in the implication $(iii) \Rightarrow (i)$ in the previous proof. And as all the blocks are infinite, tere is no constraint on blocks' sizes. We will later show a construction of so called universal cover, which for a given 0-symmetric matrix provides a forest (or a tree when $\vec{F}_M$ is connected) with the required equitable partition.

Theorem 2.3 and Corollary 2.5 immediately imply that for examining whether an weighted bidirected graph has the cycle product identity we do not have to check all (induced) cycles explicitly.

Corollary 2.6. *The problem whether a bidirected graph with positive edge weights has the cycle product identity can be solved in polynomial time.*

Finally, for a given degree matrix M we can easily find a smallest graph having this as one of its degree matrices.

Corollary 2.7. *For any degree matrix M the block sizes of a smallest graph G that has M as one of its degree matrices can be computed in polynomial time.*

Proof. We compute rational coefficients b_i as in the proof of Theorem 2.3. Let α_1 be the least common multiple of all denominators of elements b_i. Then $\mathbf{b}^* = \alpha_1 \mathbf{b}$ is the smallest integer vector of $\ker(IM)$. Now we choose the integer α_2 such that $\alpha_2 \geq \max_{1 \leq i,j \leq k}\{\frac{m_{i,i}+1}{b_i^*}, \frac{m_{j,i}}{b_i^*}\}$, where α_2 is required to be even if for some i the product $b_i^* m_{i,i}$ is odd. Then $\mathbf{b}^{**} = \alpha_2 \mathbf{b}^*$ satisfies all three conditions $(*)$, i.e., it yields the block sizes of a smallest graph G in the same way as in the proof of Theorem 2.3.

As in Corollary 2.5 the running time is $O(k^2)$ assuming unit time per arithmetic operation. It is possible that the size of G itself is exponential in $\langle \mathbf{b}^{**} \rangle$. □

We now consider the problem whether it can be justified that M is a degree matrix of G.

Degree Matrix Association (DMA)
Instance: A degree matrix M and a graph G.
Question: Is M a degree matrix of the graph G?

Theorem 2.8. *The* DMA *problem is* NP*-complete even for the fixed matrix* $M = A_{K_4}$.

Proof. If a cubic graph G allows a partition $\mathcal{B} = \{B_1, \dots, B_4\}$ witnessing the fact that M is a degree matrix of G. Then we can regard the partition into these four blocks as a coloring of G such that on every closed neighborhood all four colors are used. This coloring is equivalent with the problem K_4-LBiHom which has been shown NP-complete in [35]. We discuss it later in more details in Section 5.1. □

2.2 Degree refinement matrices

For many pairs of graphs (G, H) we can easily determine that a locally bijective homomorphism from G to H does not exist.

Definition 2.9. The *degree refinement matrix* $\mathrm{drm}(G)$ of G is the degree matrix corresponding to the canonical (as explained below) coarsest equitable partition of G, i.e., with the fewest number of blocks.

If $\mathrm{drm}(G) \neq \mathrm{drm}(H)$ then no locally bijective homomorphism exists between G and H, and this condition can be checked by computing both minimum equitable partitions by procedure MDP Construction that runs in $\mathcal{O}(n^3)$ time (cf. [2]).

MDP CONSTRUCTION
Input: A graph G.
Output: The minimal equitable partition $\mathcal{B}$.

0. Set $\mathcal{B}^0 = \{B_1^0\} = \{V_G\}$, $t = 1$.
1. For each vertex u compute the degree vector $\overrightarrow{d(u)} := \Big(|N(u) \cap B_1^t|, |N(u) \cap B_2^t|, \dots\Big)$.
2. Set $t := t + 1$ and define the new partition $\mathcal{B}^t$ of V_G such that
   - $u, v \in B_i^t$ if and only if $\overrightarrow{d(u)} = \overrightarrow{d(v)}$,
   - $u \in B_i^t$, $v \in B_{i'}^t$ with $i < i'$ if and only if
     * either $u \in B_j^{t-1}$, $v \in B_{j'}^{t-1}$ with $j < j'$,
     * or $u, v \in B_j^{t-1}$ and $\overrightarrow{d(u)} >_{\text{Lex}} \overrightarrow{d(v)}$,

   where $>_{\text{Lex}}$ is the lexicographic order on integer sequences.
3. If $\mathcal{B}^t = \mathcal{B}^{t-1}$ then set $\mathcal{B} = \mathcal{B}^t$ and stop, otherwise continue by step 1.

We modify this procedure into the efficient algorithm DRM CONSTRUCTION. Given a degree matrix M it computes a matrix M' such that $M' = \text{drm}(G)$ for any graph G with degree matrix M. Moreover, given a graph G it provides the degree refinement matrix of G when we take an adjacency matrix of G as its input. Note that in steps **2** and **3** the canonical order of the blocks is defined.

DRM CONSTRUCTION
Input: A degree matrix M.
Output: The degree refinement matrix M' of all graphs with degree matrix M.

0. Set $\mathcal{R}^0 = \{R_1^0\} = \{1, \dots, k\}$, $t = 1$.

1. For each row $r = 1, \dots, k$ compute the row-degree vector
$\overrightarrow{d(r)} := \left(\sum_{i \in R_1^t} m_{r,i}, \sum_{i \in R_2^t} m_{r,i}, \dots\right)$.

2. Set $t := t+1$ and define the new partition $\mathcal{R}^t$ of $\{1, \dots, k\}$ such that

- $r, s \in B_i^t$ if and only if $\overrightarrow{d(r)} = \overrightarrow{d(s)}$,
- $r \in B_i^t$, $s \in B_{i'}^t$ with $i < i'$ if and only if
  - either $r \in B_j^{t-1}$, $s \in B_{j'}^{t-1}$ with $j < j'$,
  - or $r, s \in B_j^{t-1}$, and $\overrightarrow{d(r)} >_{\text{Lex}} \overrightarrow{d(s)}$.

3. If $\mathcal{R}^t = \mathcal{R}^{t-1}$ then set $M' = \begin{pmatrix} \overrightarrow{d(r)} : r \in R_1^t \\ \overrightarrow{d(r)} : r \in R_2^t \\ \vdots \end{pmatrix}$ and stop,

otherwise continue by step 1.

By applying the above algorithm and Corollary 2.5 we immediately obtain the following.

Theorem 2.10. *Checking if a given matrix M is a degree refinement matrix can be done in polynomial time.*

Chapter 3

Locally constrained homomorphisms

In this thesis we are interested in homomorphisms that satisfy further 'local' restrictions. In particular we may request that for a graph homomorphisms each neighborhood of a vertex is mapped *bijectively* or *injectively* or *surjectively* onto the neighborhood of the image. This requirement is formalized in the core definition of our thesis.

Definition 3.1. For graphs G and H we denote:

- $G \xrightarrow{B} H$ if there exists a so-called *locally bijective homomorphism* $f : V_G \to V_H$ satisfying:

  $$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)) \text{ and } |f(N_G(u))| = |N_G(u)|.$$

- $G \xrightarrow{I} H$ if there exists a so-called *locally injective homomorphism* $f : V_G \to V_H$ satisfying:

  $$\text{for all } u \in V_G : |f(N_G(u))| = |N_G(u)|.$$

- $G \xrightarrow{S} H$ if there exists a so-called *locally surjective homomorphism* $f : V_G \to V_H$ satisfying:

  $$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)).$$

In the literature such mappings are also known as *(full) covering projections* (bijective) or as *partial covering projections* (injective), or as *role assignments* (surjective). Examples of locally constrained homomorphisms are depicted in Fig. 3.1.

Note that a locally bijective homomorphism is both locally injective and surjective. Hence, any result valid simultaneously for locally injective or for locally surjective homomorphisms is also valid for locally bijective homomorphisms. We provide an alternative definition of these three kinds of mappings via subgraphs induced by preimages of edges.

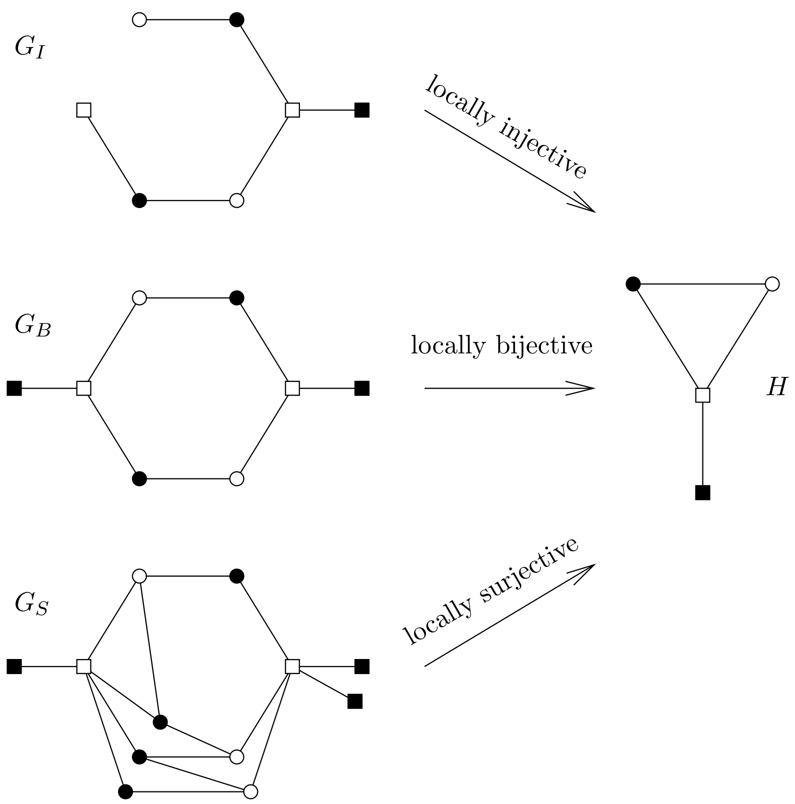


Figure 3.1: Examples of locally constrained homomorphisms.
The vertex mappings are indicated by colors and shapes.

Observation 3.2. *Let $f : G \to H$ be a graph homomorphism. If H is connected then for every edge (u, v) of H, the subgraph of G induced by $f^{-1}(u) \cup f^{-1}(v)$ is a*

- perfect matching *if and only if f is locally bijective,*
- matching and possibly isolated vertices *if and only if f is locally injective,*
- bipartite graph without isolated vertices *if and only if f is locally surjective.*

Note that for locally bijective homomorphisms the preimage classes all have the same size and for locally surjective homomorphisms all the preimage classes have size at least one. This yields the following observation:

Lemma 3.3. *If $G \xrightarrow{B} H$, for H connected and finite, then exists $k \in N$ such that $|V_G| = k \cdot |V_H|$.*

Proof. Suppose that k is size of $f^{-1}(u)$ for a particular vertex u of H, and that $e = (u, v)$ is an arbitrary edge incident with u. Since the mapping f is a local isomorphism, it means that $|f^{-1}(e)| = k$, and the constant is the same for both ends of e, i.e. $|f^{-1}(u)| = |f^{-1}(v)|$.

Due to the connectedness of the graph H, we get the equality for all vertices $u \in V_H$. □

A homomorphism discussed in the previous lemma is sometimes called with adjective k-fold.

As a corollary observe that every locally bijective homomorphism $G \xrightarrow{B} G$ is for a connected graph G an automorphisms of G.

Lemma 3.4. *(i) Any locally surjective homomorphism f from a graph G to a connected graph H is globally surjective.*
(ii) Any locally injective homomorphism f from a connected graph G to a forest H is globally injective.

Proof. (i) Suppose a vertex v of H remains (globally) uncovered. Then v is connected by a path to some covered vertex of H and we get a contradiction with the local surjectivity of f along this path.
(ii) Suppose there are two vertices u, v in G such that $f(u) = f(v)$ in H. As G is connected, u and v are connected by a path in G. This path must be mapped by f to a cycle in H (which is impossible if H is a forest). □

As a direct consequence of Lemma 3.4 we have the following two statements.

Corollary 3.5. *Any locally bijective homomorphism between two trees is an isomorphism.*

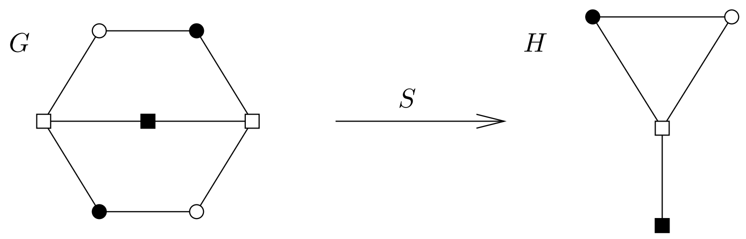


Figure 3.2: $G \xrightarrow{S} H$ but no nonempty subgraph $G' \subseteq G$ satisfies $G' \xrightarrow{B} H$.

In addition, when $G \xrightarrow{B} H$ and H is a tree then G must be a forest. When G is a finite tree, then $H \simeq G$. The finiteness is necessary, since for an counterexample we can take an infinite path mapping onto a cycle.

Corollary 3.6. *If $G \xrightarrow{S} H$, for H connected and finite, then either $|V_G| > |V_H|$ or else $G \simeq H$.*

We shall further mention a graph that allows a locally injective homomorphism can be enlarged so the homomorphism becomes locally bijective:

Proposition 3.7 (Kratochvíl, Proskurowski, Telle [38]). *Any locally injective homomorphism $G \xrightarrow{I} H$ can be extended to a locally bijective homomorphism $G' \xrightarrow{B} H$, where $G \subseteq G'$.*

Proof. Denote by $g : G \xrightarrow{I} H$ the locally injective homomorphism.

Enlarge the vertex set V_G by introducing extra new vertices into the set $V_{G'}$ and extend the mapping g into $V_{G'}$ such that $\forall v, v' \in V_H : |g^{-1}(v)| = |g^{-1}(v')|$.

For each edge $e = (v, v')$ of H, find sets $A = g^{-1}(v), B = g^{-1}(v')$ and, if necessary, insert into G' new edges, s.t. the sets A and B are connected by a perfect matching. The mapping g is locally isomorphic. Hence, $G' \xrightarrow{B} H$. □

The corresponding statement with reducing a graph that allows a locally surjective homomorphisms does not hold, an counterexample is depicted in Fig. 3.2.

Finally, we explore two special constructions on graphs and their relations to mappings $\xrightarrow{*}$. The first construction concerns the graph $G \times K_2$, called the *Kronecker double cover* [2] of G. For vertices of the Kronecker double cover we take twice the vertex set of H, i.e. $V_{H \times K_2} = V_H \times \{1, 2\}$ and $E_{H \times K_2} = \{((u, i), (v, j)) \mid (u, v) \in E_H, i \neq j\}$. Clearly, the projection to the first coordinate $\pi_1(u, i) = u$ is a locally bijective homomorphism $\pi_1 : H \times K_2 \xrightarrow{B} H$.

Lemma 3.8. *For any local constraint $* = B, I, S$ and for any pair of graphs G and H it holds that $G \xrightarrow{*} H \times K_2$ if and only if G is bipartite and $G \xrightarrow{*} H$.*

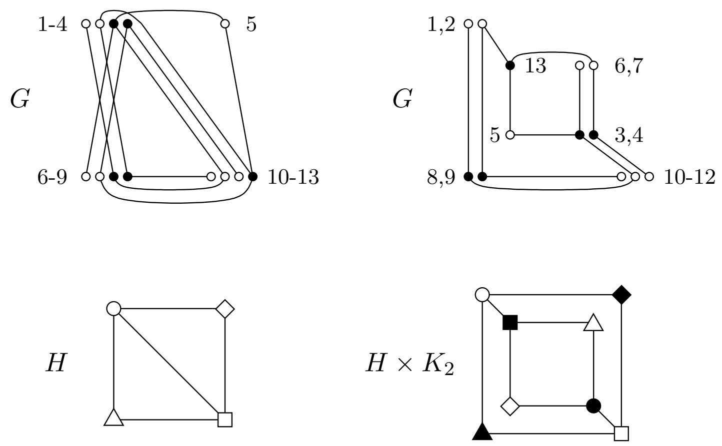


Figure 3.3: $G \xrightarrow{I} H$ and G bipartite $\iff G \xrightarrow{I} H \times K_2$.

See Fig. 3.3 for an example for the case of locally injective homomorphism.

Proof. If H is bipartite then $H \times K_2$ is isomorphic to two disjoint copies of H and the statement follows straightforwardly.

Suppose H is not bipartite and $f : G \xrightarrow{*} H \times K_2$ is the locally constrained homomorphism. Since nonbipartite graphs cannot homomorphically map to any bipartite graph like $H \times K_2$, the graph G must be bipartite. Then the composition of $\pi_1 \circ f$ is the required homomorphism $G \xrightarrow{*} H$ as $\xrightarrow{*}$ is transitive for any local constraint $* = B, I, S$.

Now suppose G is bipartite and $f : G \xrightarrow{*} H$ is a locally constrained homomorphism. Let $g : V_G \to V_{K_2}$ be the homomorphism derived from the bipartition of G. Then $h : V_G \to V_{H \times K_2}$ given by $h(u) = (f(u), g(u))$ is the desired mapping $G \xrightarrow{*} H \times K_2$.

This can be seen as follows. Let u be a vertex in G with $h(u) = (f(u), g(u)) = (x, i)$. Then vertices in $N_H(x)$ are in one-to-one correspondence with vertices in $N_{H \times K_2}(x, i)$ as $N_{H \times K_2}(x, i) = N_H(x) \times \{j\}$ where $j \neq i$. Hence, if f acts bijectively (injectively, surjectively, resp.) on $N_H(x)$ then so does h on $N_{H \times K_2}(x, i)$. $\square$

For the other special construction denote by $G^{:t}$ the graph that arises from G by subdividing each edge by $t - 1$ extra new vertices, where t is a positive integer. We come to the following observation.

Lemma 3.9. *For any positive integer t and graphs G and H the following equivalence holds for each local constraint $* = B, I, S$:*

$$G \xrightarrow{*} H \qquad \iff \qquad G^{:t} \xrightarrow{*} H^{:t}$$

Proof. The forward implication is straightforward since any locally constrained homomorphism $f : G \to H$ can be extended to vertices alled along the edges in unique way: if $u, v \in V_G$ then we map the u, v path in $G^{:t}$ bijectively onto the unique $f(u), f(v)$ path in $H^{:t}$.

For the opposite direction we assume without loss of generality that G and H are connected or, alternatively, restrict our attention to the particular components of $G^{:t}$ and $H^{:t}$ between which the mapping is defined, and to the corresponding components of G and H. We now treat each local constraint separately.

Assume that $G^{:t} \xrightarrow{B} H^{:t}$ and that $G^{:t}$ contains a vertex u of degree different from two. Then $u \in V_G$ as well as $f(u) \in V_H$ since $\deg_{H^{:t}}(f(u)) \neq 2$. Every vertex at a distance divisible by t from u maps to a vertex at distance divisible by t from $f(u)$, i.e. the mapping $f : G^{:t} \xrightarrow{B} H^{:t}$ restricted to V_G maps onto vertices of H. It is also a valid locally constrained homomorphisms, since different paths of lenght t stemming from $f(u)$ end up in different vertices, as H is a simple graph.

It remains to discuss the situation when $G^{:t}$, and consequently G, $H^{:t}$ and H, are cycles. Then by Lemma 3.3 we get that $|V_{G^{:t}}| = k \cdot |V_{H^{:t}}|$ as well as $|V_G| = k \cdot |V_H|$, which is a sufficient condition for the existence of a locally bijective homomorphisms betwen two cycles.

For the locally injective homomorphism $f : G^{:t} \xrightarrow{I} H^{:t}$ consider first the case when $G^{:t}$ contains a vertex u of degree at least three. Then $deg_{H^{:t}}(f(u)) > 2$ and consequently $f(u) \in V_H$. We then argue on the vertices at a distance divisible by t from u as in the case of locally bijective homomorphisms and get that $G \xrightarrow{I} H$.

Assume now that $G^{:t}$ is a cycle. Then the image of $G^{:t}$ in $H^{:t}$ is a closed walk whose every t-th vertex belongs to H. We use these vertices of H as images for the verties of G and following the walk we find a homomorphism witnessing that $G \xrightarrow{I} H$. Similarly, if $G^{:t}$ is a path then its image in $H^{:t}$ is an open walk and we use vertices of H from this walk, exteneded at both edns by at most t vertices, to show that $G \xrightarrow{I} H$.

For the case of locally surjective homomorphisms assume first that $H^{:t}$ contains a vertex x of degree at least three. As by Lemma 3.4 any locally surjective homomorphism is globally surjective on connected targets, the graph $G^{:t}$ contains a vertex u of degree at least three such that u maps onto x. Then $u \in V_G$ and any other vertex at distance divisible by t from u maps onto a vertex of H as was shown in the previous two cases.

If H is a cycle, then we may first compose the mapping $f : G^{:t} \xrightarrow{S} H^{:t}$ with an appropriate automorphism π of $H^{:t}$ such that their composition $\pi \circ f$ maps vertices of G onto vertices of H. Restriction of this composed mapping onto V_G gives the desired locally surjective homomorphism $G \xrightarrow{S} H$.

It remains to discuss the case when H is a path. A necessary condition for the existence of any homomorphism between G and H is that G is bipartite. Choose black and white colors corresponding to the bi-partition of G and

H arbitrarily. Assume now that f is the locally surjective homomorphism witnessing $G^{:t} \xrightarrow{S} H^{:t}$ and that the image of the restriction of f to V_G is disjoint with V_H. In such case we adjust the mapping f such that every vertex $u \in V_G$ is mapped to the vertex of the same color which is closest to $f(u)$. The image of the added vertices is then given uniquely, since they lie on a path of length t between a black and a white vertices. Straightforwardly, the modified mapping is a locally surjective and its restriction on V_G provides a proof of $G \xrightarrow{S} H$. □

Observe that in the case of locally bijective homomorphisms, the only graphs that can be mapped onto subdivided targets must be also subdivided.

Observation 3.10. *For every H, G and a positive t holds that whenever $G \xrightarrow{B} H^{:t}$ then there exists a graph F such that $G = F^{:t}$.*

This assertion follows from a simple fact that preimage of a path of length t is a collection of paths of length t.

An analogous claim holds also for locally injective homomorphisms

Observation 3.11. *For every H, G and a positive t holds that whenever $G \xrightarrow{I} H^{:t}$ then there exists a graph F such that $G \subseteq F^{:t}$ and $F^{:t} \xrightarrow{I} H^{:t}$*

Proof. The length of every cycle in $H^{:t}$ is a multiple of t, hence the same property holds for cycles in G. Similarly, vertices of degree at least three in G must be at a distance divisible by t. The only exception from this rule are vertices of degree one. In this case, we extend the pending path such that its length is divisible by t. By repeating this procedure on all vertices of degree one we obtain the graph F. □

3.1 Universal cover

Motivated by construction of universal covering spaces in topology, an analogous notion for graphs was as follows (see, e.g., Massey [46]).

Definition 3.12. For a connected graph G the *universal cover* is the only (possibly infinite) tree T_G that allows a locally bijective homomorphism $T_G \xrightarrow{B} G$.

The vertices of T_G can be represented as walks in G starting in a fixed vertex u_1 that do not traverse the same edge in two consecutive steps. Edges in T_G connect those walks that differ in the presence of the last edge. The mapping $f_0 : T_G \xrightarrow{B} G$ sending a walk in V_{T_G} to its last vertex is a locally bijective homomorphism.

By this definition T_G is a tree, since from each vertex $A \in T_G$ there is a unique path to the initial vertex $(u_1) \in V_{T_G}$, where edges along the path are in one-to-one correspondence to the edges of A. Observe also that unless G

itself is a finite tree (in which case T_G is finite and isomorphic to G), T_G is an infinite graph.

The definition of T_G is independent up to an isomorphism of the choice of the initial vertex u_1: If we initiate the construction of the universal cover T'_G in another vertex v_1 we take an arbitrary walk B from v_1 to u_1 in G and define an isomorphism $f : T_G \to T'_G$ by $f(A) = B \circ A$. Here $\circ$ denotes concatenation of the two walks where we further recursively remove edges that were traversed in two consecutive steps.

The following theorem of Leighton shows that the universal cover essentially describes the degree structure of a graph as its degree refinement matrix.

Theorem 3.13 (Leighton [42]). *Given any two finite connected graphs G and H, the following conditions are equivalent:*

(i) graphs G and G' have the same degree refinement matrix $\mathrm{drm}(G) = \mathrm{drm}(G')$

(ii) G and G' isomorphic universal cover $T_G \simeq T_H$

(iii) G and G' share a common finite cover, i.e. there exists a finite graph H s.t. $H \xrightarrow{B} G, F \xrightarrow{B} G'$.

Before prooving the theorem we settle a special case.

Lemma 3.14. *For any pair of k-regular graphs G and G' there exists a graph H that allows a locally bijective homomorphism to both of them.*

Proof. Assume without loss of generality that both G and G' are bipartite, otherwise take the Kronecker double product(s) instead. Denote the two blocks of bipartition of G by A, B and similarly A', B' for G'. It is well known that k-regular bipartite graphs are k-edge colorable [28].

Consider that edges of both graph are colored and define colored product as the bipartite graph $H = G \dot{\times} G'$ with vertices as $V_H = A \times A' \cup B \times B'$. The edge set of H is taken as $E_H = \{((u, u'), (v, v')) \mid (u, v) \in E_G, (u', v') \in E_{G'}$ and $(u, v), (u', v')$ have the same color$\}$ Any vertex in G or in G' is incident with precisely one edge of every color, hence the resulting graph H is also k-regular. In addition, $H \xrightarrow{B} G$ by the projection to the first coordinate $\pi((u, u')) = u$. Similarly $H \xrightarrow{B} G'$ via $\pi'((u, u')) = u'$ □

We are now ready to show the construction for the implication $(i) \Rightarrow (iii)$ of Theorem 3.13.

Sketch of the proof of Theorem 3.13. Let $\mathcal{B} = (B_1, \ldots, B_k)$ be the equitable partition of G corresponding to the matrix M. Denote by $G_{i,j}$ the subgraph of G induced by edges connecting blocks B_i and B_j. Use analogous notation for G'.

From Lemma 3.14 follows that we are able to construct graphs $H_{i,j}$ that allow locally bijective homomorphisms to both $G_{i,j}$ and $G'_{i,j}$ for any i, j when $m_{i,j} \neq 0$. By the induction hypothesis, suppose that we are able to construct a common cover $H_{\overline{i,j}}$ of graphs $G \setminus E_{G_{i,j}}$ and $G' \setminus E_{G'_{i,j}}$ for some $m_{i,j} \neq 0$. Then, we use multiple copies of $H_{i,j}$ and $H_{\overline{i,j}}$ until the i-th blocks of $H_{i,j}$ and $H_{\overline{i,j}}$ have the same size. Immediately, the equality holds also for the j-th blocks. By collapsing a matching between vertices of the i-th blocks of $H_{i,j}$ and $H_{\overline{i,j}}$ that have the same images in both G and G' and in the same way also for the j-th blocks, we get the desired finite graph that allows a locally bijective homomorphisms to both G and G'.

Now we show a construction of the graph $H_{i,j}$. Due to Lemma 3.14, we suppose that $m_{i,j} \neq m_{j,i}$. For each vertex $u \in V_{G_{i,j}}$, fix an injective labeling of the incident edges by numbers from $[\deg(u)]$. We use the symbol $c(u, e)$ for the label of edge e incident with the vertex u. We perform the same procedure for the graph $G'_{i,j}$. Now take

$$V_{H_{i,j}} = B_i \times B'_i \times [m_{i,j}]) \cup B_j \times B'_j \times [m_{j,i}]).$$

Two vertices (u, u', k) and (v, v', l) are adjacent if and only if

$$\begin{aligned} & (u, v) \in E_G, \quad (u', v') \in E_{G'}, \\ k &\equiv c(u, (u, v)) - c'(u', (u', v')) \pmod{m_{i,j}}, \text{ and} \\ l &\equiv c(v, (u, v)) - c'(v', (u', v')) \pmod{m_{j,i}}. \end{aligned}$$

Projections $f : (u, u', k) \to u$ and $f' : (u, u', k) \to u'$ are the required locally bijective homomorphisms $H_{i,j} \xrightarrow{B} G_{i,j}$ and $H_{i,j} \xrightarrow{B} G'_{i,j}$. Observe also that in the i-th block of $H_{i,j}$ exactly $m_{i,j}$ vertices have the same image in the two homomorphisms f, f'. This is necessary and sufficient for the construction of the matching used to compose graphs $H_{i,j}$ and $H_{\overline{i,j}}$ together. □

Note, that in view of Theorem 3.13 we can also define the universal cover T_M associated with a degree refinement matrix M as the universal cover $T_G = T_M$ of any graph G with $\mathrm{drm}(G) = M$.

Besides the essential importance of the notion of universal cover for locally bijective homomorphisms we show, that the notion of universal cover is valuable also for homomorphisms of the other two kinds of constraint (even though these do not maintain the degree structure). We start with a simple observation:

Lemma 3.15. *Let G and H be two connected graphs. From any $f : G \xrightarrow{*} H$ a locally constrained homomorphism $f' : T_G \to T_H$ can be derived, where $*$ indicates the appropriate local constraint $* = B, I$ and S.*

Proof. To validate this statement it is sufficient to observe that for an arbitrary walk $A = (u_1, \dots, u_n)$ in G, its neighborhood $N_{T_G}(A)$ is in one-to-one correspondence to the neighborhood $N_G(u_n)$ of the vertex u_n in G.

We discuss the case of locally bijective f first. Here the neighborhood $N_G(u_n)$ maps bijectively onto $N_H(f(u_n))$, by the definition of f. Moreover $N_{T_G}(A)$ is in one-to-one correspondence with $N_G(u_n)$. Similarly $N_{T_H}(f'(A))$ is in one-to-one correspondence with $N_H(f(u_n))$. The composition of these three relations provides a bijection between sets $N_{T_G}(A)$ and $N_{T_H}(f'(A))$, i.e., the mapping f' is locally bijective.

By replacing above the word 'bijective' with 'injective' or 'surjective' we get the proof also for the other two constraints

$$N_{T_G}(A) \longleftrightarrow N_G(u_n) \xrightarrow{*} N_H(f(u_n)) \longleftrightarrow N_{T_H}(f'(A))$$

□

3.2 Cantor-Bernstein type theorem

At this point we would like to highlight two structural theorems that provide an important tool for proving an NP-hardness reduction for the decision problems on the existence of locally constrained homomorphisms.

Theorem 3.16 (Fiala, Kratochvíl [18]). *Let G be a finite graph and let H be a finite connected graph. Suppose there is a locally bijective homomorphism from G to H. Then any locally injective homomorphism from G to H is locally bijective.*

Theorem 3.17 (Kristiansen, Telle [41]). *Let G be a finite graph and let H be a finite connected graph. Suppose there is a locally bijective homomorphism from G to H. Then any locally surjective homomorphism from G to H is locally bijective.*

The next theorem motivated by the celebrated Cantor-Bernstein theorem showing that the simultaneous existence of a surjective and injective mapping between two set provides a sufficient condition for the existence of a bijection between these sets. We link the above theorems together and show that an analogous statement to Cantor-Bernstein theorem holds also for locally constrained homomorphisms:

Theorem 3.18. *Let G be a graph and let H be a finite connected graph. Suppose there is a locally surjective homomorphism $g : G \to H$ and a locally injective homomorphism $h : G \to H$. Then both g and h are locally bijective.*

Let G and H be two simple connected graphs and let f be an arbitrary homomorphism $G \to H$. Without loss of generality assume that vertices of T_G are walks starting in the vertex u_1, and that the vertices of T_H are walks emanating from $f(u_1)$. We define a derived mapping $f' : V_{T_G} \to V_{T_H}$ by $f'(A) = [(f(u_1), f(u_2), \ldots, f(u_n))]_\sim$, for $A = (u_1, u_2, \ldots, u_n)$.

We now give a proof of Theorem 3.18, which is moreover shorter than the original proofs of Theorems 3.16 and 3.17. The next lemma contains the core argument for the proof.

Lemma 3.19. *Let G be a connected graph and let H be a finite connected graph. Suppose there is a locally surjective homomorphism $g : G \to H$ and a locally injective homomorphism $h : G \to H$. Further let g' and h' be defined as above. Then*

- *g and h are locally bijective homomorphisms, and*
- *g' and h' are isomorphisms between the corresponding universal covers T_G and T_H.*

Proof. Consider the universal covers T_G and T_H for G and H. Due to the previous lemma, the derived mapping g' is a local surjection from T_G to T_H, and h' is a local injection between the same trees.

According to Lemma 3.4 g' is globally surjective and h' is globally injective.

Let d be the diameter of H. For a universal cover T and a vertex $A \in V_T$ we denote by $M(A)$ the set of all vertices that are at distance at most $d+1$ from A. For any A the set $M(A)$ induces a finite subtree of T.

Select a vertex $B \in V_{T_H}$ such that $|M(B)|$ is maximal. Due to the global surjectivity of g' there exists a vertex $A \in V_{T_G}$ such that $g'(A) = B$. Also denote $h'(A) = C$.

Now we get

$$|M(A)| \geq |M(g'(A))| = |M(B)| \geq |M(C)| = |M(h'(A))| \geq |M(A)| \quad (3.1)$$

The first inequality follows from the local surjectivity of g', the second from the choice of B, and the third from the injectivity of h'.

Since both sides of the inequality are the same, we get in fact equality.

From $|M(A)| = |M(g'(A))|$ follows that g' acts as an isomorphism between $M(A)$ and $M(g'(A))$. The same holds for h' as well. Observe that the set $M(A)$ was selected such that every vertex v of H appears as the last vertex of some walk $A_v \in M(g'(A))$. Moreover the particular A_v can be chosen such that all its neighbors are still inside $M(g'(A))$.

Since g' restricted to $M(A)$ was shown to be an isomorphism, we get that for some $A_u \in M(A)$ with $g'(A_u) = A_v$ the projection g' maps bijectively $N(A_u)$ onto $N(A_v)$. From this immediately follows that the mapping g acts as a bijection between $N(u)$ and $N(v) = N(g(u))$.

The last argument holds for any vertex u. More precisely, for an arbitrary $u \in V_G$ we can choose A for (3.1) such that $A_u \in M(A)$, where A_u represents a walk in G ending by u. Then we have shown that$g$ acts bijectively on $N(u)$.

It means that g is a locally bijective homomorphism between G and H. By a substitution of h instead of g the same result can be derived for the mapping h.

The second assertion of the lemma follows directly from Lemma 3.4 as any locally bijective homomorphism between two trees is an isomorphism. □

Theorem 3.18 follows by applying Lemma 3.19 separately on each component of G. By a similar argument we obtain the following corollary.

Corollary 3.20. *Let G and H be connected graphs and let H be finite. If the two universal covers T_G and T_H of G and H are isomorphic then any locally injective or locally surjective homomorphism from G to H is locally bijective.*

Proof. The statement follows from the fact that any locally surjective or locally injective graph homomorphism between the two isomorphic trees T_G and T_H is locally bijective.

The proof of this assertion mimics the proof of Lemma 3.19. We select $B \in V_{T_H}$ which maximizes $|M(B)|$ and adjust equation (3.1) so it uses the tree isomorphism instead one of the two locally constrained homomorphisms. The remaining argument follows the same guidelines. □

Observe that the finiteness of the graph H is necessary. A counterexample can be constructed as follows: Take the infinite path $G = H = (\mathbb{N}, \{(i, i+1) : i \in \mathbb{N}\})$ and the mapping $h : V_G \to V_H$ such that $h(i) = i + 1$. Then h is locally injective but not locally bijective.

It does not make any sense to consider the case when G is finite and H is infinite and connected since then no locally surjective homomorphism from G to H exists. This follows immediately from Lemma 3.4 (i).

Chapter 4

Orders

4.1 Partial orders on connected graphs

It is well-known that graph homomorphisms define a quasiorder on the class of all graphs, which can be factorized into a partial order. For an overview of these results see a recent monograph of Hell and Nešetřil [30]. We show that a similar interesting structure exists on the class offinite connected graphs $\mathcal{G}^c$ for locally constrained homomorphisms. For this purpose we view $\xrightarrow{B}$, $\xrightarrow{I}$ and $\xrightarrow{S}$ as binary relations on $\mathcal{G}^c$, denoted by $(\mathcal{G}^c, \xrightarrow{*})$ if necessary, where $*$ indicates the appropriate local constraint. We show that $(\mathcal{G}^c, \xrightarrow{*})$ is a partial order for any of the three local constraints $* = B, I, S$.

Observe first that for any $G \in \mathcal{G}^c$, the identity mapping $i : V_G \to V_G$ clarifies that all three relations $\xrightarrow{*}$ are *reflexive*.

The composition of two graph homomorphisms with the same local restriction (B, I, S) is again a graph homomorphism of the same kind. Hence each $\xrightarrow{*}$ is also *transitive*.

To sow that these relations are antisymmetric, suppose that $f : G \xrightarrow{*} H$, $g : H \xrightarrow{*} G$, where f, g are of the same local constraint and $G, H \in \mathcal{G}^c$. For $* = B, S$ we can invoke Corollary 3.6 to conclude that $G \simeq H$.

Recall Theorem 3.18 stating that if G allows both a locally injective and a locally surjective homomorphism to H, then both these homomorphisms are locally bijective. For $* = I$ we have $g \circ f : G \xrightarrow{I} G$ and $G \xrightarrow{S} G$ by the identity mapping. By Theorem 3.18 the mapping $g \circ f$ is locally bijective. Since G is connected, we have that $(g \circ f)(V_G) = V_G$. Hence, f is globally injective. By the same kind of arguments we deduce that g is injective. This means that f is surjective, and hence f is a graph isomorphism from G to H. Hence, all three relations are *antisymmetric*. We note here that the antisymmetry of $\xrightarrow{I}$ also follows from an iterative argument of Nešetřil [54].

Combining the results above with Theorem 3.18 yields the following result.

Theorem 4.1. *All three relations $(\mathcal{G}^c, \xrightarrow{B})$, $(\mathcal{G}^c, \xrightarrow{I})$ and $(\mathcal{G}^c, \xrightarrow{S})$ are partial* *orders with* $(\mathcal{G}^c, \xrightarrow{B}) = (\mathcal{G}^c, \xrightarrow{I}) \cap (\mathcal{G}^c, \xrightarrow{S})$.

4.2 The lattice of equitable partitions

In section 2.2 we have shown that a single graph may allow several equitable partitions. In particular, the coarsest equitable partition corresponding to the degree refinement matrix can be always obtained by a suitable mergence of some blocks of the partition.

This observation yields a natural comparison of two equitable partitions of the same graph, by means of coarser/finer partitions.

Definition 4.2. We say that an equitable partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ of a graph G is *coarser* than an equitable partition $\mathcal{C} = \{C_1, \ldots, C_l\}$ of the same graph if there exists a partition $R_1, \ldots, R_k$ of the index set $\{1, 2, \ldots, l\}$ such that for every $i = 1, \ldots, k : B_i = \bigcup_{j \in R_i} C_j$. In such a case we write $\mathcal{B} \preceq \mathcal{C}$.

Let us denote $\mathcal{Q}_G$ the set of all equitable partitions of G factorized upto permutations of blocks inside the partitions.

As the refinement can be iterated, the relation $\preceq$ is transitive and the structure $(\mathcal{Q}_G, \preceq)$ is a partial order. Moreover, the degree refinement is the minimal element in $(\mathcal{Q}_G, \preceq)$ and the partition into singletons is its maximal element.

It is well known that the order $\preceq$ is indeed a lattice on $\mathcal{Q}_G$:

Theorem 4.3 (folklore [50]). *For an arbitrary graph G it holds that the partial order $(\mathcal{Q}_G, \preceq)$ is a lattice.*

To keep the thesis self-contained we provide a sketch of the proof.

Proof. For given two partitions $\mathcal{C}$ and $\mathcal{B}$ of the same graph G we construct the two unique elemets $\mathcal{B} \wedge \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$. We involve similar arguments like in the example in Section 1.1 of the lattice of equivalence relations.

The coarsest equitable partition that is finer than both $\mathcal{B}$ and $\mathcal{C}$ must be finer than the following partition $\mathcal{D}$ of V_G defined as $\mathcal{D} = \{B_i \cap C_j \mid i = 1, \ldots, k, j = 1, \ldots, l\}$. It may happen that $\mathcal{D}$ is not an equitable partition. In such a case we involve Algorithm MDP Construction of Section 2.2 starting at step 1. with the initial setting $\mathcal{B}^0 = \mathcal{D}$. The resulting partition is the unique equitable partition that is finar than $\mathcal{D}$, i.e. the desired supremum $\mathcal{B} \vee \mathcal{C}$.

For the infimum $\mathcal{B} \wedge \mathcal{C}$ we involve an auxiliary relation $\sim$ on V_G such that $u \sim v$ if and only if u and u' belong to the same block of $\mathcal{B}$ or of $\mathcal{C}$. We then take the transitive closure $\sim'$ of $\sim$, i.e. the inclusion wise minimal equivalence relation that contains $\sim$. The partition $\mathcal{D}$ of V_G whose blocks are equivalence classes of $\sim'$ is clearly the finest partition that is coarser than both $\mathcal{B}$ and $\mathcal{C}$. It only remains to show that it is an equitable partition.

Take two vertices $u \sim u'$. We prove that for an arbitrary block $D \in \mathcal{D}$ the two vertices u and u' have the same number of neighbors in D. If u, u' are in the same block of $\mathcal{B}$ then we use the decomposition of D as the union of some blocks of $\mathcal{B}$, i.e. $D = B_{j_1} \cup B_{j_2} \cup \cdots \cup B_{j_t}$. We now have $|N_G(u) \cap D| = |N_G(u) \cap B_{j_1}| + \cdots + |N_G(u) \cap B_{j_t}| = |N_G(u') \cap B_{j_1}| + \cdots + |N_G(u') \cap B_{j_t}| = |N_G(u') \cap D|$. In the other case, i.e., when u and u' belong to the same block of $\mathcal{C}$, we build D from some blocks of $\mathcal{C}$. In both cases we get that the quantity $|N_G(u) \cap D|$ is constant for all u of the same block of $\mathcal{D}$. Therefore $\mathcal{D}$ is an equitable partition and it is equal to the infimum $\mathcal{B} \wedge \mathcal{C}$. □

4.3 Matrix order induced by equitable partition

We have explored that a single graph may have many equitable partitions, with the unique coersest partition corresponding to the degree refinement matrix of the graph. In this section we show that the corresponding matrices can be arranged in a quasiorder respecting the refinement of the equitable partitions.

Definition 4.4. For two degree matrices M of order k and N of order l we write $N \sqsubseteq M$ if there exists a partition $R_1, \ldots, R_l$ of the index set $\{1, 2, \ldots, k\}$ where $i, i' \in R_r$ implies that for all $s = 1, \ldots, l$ holds

$$\sum_{j \in R_s} m_{i,j} = \sum_{j \in R_s} m_{i',j} = n_{r,s}$$

For example the following two matrices

$$M = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \qquad\qquad N = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 2 \end{pmatrix}$$

are comparable $N \sqsubseteq M$ due to the partition $R_1, R_2, R_3 = \{1\}, \{2, 3\}, \{4\}$.

We now explore some properties of this quasiorder. Observe first that two matrices satisfy $M \sqsubseteq N$ and $N \sqsubseteq M$ if they are of the same order and differ only by a permutation of their rows and columns. Hence, a partial order can be obtained by representing all such matrices by a unique element (see Section 1.1).

Furher observe that for any pair of matrices M and N satisfying $M \sqsubseteq N$ there exists a graph such that both M and N are degree matrices of G, where the equitable partition for M can be obtained from a partition for N by further subdivision of some sets. Moreover, it follows straightforwardly that $\mathrm{drm}(G) \sqsubseteq M$, $\mathrm{drm}(G) \sqsubseteq N$ and that $\mathrm{drm}(G)$ is the unique matrix obtained by the DRM Construction procedure either from M or from N.

In contrary to the fact that for any graph G the parition order $(\mathcal{Q}_G, \preceq)$ is a lattice the order $(\mathcal{M}_G, \sqsubseteq)$ is not a lattice, even if it can be obtained from the partition order by representing equitable partitions with the same degree matrix by the matrix itself.

The counterexample is due to Jan Kratochvíl [personal communication, 2006]. Take the following degree matrices of the complete graph K_5:

$$\mathbf{M}_1 = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \quad \mathbf{M}_2 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

$$\mathbf{N}_1 = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \quad \mathbf{N}_2 = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$$

These four matrices satisfy $M_1 \sqsubseteq N_1$ for the partition $R_1, R_2 = \{1\}, \{2, 3\}$, and furthermore $M_1 \sqsubseteq N_2$ for $\{1, 2\}, \{3\}$, $M_2 \sqsubseteq N_1$ for $\{1, 3\}, \{2\}$, and $M_2 \sqsubseteq N_2$ for $\{1, 2\}, \{3\}$. Obviously, the pair M_1, M_2 does not have infimum and also N_1, N_2 do not have supremum in $\sqsubseteq$.

The quasiorder $\sqsubseteq$ is a disjoint union of several quiasiorders, where each such suborder contains only matrices comparable with the same degree refinement matrix. Degree refinement matrices are the minimum elements in these suborders.

The following lemma shows that the quasiorder $\sqsubseteq$ can be useful in description of all degree matrices of a given graph.

Lemma 4.5. *For any finite graph G it holds that*

$$\mathcal{M}_G = \{M \mid M \sqsubseteq A_G\}$$

Proof. Any equitable partition of V_G is in one-to-one correspondence with a partition of rows of the adjacency matrix of G. Assume that vertices of G are numbered by $v_1, \ldots, v_n$ and $B_1, \ldots, B_k$ is the equitable partition of G yielding a matrix M.

Then for any $v_i \in B_r$ it holds that $\sum_{v_j \in B_s} (A_G)_{i,j} = |N(v_i) \cap B_s| = m_{r,s}$, which concludes the proof. □

We can further explore the set of matrices above a given adjacency matrix:

Lemma 4.6. *For any two finite and connected graphs G and H it holds that*

$$G \xrightarrow{B} H \qquad \Longleftrightarrow \qquad A_H \sqsubseteq A_G.$$

In particular, for any finite connected H

$$\{M \mid A_H \sqsubseteq M\} = \{A_G \mid G \xrightarrow{B} H\}$$

Proof. Let H be a graph on k vertices $x_1, \ldots, x_k$. If $A_H \sqsubseteq M$ for some degree matrix M, then the matrix M must be 0,1-valued. Hence M is an adjacency matrix of a graph G. The patrition $B_1, \ldots, B_k$ of V_G satisfies the following properties:

- Each vertex in B_i has at most one neighbor in any other class B_j and no neighbor in B_i.
- The degree of a vertex in B_i is equal to the degree of x_i

We may define a mapping $f : V_G \to V_H$ as follows: if $u \in B_i$ then $f(u) = x_i$. If (u, v) is an edge of G, we get that $(f(u), f(v))$ is an edge of H. Together with the two above properties we get that f is in fact a locally bijective homomorphism $G \xrightarrow{B} H$.

In the opposite direction it is straightforward to verify that any locally bijective homomorphism $f : G \xrightarrow{B} H$ provides a partition of rows of A_G. Here the blocks are correspond to the vertices with the same image and the resulting degree matrix is A_H. □

As a consequence of the above lemma we see that the matrix quasiorder $(\mathcal{M}^c, \sqsubseteq)$ contains as a suborder the order $(\mathcal{G}^c, \xrightarrow{B})$. It can be obtained by restriction onto adjacency matrices of nonisomorphic graphs.

4.4 Matrix orders induced by locally constrained homomorphisms

We again recall Theorem 3.13 that a locally bijective homomorphism from a graph G to a graph H may exist only if G and H have the same degree refinement matrix. As we have already argued in the previous section, the symmetric and transitive closure of the partial order $(\mathcal{G}^c, \xrightarrow{B})$ is an equivalence relation whose classes can be naturally represented by degree refinement matrices. It is natural to ask if the other two kinds of locally constrained homomorphisms are also conditioned by the existence of a well-defined relation on the degree refinement matrices. Here, we prove that such a relation exists and moreover, that it is a quasi order.

Definition 4.7. We define three relations $\overset{B}{\leqslant}$, $\overset{I}{\leqslant}$ and $\overset{S}{\leqslant}$ on $\mathcal{M}^c$ as follows: For two matrices $M, N \in \mathcal{M}^c$ we have $M \overset{*}{\leqslant} N$ if there exist graphs $G \in \mathcal{G}^c$ with $M \in \mathcal{M}_G$ and $H \in \mathcal{G}^c$ with $N \in \mathcal{M}_H$ such that $G \xrightarrow{*} H$ holds for the appropriate local constraint.

Observe that the relation $N \sqsubseteq M$ implies directly relations $M \overset{*}{\leqslant} N$ and $N \overset{*}{\leqslant} M$. Here one can take any graph G with $M \in \mathcal{M}_G$ and its isomorphism as the localy constrained homomorphism. Then it is enough to take one partition for M and refine it further to obtain N. Hence, $\overset{B}{\leqslant}$ is in fact an

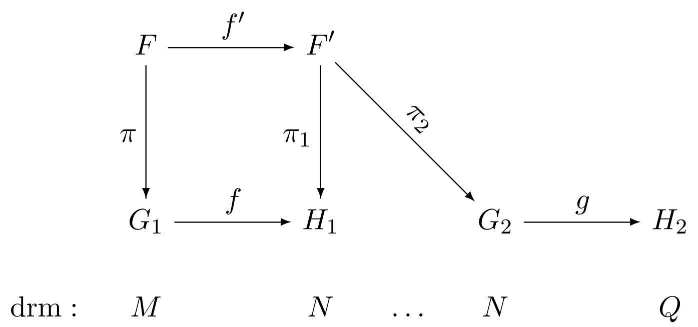


Figure 4.1: Commutative diagram for transitivity of $\xrightarrow{I}$ where horizontal mappings are injective and others are bijective.

equivalence relation. Here each equivalence class is uniquely represented by the degree refinement matrix of any graph with a degree matrix in this class.

For the remaining two constraints $* = I, S$ the *reflexivity* of the relation $\overset{*}{\leqslant}$ follows directly from the existence of the identity mapping on any underlying graph (where at least one must exist to assert the membership of the matrix in $\mathcal{M}^c$).

The fact that the relation $\overset{I}{\leqslant}$ is transitive follows directly from the next lemma.

Lemma 4.8. *Let $G_1, G_2, H_1, H_2 \in \mathcal{G}^c$ be such that $G_1 \xrightarrow{I} H_1$ and $G_2 \xrightarrow{I} H_2$, where H_1 and G_2 share the same degree refinement matrix. Then there exists a graph $F \in \mathcal{G}^c$, such that $F \xrightarrow{I} H_2$ and $F \xrightarrow{B} G_1$.*

Proof. By using Theorem 3.13 we first construct a finite graph F', such that $F' \xrightarrow{B} H_1$ and $F' \xrightarrow{B} G_2$. The projection $\pi_2 : F' \xrightarrow{B} G_2$ composed with the locally injective homomorphism $g : G_2 \xrightarrow{I} H_2$ gives that $F' \xrightarrow{I} H_2$. See Fig. 4.1.

As $F' \xrightarrow{B} H_1$ via projection π_1, it follows from Observation 3.2 that the preimage $\pi_1^{-1}(x)$ has the same size for all vertices $x \in V_{H_1}$, say k. Denote the vertices of F', which map onto a vertex x, by $\{x_1, x_2, \ldots, x_k\}$.

The vertex set of the desired graph F is the Cartesian product $V_{G_1} \times \{1, \ldots, k\}$. For simplicity we abbreviate (u, i) as u_i. Define the edges of F as follows:

$$(u_i, v_j) \in E_F \iff (u, v) \in E_{G_1} \text{ and } (f(u)_i, f(v)_j) \in E_{F'}.$$

We define two mappings $f' : u_i \to f(u)_i$ and $\pi : u_i \to u$. According to Observation 3.2, f' is a locally injective homomorphism from F to F' and π is a locally bijective homomorphism from F to G_1. Then the following commutative diagram holds: $\pi \circ f = f' \circ \pi_1$, as illustrated in Fig. 4.1. The mapping $g \circ \pi_2 \circ f'$ is a locally injective homomorphism $F \xrightarrow{I} H_2$. $\square$

The same assertion can be proven for the order $\overset{S}{\leqslant}$ with exactly the same arguments, the only difference is that the preimage in F of any edge $(x_i, y_j) \in E_{F'}$ is a spanning bipartite graph.

Corollary 4.9. *For any constraint $* = I, S$ the relation $(\mathcal{M}^c, \overset{*}{\leqslant})$ is a quasiorder. The relation $(\mathcal{M}^c, \overset{B}{\leqslant})$ is an equivalence relation.*

In Proposition 3.7 we have showed that any locally injective homomorphism $G \xrightarrow{I} H$ can be extended to a locally bijective homomorphism $G' \xrightarrow{B} H$, where $G \subseteq G'$. This yields an alternative definition of the order $(\mathcal{M}^c, \xrightarrow{I})$: For matrices M, N it holds that $M \overset{I}{\leqslant} N$ if and only if there exists graphs G and H with degree refinement matrices M and N, respectively, such that G is a subgraph of H. This straightforwardly implies the first claim of the observation below. The second claim (and the first claim as well) follows by Proposition 3.15 and a simple inductive argument on the two trees T_M and T_N.

Observation 4.10. *For any degree matrices $M, N \in \mathcal{M}^c$ it holds that if $M \overset{I}{\leqslant} N$ then $T_M \subseteq T_N$, and if $M \overset{S}{\leqslant} N$ then $T_N \subseteq T_M$.*

The reverse is not true: for $\overset{S}{\leqslant}$ take $M = \mathrm{drm}(P_4)$ and $N = \mathrm{drm}(P_3)$. The counterexample for $\overset{I}{\leqslant}$ requires a bit more effort, see section 8.1.1.

4.5 Orders on degree refinement matrices

In this section we show that if we further restrict the universe onto degree *refinement* matrices, the structure of $\overset{*}{\leqslant}$ becomes even simpler.

Denote by $\mathcal{RM}^c$ the set of degree refinement matrices of finite connected graphs. As $\mathcal{RM}^c \subset \mathcal{M}^c$ we obtain orders $(\mathcal{RM}^c, \xrightarrow{*})$ in a natural way for any local constraint $* = B, I, S$.

Observe that $(\mathcal{RM}^c, \xrightarrow{B})$ contains only reflexive pairs, since no locally bijective homomorphism can exist between graphs with different degree refinement matrices.

For proving *antisymmetry* we involve the notion of universal cover. Assume that $M \overset{I}{\leqslant} N$ and $N \overset{I}{\leqslant} M$. By Proposition 3.15, there exist locally injective homomorphisms $f' : T_M \to T_N$ and $g' : T_N \to T_M$. Recall definition of the universal cover from Sect. 3.1 and consider a locally bijective homomorphism $f_0 : T_M \to G$ for some graph G with degree matrix M. As in the previous section we now invoke Theorem 3.18 to conclude that $f_0 \circ g' \circ f' : T_M \xrightarrow{I} G$ is locally bijective. This implies that both f' and g' are locally bijective, and consequently the universal covers T_M and T_N are isomorphic. Hence $M = N$ due to Theorem 3.13.

The antisymmetry of $\xrightarrow{S}$ can be proven according to exactly the same arguments.

Theorem 4.11. *For any constraint* $* = B, I, S$ *the relation* $(\mathcal{RM}^c, \overset{*}{\leqslant})$ *is a partial order. It arises as a factor of the order* $(\mathcal{G}^c, \overset{*}{\to})$*, when we unify the graphs that have the same degree refinement matrices.*

Theorem 3.18 can now be translated to matrices. If two degree refinement matrices satisfy $M \overset{I}{\leqslant} N$ and $M \overset{S}{\leqslant} N$, then $M \overset{B}{\leqslant} N$, i.e., $M = N$.

Corollary 4.12.

$$(\mathcal{RM}^c, \overset{B}{\leqslant}) = (\mathcal{RM}^c, \overset{I}{\leqslant}) \cap (\mathcal{RM}^c, \overset{S}{\leqslant}) = (\mathcal{RM}^c, \{(M, M) : M \in \mathcal{RM}^c\}).$$

Proof. Suppose that both $G_1 \overset{I}{\to} H_1$ and $G_2 \overset{S}{\to} H_2$ hold with $M \in \mathcal{M}_{G_i}$ and $N \in \mathcal{M}_{H_i}$ $(i = 1, 2)$. By Observation 4.10, we have that $T_M \subseteq T_N$ and $T_N \subseteq T_M$. We represent these inclusions by locally injective homomorphisms $f' : T_M \to T_N$ and $g' : T_N \to T_M$. By the same arguments as in the proof of antisymmetry of $\overset{I}{\leqslant}$ we conclude that $M = N$. □

Chapter 5

Computational complexity

From the point of view of computational complexity we are interested in the decision problem whether for given graphs G and H there exists a locally constrained homomorphism from G to H. If both graphs are part of the input, then the problem is trivially NP-complete for all three kinds of constraints. By selecting $H = K_4$ we can test the existence of a proper 4-coloring of a cubic graph G, such that on the closed neighborhood of every vertex, all four colors are used [35].

We use a similar approach, as is used for the testing the existence of an ordinary graph homomorphism (i.e., the H-Coloring problem) and define a class of decision problems, where each problem corresponds to a specific graph H:

> H-Locally Bijective Homomorphism (H-LBiHom)
> *Instance:* A graph G.
> *Question:* Does there exists a locally bijective homomorphism $G \xrightarrow{B} H$?

The same setting is used for locally injective and surjective mappings:

> H-Locally Injective Homomorphism (H-LInHom)
> *Instance:* A graph G.
> *Question:* Does there exists a locally injective homomorphism $G \xrightarrow{I} H$?

> H-Locally Surjective Homomorphism (H-LSurHom)
> *Instance:* A graph G.
> *Question:* Does there exists a locally surjective homomorphism $G \xrightarrow{S} H$?

Without lost of generality, we suppose that the input graph G is connected, since each block of connectivity of G have to allow a homomorphism to H of the particular local constraint to obtain the homomorphism from the entire graph G.

Recall that the existence of a locally bijective homomorphism $G \to H$ implies that these two graphs have equal degree refinement matrices $\mathrm{drm}(G) = \mathrm{drm}(H)$. If these graphs are connected, then they have isomorphic universal covers by Theorem 3.13.

We will strengthen the theorem 3.18 to obtain a result that allows us to closely relate the decision problems H-LBiHom, H-LInHom and H-LSurHom. As a direct consequence of Corollary 3.20 we have:

Theorem 5.1. *Let H be a connected finite graph. If for a graph G holds that $\mathrm{drm}(G) = \mathrm{drm}(H)$ then any locally injective or locally surjective homomorphism is in fact locally bijective.*

This directly translates to general statement about computational complexity of locally constrained homomorphisms:

Theorem 5.2. *For any finite connected graph H we have H-LBiHom $\propto$ H-LInHom and H-LBiHom $\propto$ H-LSurHom. In particular, if for a graph H the H-LBiHom problem is NP-complete then also problems H-LInHom and H-LSurHom are NP-complete too.*

The last theorem implies the result of Nešetřil [54] claiming that every partial cover $G \xrightarrow{I} G$ of a connected graph G is its automorphism.

As a straightforward condequence of Lemma 3.9 and Observations 3.10 and 3.11 we derive another relations in the complexity characterization.

Proposition 5.3. *For any graph H and a positive integer t it holds that*

- *problems H-LBiHom and $H^{:t}$-LBiHom are polynomially equivalent,*
- *H-LInHom and $H^{:t}$-LInHom are polynomially equivalent as well,*
- *the problem H-LSurHom is polynomially reducible to $H^{:t}$-LSurHom.*

5.1 Complexity of H-LBiHom problems

Here we briefly review known results on the computational complexity for the H-LBiHom problem.

Graphs satisfying $\mathrm{drm}(H) = A_H$ have exactly one vertex in each class of degree refinement. They do not allow a locally bijective homomorphism to any smaller graph, since there is no possibility to map two vertices on the same target. On the other hand, the H-LBiHom problem is solvable in polynomial time. It is sufficient to test whether $\mathrm{drm}(G) = \mathrm{drm}(H)$ and if this is satisfied the only homomorphism $G \xrightarrow{B} H$ is uniquely identified by the classes of degree refinement in G.

This approach was extended by Kratochvíl, Proskurowski and Telle [38]. They provided a polynomial algorithm for the H-LBiHom problem for simple graphs H which have at most two vertices in each class of the degree refinement. The algorithm used a reduction to the 2-Sat problem.

Theorem 5.4 (Kratochvíl, Proskurowski, Telle [38]). *If all sets of the equitable partition of a simple graph H have at most two vertices, then the H-LBiHom problem is solvable in polynomial time.*

Sketch of the proof. For the instance G we firstly verify the necessary condition, whether G has the same degree refinement matrix.

If any $f : G \xrightarrow{B} H$ exists, then all vertices of G, that correspond to one-vertex sets in H, have uniquely determined image under f. Therefore, the difficult task is to define the mapping f on vertices that correspond to the two vertex-sets $B_i(H) = \{a_i, b_i\}$.

For each vertex $u \in B_i(G)$, introduce a Boolean variable x_u, which will be assigned the truth value, when u is mapped onto a_i, and x_u is set to false, whenever $f(u) = b_i$.

We construct a 2-Sat formula Φ, which allows a satisfying assignment if and only if $G \xrightarrow{B} H$.

- If two distinct vertices u and v belongs to the same block $B_i(G)$ and if they are adjacent or have a common neighbor, then let Φ contain $(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v)$ as a subformula.
- If $(a_i, a_j), (b_i, b_j)$ are the only edges between $B_i(H)$ and $B_j(H)$, then let Φ include conjunction $(x_u \vee \neg x_v) \wedge (\neg x_u \vee x_v)$ as a subformula, for all pairs of vertices $u \in B_i(G), v \in B_j(G)$.
- If $(a_i, b_j), (b_i, a_j)$ are the only edges that connects $B_i(H)$ and $B_j(H)$, then let Φ contains $(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v)$, for all $u \in B_i(G), v \in B_j(G)$.

These three types of clauses in Φ force, that whenever a satisfying assignment for Φ exists, then the corresponding projection is locally injective (and by the degree constraint it must be locally bijective). In the other direction, every mapping $f : G \xrightarrow{B} H$ can be transformed to a satisfying assignment of Φ.

We proved that for all graphs H, which blocks of equitable partition have at most two vertices, the H-LBiHom problem is polynomially reducible to the 2-Sat problem, which is known to be polynomially solvable. □

Both H-LBiHom and H-LInHom problems are polynomially solvable for trees, even if H became a part of the input. Then, the H-LBiHom problem is equivalent to the tree-isomorphism problem. If the tree H is fixed, then the tree isomorphism testing is solvable in constant time.

The H-LInHom problem is solvable in constant time too, because we can ask whether an input graph G is a subtree of H. If H is fixed, then it has only finitely many subtrees, and we can try each case separately.

In addition, both problems are solvable in polynomial time for graphs H that have only one cycle. If a connected graph G alows a locally bijective homomorphism to a unicyclic H, then the graph G has exactly one cycle, and its length is multiple of girth(H). We denote the multiplicity by k, and build the unique graph G' on $k|V_H|$ vertices such that $G' \xrightarrow{B} H$. Finally, we perform the test whether the two unicyclic graphs G and G' are isomorphic. This can be done by a slight modification of the tree isomorphism algorithm:

Proposition 5.5. *The* H-LBiHom *and* H-LInHom *problems are solvable in polynomial time for every graph* H *with at most one cycle in each component of connectivity.*

The result of Abello et al. [1] stated, that there are many graphs H, such that the H-LBiHom problem is NP-complete, even if the construction was based on highly symmetric graphs, i.e., graphs with a rich group of automorphisms. We recall here a general result characterizing the computational complexity of the H-LBiHom problem for k-regular graphs. It disproves the expectation of Abello et al. [1] that the H-LBiHom problem should allow a polynomial algorithm for k-regular graphs with poor (or trivial) automorphism group.

Theorem 5.6. *The* H-LBiHom *problem is* NP*-complete for all* k*-regular graphs* H *with* $k \geq 3$.

The theorem is a direct consequence of Lemma 3.8 combined with the the following proposition:

Proposition 5.7 (Kratochvíl, Proskurowski, Telle [38]). *The* H-LBiHom *problem is NP-complete when the graph* H *is* k*-regular* $\lfloor \frac{k+1}{2} \rfloor$*-edge connected or* k*-regular* k*-edge-colorable, for every* $k \geq 3$.

Proof of Theorem 5.6. If H is bipartite k-regular graph, then by König-Hall marriage theorem it is k-edge colorable and we apply Proposition 5.7 directly.

In the other case we assume that H is connected and not bipartite. We construct the Kronecker double cover $\tilde{H} = H \times K_2$. This graph $\tilde{H}$ is bipartite and hence k-edge colorable.

Now, Lemma 3.8 provides a straightforward reduction of the $\tilde{H}$-LBiHom problem to the H-LBiHom problem. Assume that a graph G is an instance of the $\tilde{H}$-LBiHom problem. Lemma 3.8 implies that $G \xrightarrow{B} \tilde{H}$ if and only if G is bipartite and $G \xrightarrow{B} H$ holds. □

As a consequence of Proposition 5.5 and Theorem 5.6, we get that the computational complexity of the H-LBiHom problem is fully classified for all regular graphs.

5.2 Colored directed multigraphs

In this section we show that for the existence of a locally bijective homomorphisms between two graphs we may compress all vertices of degree at most two.

If a general graph H contains a cutvertex whose removal gives a tree as a block, then every preimage of this cutvertex in G separates an isomorphic tree whenever $G \xrightarrow{B} H$. An analogous condition holds for a cutvertex which separates a forest.

This observation gives us an idea of how to concentrate our attention only to graphs without leaves: If there is a leaf in the graph, remove it and maintain a code that the leaf was removed together with the code of the leaf. This gives us a graph without leaves, where some vertices are labeled. For simplicity, we view different labels as different vertex colors.

At the second step, we remove all vertices of degree two from the graph: Consider a path connecting two vertices of degree at least three, whose all internal vertices have degree two. We replace the path by a single edge and maintain the code of the number, order and colors of vertices of the replaced path. Due to a similar reason, we call the code of the path the edge color. Since the removed path is not necessarily symmetric, we assign an orientation to the edge. Note, that we can uniquely reconstruct the original path from the color and the orientation of the edge.

It is possible, that the path replacement create a multigraph with loops and multiple edges.

We now represent each graph by a directed, edge and vertex colored multigraph of minimum degree three, and with the following property: Two simple graphs allow a locally bijective homomorphism if and only if there exist a locally bijective homomorphism between the two derived directed colored multigraphs, such that the homomorphism maintains edge directions as well as edge and vertex colors [37].

In the section 2.2 we have constructed a degree refinement and the matrix of degree refinement as a tool, that allows us to partially determine the image of a vertex under hypothetical locally bijective homomorphism between simple graphs. A similar procedure can be performed for a colored directed multigraph G:

First fix an ordering of all edge and vertex colors — this is necessary for the unique definition of the degree refinement matrix. Suppose that the undirected edges are colored by $1, \dots, p$, while directed edges by $1', \dots, q'$.

Denote by $\vec{d}(u)$ the degree vector of the following form: $\vec{d}(u) = (c_v(u), \deg_1(u), \dots, \deg_p(u), \mathrm{indeg}_{1'}(u), \mathrm{outdeg}_{1'}(u), \dots, \mathrm{indeg}_{q'}(u), \mathrm{outdeg}_{q'}(u))$, where $c_v(u)$ is the vertex color of u, $\deg_i(u)$ is the number of edges of color i incident to u, and the symbols $\mathrm{indeg}_{i'}(u)$ and $\mathrm{outdeg}_{i'}(u)$ have a similar meaning — the number of oriented edges of color i' incident to u.

The first step of the degree refinement procedure consists of splitting V_G into sets R'_i, such that vertices in the same set have the same degree vector. Sort sets by the lexicographical order of their representatives.

Then refine the partition, as shown in section 2.2 until all vertices from the same set have the same number in neighbors in each set R_i.

Without loss of generality we can use different colors on vertices from different blocks of the equitable partition. We can also use distinct colors for edges, that connect different pairs of blocks. Therefore, we may assume that the edge colors used inside a single block are unique as well as colors of edges that connect a particular pair of blocks. For this purpose, we introduce extra new colors if necessary.

By using the same argument, we separate oriented edges leaving a block from the incoming edges. Hence, we assume that the oriented edges appear only inside blocks of the degree refinement.

Kratochvíl, Proskurowski and Telle [39] designed a complete catalogue of H-cover instances of all simple graphs H with at most six vertices. They showed that out of 208 parameter graphs H 36 cases are NP-complete, and a non-trivial polynomial reduction was shown for about 100 graphs H.

The same group of authors introduced in the sequel paper [37] the colored directed multigraphs as a structure, which exclude vertices of degree at most two, and they gave a complete characterization for colored directed multigraphs graphs with at most two vertices.

The proof technique of Theorem 5.4 was extended to the case of colored directed multigraphs as follows:

Proposition 5.8 (Kratochvíl, Proskurowski, Telle [36]). *The H-LBiHom is a polynomially solvable problem if H is a colored directed multigraph, whose classes of degree refinement (with respect to vertex and edge color) have 1, 2 or 4 vertices, and further two conditions are satisfied:*

- *Each block of degree refinement restricted to the edges of the same color is one the following type:*
  - *a disjoint union of (directed) loops or (directed) multiple edges,*
  - *the graph depicted in Fig. 5.1 or two disjoint copies of this graph,*
  - *the cycle C_4,*
  - *C_4 whose all edges are replaced by a multiple directed edges, all in the same direction and with the same multiplicity,*
  - *C_4 whose all edges are replaced by a directed C_2.*
- *Moreover, the edges of the same color, that join a pair of distinct blocks, induce a undirected subgraph of one of the following type:*
  - *a disjoint union of multiple edges,*
  - *$K_{2,1}$ or a disjoint union of two $K_{2,1}$,*
  - *$K_{2,2}$ or a disjoint union of two $K_{2,2}$.*

Recall, that vertices forming a block of degree refinement have the same degree, indegree and outdegree with respect to an arbitrary edge color. If a block is a disjoint union of more components (the fist and the second case), then the degree is the same, for all vertices from the block of the degree refinement.

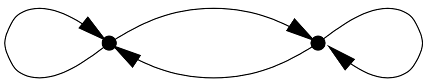


Figure 5.1: A subgraph type for the tractable H-LBiHom problem.

Chapter 6

Complexity of H-LInHom problems

As follows from Theorem 5.1 the H-LInHom problem becomes NP-complete for any graph H such that the H-LBiHom problem is NP-complete. In particular, this applies for all regular graphs with valency at least three.

Recall also Proposition 5.5, which states that the H-LInHom problem is polynomially solvable for graphs with at most one cycle in each component of connectivity.

We have already pointed that graphs whose H-LBiHom problem can be solved in polynomial time can bring a nontrivial characterization for the corresponding H-LInHom problem. In this chapter, we focus on three classes of simple graphs, such that the H-LBiHom problem allows a polynomial time algorithm due to Proposition 5.8.

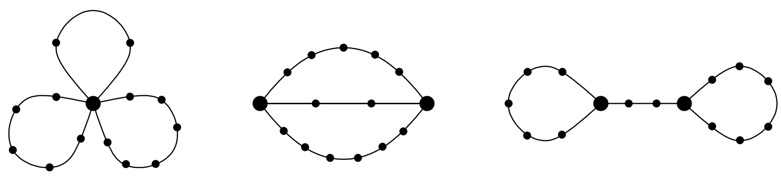
Definition 6.1. Let $(a_1, \ldots, a_k)$ be a k-tuple of positive integers, $k \geq 2$. The *flower graph* $F(a_1, \ldots, a_k)$ is the only graph (upto an isomorphism) whose 2-connected components are cycles of length $a_1, \ldots, a_k$, all intersecting in a common single vertex.

Definition 6.2. For a k-tuple of positive integers $(a_1, \ldots, a_k)$, $k \geq 3$, we define the *theta graph* $\Theta(a_1, \ldots, a_k)$ as the unique graph consisting of two vertices of degree k joined by k paths of length $a_1, \ldots, a_k$.

Definition 6.3. For a triple (a_1, a_2, a_3) of positive integers we define the *barbell graph* $B(a_1, a_2, a_3)$ as the graph containing exactly two cycles of length a_2 and a_3 that are connected by a path of length a_1.

Examples of the above defined graphs are depicted in Fig. 6.1.

As a corollary of Proposition 5.3 we get that the $B(ta, tb, tc)$-LInHom problem is polynomially equivalent to the $B(a, b, c)$-LInHom problem. Similarly, the $F(ta_1, \ldots, ta_k)$-LInHom, and the $\Theta(ta_1, \ldots, ta_k)$-LInHom problem are polynomially equivalent to the $F(a_1, \ldots, a_k)$-LInHom and to the



$F(3,6,7)$ $\Theta(3,6,7)$ $B(3,6,7)$

Figure 6.1: Examples of flower, theta and barbell graphs with parameters $(3, 6, 7)$.

H-LINHOM	Polynomially solvable	NP-complete
$H = F(a^i, b^j)$	for all a and b and every i, j.	
$H = \Theta(a^i, b^j)$	when a and b are divisible by the same power of two	otherwise
$H = B(a, b, b)$	whenever b is divisible by a strictly higher power of two than a	otherwise

Table 6.1: Complexity characterization for some H-LINHOM problems

$\Theta(a_1, \ldots, a_k)$-LINHOM, respectively. Hence, from now we assume that the parameters have altogether no common nontrivial divisor.

6.1 Special graphs with two parameters

We first consider the situation, when at most two distinct parameters a, b appear in the specification of flower, theta and barbell graphs. Instead of $F(a, a, .., a, b, \ldots, b)$, we write $F(a^i, b^j)$, where i and j denote the multiplicity of the parameters a and b, respectively. For simplicity, we drop the zero exponent term in our notation, i.e., $F(a^i) = F(a^i, b^0) = F(b^0, a^i)$. The same notation we use for theta graphs.

The full computational complexity characterization of the H-LINHOM problem on flower, theta and barbell graphs with only two distinct parameters is summarized in the Table 6.1 [16].

Surprisingly, while the problems $F(a^i)$-LINHOM and $\Theta(a^i)$-LINHOM are solvable in polynomial time, the $B(a, a, a)$-LINHOM problem is NP-complete.

The polynomial algorithm are based on the technique developped in the following section.

6.1.1 Flag factors

Let G be a multigraph. A *flag* is a pair $[u, e]$, where u is a vertex of G and e is an edge incident with u. We denote by $F_G = \{[u, e] : u \in e \in E_G\}$ the multiset of flags of G. Note that loops in G give rise to two flags each.

Suppose we are given sets of nonnegative integers I_u, for every vertex u, and for every edge $e = (u, v)$, a direction is chosen (say $[u, v]$) and a set $J_e \subseteq \{0, 1\} \times \{0, 1\}$ is given.

Definition 6.4. Given multigraph G, a set $I_u \subset \mathbb{N} \cup \{0\}$ for each $u \in V_G$ and a set $J_e \subseteq \{0, 1\} \times \{0, 1\}$ for each $e \in E_G$, we say that a set $S \subseteq F_G$ of flags is a *flag factor satisfying constraints I_u and J_e* if the following two conditions hold.

1. the number of flags of S emanating from a vertex u is in I_u, for every vertex u,
2. for every directed edge $e = [u, v]$ holds that $[|\{[u, e]\} \cap S|, |\{[v, e]\} \cap S|] \in J_e$.

In other words, the sets I_u represent permitted degrees of vertices in the 'subgraph' determined by S. In addition, the sets J_e contain permitted characteristic vectors of S reduced to the flags that arise from the edge e. We refer to the problem of deciding the existence of S as the Flag Factor problem.

Flag Factor (FF)
Instance: Multigraph G with sets I_u, J_e for each $u \in V_G$, $e \in E_G$.
Question: Does G allow a flag factor $S \subseteq F_G$ respecting all vertex constraints I_u as well as all edge constraints J_e?

We show that the FF problem is polynomially solvable if the permitted degree sets are intervals, while the sets J_e may be arbitrary:

Lemma 6.5. *The* Flag Factor *problem is solvable in polynomial time if all the sets $I_u, u \in V_G$, are intervals of integers.*

Proof. When $J_e = \{[0, 0], [1, 1]\}$, then a flag factor contains either both or none flags $[u, e], [v, e]$. If this holds for every edge in G then a flag factor is a spanning subgraph of $G' \subset G$, such that $\deg_{G'}(u) \in I_u$ for every vertex u. If all I_u's are intervals, then this problem can be solved in polynomial time by using the maximum matching technique [45, exercise 10.2.2].

We show that the other instances can be transformed to the above case by performing local reductions at edges of G. In the transformed graph $\tilde{G}$ holds sets $\tilde{J}_e = \{[0, 0], [1, 1]\}$ for every edge $e \in E_{\tilde{G}}$. Hence, we only define the sets $\tilde{I}_u$ according to the folowwing rules:

1. Each vertex u of G remains a vertex of $\tilde{G}$ with $\tilde{I}_u := I_u$.
2. Keep all edges $e \in E_G$ such that $J_e = \{[0, 0], [1, 1]\}$ unchanged.

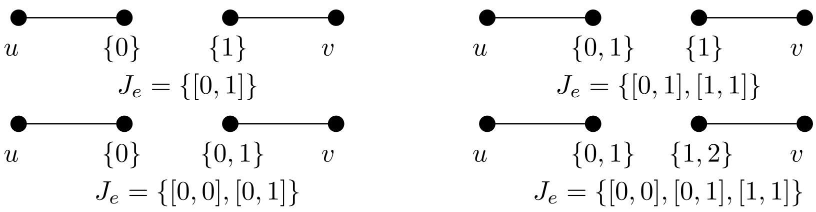


Figure 6.2: The four cases of rule 4.

3. If the set J_e contains none or both the asymmetric pairs $[0,1]$ and $[1,0]$, then subdivide the edge e by an extra new vertex x_e, and set $\tilde{I}_{x_e} = \{a+b, [a,b] \in J_e\}$.

4. Othervise assume without loss of generality that $e = (u,v)$ allows exactly one asymmetric pair, say $[0,1] \in J_e$. There are four possible cases for the set J_e. The corresponding replacement rules of $e = [u,v]$ together with the definition of the intervals $\tilde{I}$ for the new vertices are depicted in Fig. 6.2.

We claim that a flag factor S exists in G if and only if the new graph $\tilde{G}$ contains a spanning subgraph G', s.t. $\deg_{G'}(u) \in \tilde{I}_u$ for all vertices $u \in V_{\tilde{G}}$.

Suppose first that G' exists. The first two rules guarantee that the vertex constraints are fulfilled on all vertices of G as well as on all edges with $J_e = \{[0,0],[1,1]\}$.

Consider a vertex x_e which was inserted into an edge $e = (u,v)$ by the third rule. If both edges $(u,x_e),(v,x_e)$ belong to G', then $2 \in \tilde{I}_{x_e}$. This is only possible if $[1,1] \in J_e$. Thus putting both flags $[u,e],[v,e]$ into S keeps the degrees of u and v and is compatible with J_e. Analogously, if $(u,x_e),(v,x_e)$ are not present in G' then by the same argument we can leave both flags $[u,e],[v,e]$ out of S and fulfils the constraint J_e. Finally, if only one of (u,x_e) belongs to G', then we get $\{[1,0],[0,1]\} \subseteq J_e$ and we can putting $[u,e]$ into S without violating J_e.

The case analysis for the remaining rule is similar. The opposite implication, i.e., tht the existence of a flag factor implies the existence of G', is straightforward. □

6.1.2 Proofs

We now give proofs to the result presented in Table 6.1.

Definition 6.6. We say that P is a *maximal path* in G connecting (not necessarily distinct) vertices u and v if the internal vertices of P have degree two in G, while $\deg_G(u), \deg_G(v) \neq 2$.

Theorem 6.7. *The $F(a^i, b^j)$-LINHOM problem is polynomially solvable for any positive integers a, b, i, j.*

Proof. We may assume that $i+j \geq 2$, since the C_n-LINHOM problem is solvable in polynomial time by Proposition 5.5. As the order of parameters a, b does not matter, we assume without lost of generality that $i \geq j$. Now, let G be the graph for which the existence of a locally injective homomorphism to $F = F(a^i, b^j)$ is questioned.

Assume that G is connected, otherwise we perform the computation separately for each component of G. If G is a cycle, then it maps onto F if and only if its length is a nonnegative linear combination of a and b (when $i, j \geq 1$) or a multiple of a (when $j = 0$). This question can be easily tested in linear time.

Now, assume that G is not a cycle, and denote by v the central vertex of F. By the local injectivity, every vertex of G of degree at least three must be mapped onto v by any locally injective homomorphism. It remains to find the image of vertices of degree at most two. Consider a maximal path in the graph G with both endpoints of degree at least three. We decide whether none, one or both terminal edges of the path can be mapped into a cycle of length a in F. This decision can be done in constant time, since the outcome depends only on the length l of the path, and for $l > ab$ all three cases are possible. Denote the set of all possible cases by $J(l)$, more formally, put $[0, 0] \in J(l)$ if the equation $l = pa + qb$ allows a nonnegative integer solution with $q \geq 2$, put $[1, 0], [0, 1] \in J(l)$ when $p, q \geq 1$, and finally, $[1, 1] \in J(l)$ if $p \geq 2$.

In G, replace each maximal path of length l by a single edge e, and set $J_e = \{[0, 0], [0, 1], [1, 0], [1, 1]\}$, when e ends in a vertex of degree one, and set $J_e = J(l)$ otherwise. Call the new multigraph G'.

Assign $I_u = [\max(\deg(u) - 2j, 0), \min(\deg(u), 2i)]$ to every vertex u of G' and ask whether a flag factor S for G' exists, with respect to the sets I_u and J_e. Due to Lemma 6.5 the question can be answered in polynomial time. If the result is negative, then G does not allow any locally injective homomorphism to F, since the existence of a flag factor S is a necessary condition.

We argue that this necessary condition is also sufficient for the existence of a locally injective homomorphism. Suppose now that a flag factor S exists. We construct a locally injective homomorphism as follows. Vertices of degree greater than two in G will map onto vertex v, and along each path (corresponding to an edge of G') we use a locally injective homomorphism compatible with the flag factor S. E.g., if $S \cap \{[u, e], [w, e]\} = \{[u, e]\}$ for an edge $e = (u, w)$, the begining of this $u - w$ path is mapped onto a cycle of length a and its end segment (near w) is mapped onto a cycle of length b. It needs to be shown that we can really distribute the a-cycles (and b-cycles) properly, i.e., we can say onto *what* a-cycle (b-cycle) a segment of a path is mapped.

To see this, direct the cycles of H cyclically and number the a-cycles $1, \ldots, i$ and number the b-cycles $1, \ldots, j$. There is a natural correspondence between the flags in $F(G')$ and the edges of $G'^{:2}$. Let G'_a be the bipartite subgraph of $G'^{:2}$ restricted to the flags of S.

Further, let G''_a be the graph obtained from G'_a by replacing a path u, $[u, e]$, $[w, e]$, w by the edge $e = (u, w)$ if both flags $[u, e], [w, e]$ belong to S. The definition of the interval I_u guarantees that the maximum degree in G''_a is $\leq 2i$. Then G''_a has an orientation with maximum indegree as well as maximum outdegree $\leq i$ (to see this, add edges to embed G''_a into a $2i$-regular graph and direct its edges along an Eulerian circuit).

Now color the edges of G''_a so that each vertex has at most one outgoing and at most one ingoing arc of each color in the chosen orientation (this is possible by splitting each vertex into two — one being the endvertex of all incoming edges and the other one being the starting vertex of all outgoing edges — obtaining a bipartite graph of maximum degree i, which is edge-i-colorable by Petersen Theorem). This means that for each color, the subgraph of G''_a determined by edges of this color is a disjoint union of directed cycles and/or paths.

Now in G, we map the 'outer' vertices of paths corresponding to edges colored in G''_a by the color h onto the h-th a-cycle of H. E.g., if this a-cycle has vertices $v, x_1, \ldots, x_{a-1}$ and an edge $e = (u, w)$ of G''_a is directed from u to w and corresponds to a path in G of length $l = pa + bq$ where $p \geq 2$ or $q = 0$, we map the vertices along this path (from u to w) onto $v, x_1, \ldots, x_{a-1}$, then q-times onto arbitrary b-cycle (in the direction of the cycle), then $(p-1)$-times onto $x_1, \ldots, x_{a-1}$ and finally w is mapped onto v. Similarly, we handle the edges of G''_a that correspond to flags of G'. Note that if an edge of G' gives rise to only one flag in S, the locally injective homomorphism along its preimage path in G starts with a mapping onto a a-cycle and ends with a mapping onto an b-cycle.

In this way we guarantee that each vertex of degree greater than two in G has at most one neighbor mapped onto x_1 and at most one neighbor mapped onto x_{a-1}. Since this holds true for all $h = 1, 2, \ldots, i$, and since a similar procedure works for the b-cycles, we see that the mapping constructed is indeed a locally injective homomorphism. □

Theorem 6.8. *The* $\Theta(a^i, b^j)$-LINHOM *problem is polynomially solvable if a and b are divisible by the same power of 2, or if $i + j \leq 2$.*

Proof. Note that due to Proposition 5.3 the $\Theta(a^i)$-LINHOM problem is equivalent to the $\Theta(1^k)$-LINHOM problem, that is equal to the edge coloring of bipartite graphs, and can be solved in polynomial time. Hence we assume $i, j \geq 1$.

When $i + j \leq 2$ then the graph $\Theta(a^i, b^j)$ contains at most one cycle and $\Theta(a^i, b^j)$-LINHOM problem clearly allows a polynomial time algorithm. Since a and b are of the same parity, we assume $i \geq 2$.

The proof is based on a similar argument as the proof of Theorem 6.7. We point out the differences from the previous proof.

Let G be a connceted instance for the $\Theta(a^i, b^j)$-LINHOM problem. If G is a cycle, then its length l must be a nonnegative linear combination of a and b, say $ap + bq$, such that $p + q$ is even and $q = 0$ when $j = 0$, or $q \leq p$ when $j = 1$. This can be tested in constant time.

Denote by v and w the two vertices of $\Theta = \Theta(a^i, b^j)$ that have degree at least three. Observe that the graph Θ is bipartite, with vertices v and w in different classes of the bipartition. Hence, the graph G should be bipartite as well and any pair of vertices of degree at least three must map onto the same target (v or w), whenever they belong to the same class of bipartition. Otherwise no locally injective homomorphism exists. We fix one of the two possible mappings on vertices of degree at least three and call it f.

Create the graph G' and compute sets $J(l)$. When $j = 1$, then only linear combinations with parameters $p - 1 \geq q$ are allowed. Assign sets J_e and $I_u = [\max(\deg(u) - j, 0), \min(\deg(u), i)]$, and ask for a subset of flags S. As above the existence of S is a necessary and sufficient condition for the existence of a locally injective homomorphism $f : G \to \Theta$.

Consider the graph G'_a induced by flags from S and determine a proper edge coloring using at most i colors. This is always possible since G'_a is bipartite and $\Delta(G'_a) \leq i$. This coloring helps us to extend the mapping f onto the initial segments of maximal paths in G that map onto a-paths of B. Finally, perform the same procedure for the complement of S, and extend f onto the entire graph G. □

Theorem 6.9. *The $\Theta(a^i, b^j)$-LINHOM problem is NP-complete whenever $|a - b|$ is odd, $i, j \geq 1$ and $i + j \geq 3$.*

Proof. For $i \geq 2$ we show a reduction from the BW(i, j) problem (see Appendix), and we reduce the BW$(j, 1)$ problem in the case of $i = 1$.

Assume a is odd, b is even, and both parameters are relatively prime. We discuss the case $i, j \geq 2$ first. Let G be the $i + j$-regular graph whose black and white coloring is questioned. We replace each edge of G by a path of length $l = ab$ we claim that the new graph G' maps locally injectively $\Theta = \Theta(a^i, b^j)$, if and only if a proper BW(i, j) coloring of G exists.

Suppose that a locally injective homomorphism for $f : G' \xrightarrow{I} \Theta$ exists. All original vertices are mapped onto v or w, the vertices of degree at least three in Θ. Color a vertex $u \in V_G$ black, if $f(u) = v$ and color it white otherwise. There are only two ways how to express $l = ab$ as nonnegative linear combination $ap + bq$: either $p = b, q = 0$ or $p = 0, q = a$. For the parity reasons, each maximal path, which is mapped only onto b-paths of Θ, has end-vertices mapped onto distinct vertices of Θ, whereas both ends are mapped onto the same target, if the a-pattern is used. Due to the local injectivity and the fact that every vertex of G has degree $i+j$, exactly i neighbors of any vertex of degree at least three are mapped into an a-path, and exactly j neighbors are mapped into a b-path of Θ. Obviously, the black and white coloring derived from the locally injective homomorphism is a proper BW(i,j) coloring.

For the opposite direction, consider any BW(i,j) coloring of the graph G. The subgraph of G spanned by the edges connecting vertices with the same color is i-regular and we denote it by G_s. The graph $G_s^{:2}$ is bipartite with maximum degree i, and a proper edge coloring using i colors always exists. This edge coloring determines the mapping from G' into a-paths of Θ as follows: i different colors represent i different a-paths of Θ. Since the beginning segments on any maximal path connecting vertices with the same color should be mapped onto different a-paths, such mapping always exists (remember that b is even, $j \geq 2$).

Similarly, subgraph of G, spanned by the edges interconnecting the sets of white and black vertices, is bipartite and j-regular, and its edges can be colored with j colors. These edge colors represent different b-paths of Θ. For each edge e of G we define mapping of the corresponding path of length l in G' such that the mapping starts and ends in the b-path of Θ corresponding to the color of e.

The mapping defined above is locally injective on the neighborhood of every vertex of G', hence, $G' \xrightarrow{I} \Theta$.

Now, consider the $\Theta(a^i,b)$ and $\Theta(a,b^i)$-LInHom problems. We show a reduction from BW$(i,1)$ problem. The main idea and several arguments are inherited from the previous case. Let G be the $(i+1)$-regular graph whose black and white coloring is questioned. Replace every edge of G by a path of length l where

- $l = ab + (a-1)a$ for the reduction to the $\Theta(a^i,b)$-LInHom problem,
- $l = ab + (b-1)b$ for the $\Theta(a,b^i)$-LInHom.

Suppose, that the new graph $G' = G^{:l}$ satisfies $G' \xrightarrow{I} \Theta(a^i,b)$. There are only two possibilities to cover a path of length $l = ap + bq$ with both ends mapped onto vertices v and w, namely, $p = a+b-1, q = 0$ and $p = a-1, q = a$. The corresponding patterns are $l = a + a + \cdots + a$ and $l = b + a + b + a + \cdots + b$, and in the first case both ends of the path are mapped onto the same target, while at the second case, one end is mapped on v and the other onto w. Note, that it is impossible to use two b-paths consecutively, since it violates the local injectivity around vertices v or w. As in the above case, the existence of a locally injective homomorphism is a proper BW$(i,1)$ coloring. When a such coloring exists, it is possible to find a locally injective homomorphism by the edge/flag coloring argument.

Finally consider the case when $G' \xrightarrow{I} \Theta(a, b^i)$. The equation $l = ap + bq$ allows only the following solutions: $p = 0, q = a + b - 1$ and $p = b, q = b - 1$ that corresponds to a mapping of a maximal path of length l, namely by patterns $l = b + b + \cdots + b$ and $l = a + b + a + b + \cdots + a$. The only difference from the previous case is that the covering pattern that starts with a b-path corresponds to an edge connecting two vertices with the same color (observe that the number of summands is even), while the pattern with the a-path corresponds to an edge in G that connects white and black vertex. The already presented edge coloring argument shows that $G' \xrightarrow{I} \Theta(a, b^i)$ holds, whenever a proper BW$(i, 1)$ coloring exists. □

We now focus our attention to the class of barbell graphs. Hardness of the BW$(2, 1)$ problem together with Proposition 5.3 yields that the $B(a, a, a)$-LInHom problem is NP-complete. Surprisingly, there are parameters a and b that the $B(a, b, b)$-LInHom problem allows a tractable — polynomial time algorithm.

Theorem 6.10. *The $B(a, b, b)$-*LInHom *problem is polynomially solvable, when the parameter a is odd, and b is even.*

Proof. Observe that the graph $B = B(a, b, b)$ is bipartite. Hence, only bipartite graphs G can allow a homomorphism to B. Moreover, the classes of bi-partition of G determine the mapping f on vertices of degree three, as in the proof of Theorem 6.8. Denote by v, w the two vertices of degree three in W, and color a vertex $u \in V_G$ of degree three black, if $f(u) = v$, and color it white when $f(u) = w$. Thus, the 'hard' problem is to determine the mapping on vertices of degree at most two, and it can be solved by a simple procedure: For each maximal path of length l connecting two vertices of the same color, determine whether $l = ap + bq$ allows a nonnegative solution with p even and $q \geq p/2 - 1$. Any maximal path connecting vertices of different colors can cover W, when $l = p$, or if $l = ap + bq$ has a solution satisfying $q \geq (p - 1)/2 - 1$ and p is odd and greater or equal to three.

The local injectivity on vertices of degree three — namely the decision which initial segments will be mapped onto a-paths — can be tested by the flag coloring procedure described in the proof of Theorem 6.8. □

Theorem 6.11. *The $B(a, b, b)$-*LInHom *problem is* NP*-complete, if the parameter b is odd.*

Proof. We show a reduction from the BW$(2, 1)$ problem. Let G be a cubic graph whose black and white vertex coloring is questioned.

Replace each edge of G by a path of length $l = ab + (b - 1)b$ to obtain the graph $G' = G^{:l}$ and suppose that mapping f is a witness for $G' \xrightarrow{I} B = B(a, b, b)$ exists. Color vertices of G, such that a vertex u of degree three gets black color, if $f(u) = v$, and is colored white when $f(u) = w$. The length l can be expressed either as $a + a + \cdots + a$, or $b + b + \cdots + b$. Hence, each vertex has two neighbors of the same color (when the b-pattern is used in G' along the corresponding path), and exactly one vertex of the opposite color: note, that the number of summands equal to a in the expression $l = a + b + a \cdots + b + a$ is odd.

In the opposite direction assume that G allows a BW$(2,1)$ coloring. The maximal paths of G' can be partially mapped into B exactly by the same way, as was shown in the proof of $B(a,b,b)$-LInHomproblem. □

6.2 Special graphs with three parameters

If three distinct parameters are used, the computational complexity of the considered problems is not fully classified. We recall known results for theta and barbell graphs. We also present a recent technique for the $\Theta(a,b,c)$-LInHom problem where all three parameters are odd.

6.2.1 The $\Theta(a,b,c)$-LInHom problem

We first foucus our attention on Theta graphs and present several cases when the $\Theta(a,b,c)$-LInHom problem is NP-complete.

Theorem 6.12. *The $\Theta(a,b,c)$-LInHom problem is NP-complete if*

1. *$a + b$ divides c or*
2. *$a = 1$, $b = 2$ and $c \geq 3$,*
3. *$a = 1$, $b = 3$, $c \geq 4$ and c is even.*

Our argument is based on the following approach:

Definition 6.13. Let $J = \{j_1, ..., j_k\}$ be a set of distinct positive integers. We say that a number m has a *path pattern* with respect to J of type (a,b) and length l if there exist integers x_i for $i \in [l]$ satisfying

- $m = x_1 + \cdots + x_l$;
- $x_i \in J$ for every $i \in [l]$;
- $x_1 = a$ as well as $x_l = b$;
- $x_{p-1} \neq x_p \neq x_{p+1}$ whenever x_{p-1} or x_{p+1} are defined.

Note that whenever m has a solution of type (a,b), then it can be transformed into a solution of type (b,a) of the same length. Hence, the type of a solution is an unordered pair.

Lemma 6.14. *The $\Theta(a,b,c)$-LInHom problem is NP-complete if there exists an m, which has a path pattern of type (c,c) of an odd length, and a pattern of type (a,b) of an even length, and no other path patterns exist with respect to $J = \{a,b,c\}$.*

Proof. We show a reduction from the BW$(2,1)$ problem. Let G be a cubic graph, whose black and white coloring is questioned. We replace each edge of G by a path of length m, and show that the new graph $G' = G^{:m}$ allows a locally injective homomorphism to $\Theta = \Theta(a,b,c)$ if and only if G has a proper BW$(2,1)$ coloring.

Denote by v, w the two vertices of degree three in the graph Θ, and assume that $f : G' \xrightarrow{I} \Theta$ exists. Then every vertex of degree three in G' is mapped either on v or w. Color each vertex $u \in V_G$ black, if $f(u) = v$, and color it white otherwise. The mapping f is locally injective on the neighborhood of any u in G'. Hence, one of the incident edges (u,x) is mapped into a c-path. The maximal path of length m that starts with the exposed edge can be mapped only according to the path pattern of type (c,c). The odd length of the path pattern implies that the opposite end u' of the maximal path will be mapped onto the other vertex of degree three in B, causing that u' gets a different color from the color of u.

By the same argument we can show that the even length of the path pattern of type (a,b) implies that every vertex of G has two neighbors colored by the same color.

In the opposite direction, assume that G has a valid BW$(2,1)$-coloring. A locally injective homomorphism can be found by the technique already described in the proof of NP-completeness of $\Theta(a,b,b)$-LINHOM problem (Theorem 6.9). □

Proof of Theorem 6.12. We apply Lemma 6.14 directly:

1. Set $m = c$. The only path patterns are $m = c$ of type (c,c) (odd length) and $m = a+b+a+b+\cdots+a+b$ of type (a,b) (even length).

2. If $c = 3k$, then the result follows directly from the previous case. When $c = 3k+1$, then setting $m = c+1$, we get the following path patterns $m = 2+1+2+1+2\cdots+2$ (odd length) and $m = c+1$ (even length). Similarly, for $c = 3k+2$ we set $m = c+2$ to get patterns $m = 1+2+\cdots+1 = 1+c+1$ (odd length) and $m = c+2$ (even length).

3. If $c = 4k$, then the result follows directly from the first case. When $c = 4k+2$, then setting $m = c+1$, we get the following path patterns $m = 3+1+3+1+3\cdots+3$ (odd length) and $m = c+1$ (even length).

□

6.2.2 A geometric approach to $\Theta(a,b,c)$-LINHOM

If all the three parameters a, b and c are odd, then the theta graph $\Theta(a,b,c)$ is bipartite, hence any input graph G allowing $G \xrightarrow{I} \Theta(a,b,c)$ must be bipartite. Consequently, vertices of degree three of G map the same target if and only if they belong to the same class of bipartition, which can be easily computed. We now present a NP-hardness result of the $\Theta(1,3,5)$-LinHom problem for any triple of distinct odd positive integers a, b and c.

Theorem 6.15. *For every three distinct odd positive integers a, b and c, the $\Theta(a,b,c)$-LinHom problem is NP-complete.*

The core of the NP-hardness reduction is established in the following lemma.

Lemma 6.16. *If $a < b < c$ and m are odd positive integers such that*

(i) There is no integer solution $x, y, z \geq 0$ of the equation $xa+yb+zc = m$ such that x, y and z satisfy the triangle inequalities $x+y \leq z$, $x+z \leq y$ and $y+z \leq x$;

(ii) There is an integer solution $x, y, z \geq 0$ of the equation $xa+yb+zc = m$ such that $x+y = z-1$;

(iii) There is an integer solution $x, y, z \geq 0$ of the equation $xa+yb+zc = m$ such that $y+z = x-1$;

(iv) There is an integer solution $x, y, z \geq 0$ of the equation $xa+yb+zc = m$ such that $z+x = y-1$,

then the $\Theta(a,b,c)$-LinHom problem is NP-complete.

Proof. We reduce the edge precoloring extension problem (EPE) to $\Theta(a,b,c)$-LinHom.

Given a cubic bipartite graph G with some edges precolored by two colors, say amber and black (the third color will be cyan), we construct G' from G by replacing every amber edge by a path of length a, every black edge by a path of length b, and every edge which is not precolored by a path of length m. As the problem parameters a, b, c, and m are constant, the size of the graph G' is linear in the size of G. If m satisfies the above stated properties, then G' allows a locally injective homomorphism into $\Theta(a,b,c)$ if and only if the edge precoloring of G can be extended to a proper 3-edge-coloring of the whole graph. This follows from the fact that the vertices of degree three in G' must map onto u or v and the paths joining them must each map onto a sequence of paths of length a, b, c with no two consecutive paths having the same length. (See Figure 6.3 for an example.)

If x, y, z are the numbers of occurrences of the lengths a, b, c (respectively) in such a sequence for a path of length m, the condition (i) implies that the lengths of the initial and last segments in each such sequence are the same, and conditions (ii-iv) guarantee that the path of length m can have both the initial and the last segment mapped onto the path of length a (and both onto the path of length b, and as well c). Hence these three options encode the colors (a = amber, b = black and c = cyan).

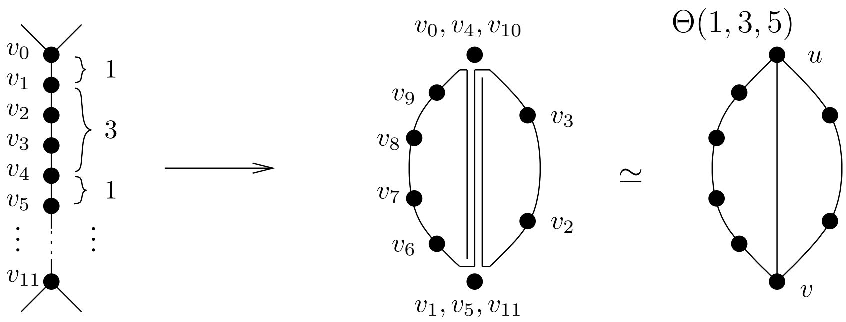


Figure 6.3: Example of a mapping of a path of length $m = 11$ into $\Theta(1, 3, 5)$ according to the pattern $1 + 3 + 1 + 5 + 1 = 11$.

A locally injective homomorphism from G' into $\Theta(a, b, c)$ thus corresponds to a proper 3-edge-coloring of G, since both vertices of degree three vertices of $\Theta(a, b, c)$ are incident with exactly one path of length a, one path of length b and one path of length c. And this coloring must extend the precoloring of G, since a path of length a in G' can only map onto the path of length a in $\Theta(a, b, c)$ (and similarly for the paths of length b). □

The geometric meaning of the condition of Lemma 6.16 is illustrated in Figure 6.4. The triangle determined in the plane $xa + yb + zc = m$ by the triangle-inequalities cone must contain no integer points, but each segment parallel with one of its sides and shifted by 1 away must contain at least one integer point. It turns out that after performing a rotation of the coordinate axes such that this triangle is transformed into the whole triangle determined on $xa + yb + zc = m$ by the coordinate planes, the statement can be proved by an essentially elementary geometric argument.

Theorem 6.17. *Let A, B, C be distinct positive integers. Then a positive integer M exists such that*

(I) There is no integer solution $X, Y, Z \geq 1$ of the equation $XA + YB + ZC = M$;

(II) There is an integer solution $X, Y \geq 1$ of the equation $XA + YB = M$;

(III) There is an integer solution $Y, Z \geq 1$ of the equation $YB + ZC = M$;

(IV) There is an integer solution $Z, X \geq 1$ of the equation $XA + ZC = M$.

The geometric meaning of this theorem is that there always exists a shift of the plane $XA + YB + ZC = 0$ such that the triangle (further referred to as Δ_M) determined in the translated plane by the halfspaces $X \geq 0, Y \geq 0, Z \geq 0$ contains at least one integer point inside each of its sides, but none inside the triangle. (See Figure 6.5.)

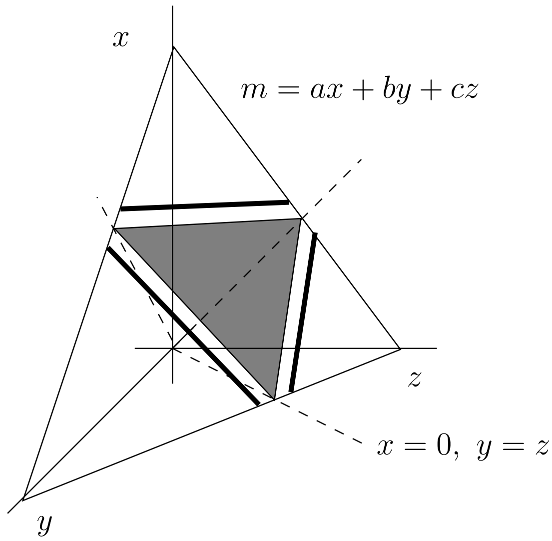


Figure 6.4: The geometric meaning of Lemma 6.16. The thick segments shall contain an integer point.

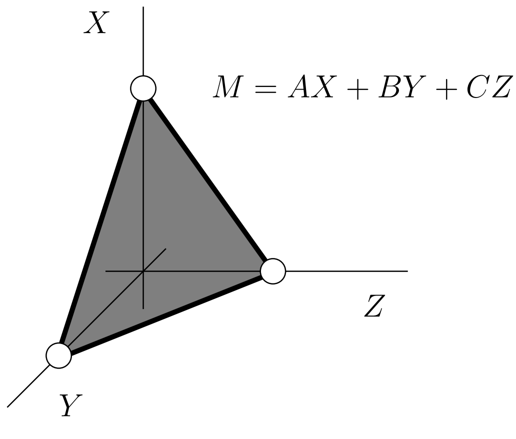


Figure 6.5: The triangle Δ_M.

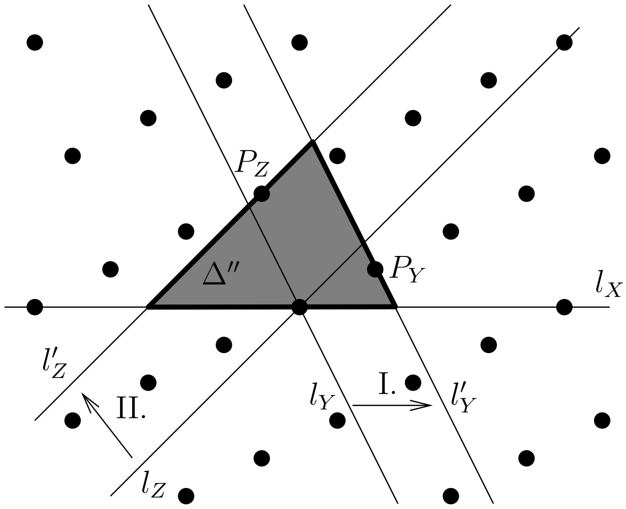


Figure 6.6: Finding points P_Y and P_Z in the plane π.

Proof. Let π be the plane of points (X, Y, Z) for which $XA+YB+ZC = 0$. Denote by L the 2-dimensional lattice that is the intersection of π and the 3-dimensional lattice of integer points. Every translate of π intersects the 3-dimensional integer lattice either in a translate of L, or in the empty set. Let l_X be the intersection line of π and the coordinate plane $X = 0$. Define similarly l_Y and l_Z. Note that l_X, l_Y and l_Z are parallel to the sides of the triangle Δ_M for every $M \neq 0$. The lines l_X, l_Y and l_Z intersect in the origin.

Let P_Y be (one of) the lattice point(s) of L lying in the angle determined by l_X and l_Z and being on the closest line parallel to the line l_Y (for every line parallel to l_Y, this triangle contains only finite number of integer points). Shift the line l_Y into l'_Y that passes through P_Y, thus obtaining a triangle Δ' with lattice points inside the sides lying on l_X and l'_Y, but with no lattice points in its interior. Similarly, let P_Z be a point of L lying in the angle determined by l_X and l'_Y and closest to the line l_Z. Shift l_Z to l'_Z passing through P_Z, obtaining a triangle Δ'' with lattice points inside each of its sides, but with no lattice points in its interior. (See Figure 6.6.)

Let $P_Y = (y_1, y_2, y_3)$ and $P_Z = (z_1, z_2, z_3)$ be the coordinates of these points. For $M = -By_2 - Cz_3$ (an integer), the triangle Δ_M is the translate of Δ'' by the integer vector $(0, -y_2, -z_3)$, and hence it contains the integer point $(0, -y_2, -z_3)$ on its side parallel to l_X, the integer point $(y_1, 0, y_3 - z_3)$ on its side parallel to l_y and the integer point $(z_1, z_2 - y_2, 0)$ on its side parallel to l_Z, but no integer point in the interior. Thus $M = -By_2 - Cz_3$ satisfies (I-IV). □

We now focus on the only missing element in the proof of Theorem 6.15. I.e. that for every three distinct odd positive integers a, b, c, there exists a positive odd integer m such that the conditions (i-iv) of Lemma 6.16 are satisfied. Given $a < b < c$, set

$$A = b + c, \qquad B = a + c, \qquad C = a + b.$$

Let M be the number guaranteed by Theorem 6.17.

Lemma 6.18. *This M satisfies (I-IV) of Theorem 6.17 if and only if $m = M - a - b - c$ satisfies (i-iv) of Lemma 6.16.*

Proof. Note first that since a, b, c are odd, A, B, C are all even and so is M. It follows that $m = M - a - b - c$ is odd. Consider the dual transformations given by

$$(X, Y, Z) \to (x = Y + Z - 1,\ y = Z + X - 1,\ z = X + Y - 1)$$

and

$$(x, y, z) \to (X = \frac{y + z - x + 1}{2},\ Y = \frac{z + x - y + 1}{2},\ Z = \frac{x + y - z + 1}{2}).$$

A simple calculation shows that

$$AX + BY + CZ =$$
$$\frac{(b + c)(y + z - x + 1)}{2} + \frac{(a + c)(z + x - y + 1)}{2} + \frac{(a + b)(x + y - z + 1)}{2} =$$
$$a + b + c + xa + yb + zc$$

and hence

$$AX + BY + CZ = M \qquad \iff \qquad ax + by + cz = m.$$

Obviously, x, y, z are integers if X, Y, Z are. On the other hand, if x, y, z are integers solving $xa + yb + zc = m$, then $x + y + z \equiv 1 \mod 2$ and X, Y, Z are also integers. Thus the transformations provide a bijection among integer solutions of $xa + yb + zc = m$ and $XA + YB + ZC = M$.

It is straightforward that $X = 0$ if and only if $y + z = x - 1$, and that under this assumption $Y \geq 1, Z \geq 1$ imply $x \geq 0, y \geq 0$ and $z \geq 0$, as well as $x, y, z \geq 0$ imply $Y = \frac{z+x-y+1}{2} = \frac{2z+1}{2} > 0$ and $Z = \frac{x+y-z+1}{2} = \frac{2y+1}{2} > 0$. Hence the conditions (ii-iv) of Lemma 6.16 of and (II-IV) of Theorem 6.17 are equivalent.

Similarly, $X > 0$ if and only $y + z > x - 1$, and since all the involved variables are integers, this means that $X \geq 1$ if and only if $y + z \geq x$. Since the inequalities are symmetric, the equivalence of the conditions (i) of Lemma 6.16 and (I) of Theorem 6.17 follows. □

6.2.3 The $B(a,b,c)$-LInHom problem

Theorem 6.19. *The $B(a,b,c)$-LInHom problem is NP-complete, whenever a is a common multiple of b and c.*

In particular, the hardness result holds for parameters $b = 1$, $c = 2$, and all even a greater than three.

For the proof of Theorem 6.19 consider a maximal path of length m (with both ends of degree three) in a graph G that allows a locally injective homomorphism to $B = B(a,b,c)$. Then m can be expressed as a sum $x_1 + \cdots + x_l$ satisfying:

- $x_i \in \{a,b,c\}$ $1 \leq i \leq l$,
- if $x_i = a$ then $x_{i-1} \neq a$ and $x_{i+1} \neq a$, whenever x_{i-1} or x_{i+1} are defined,
- if $x_i = x_j \in \{b,c\}, i < j$, then the number of summands among $x_{i+1}, ..., x_{j-1}$ equal to a is even.

The above properties also imply that the number of a elements among $x_{i+1}, ..., x_{j-1}$ is odd in the case when $x_i = b$ and $x_j = c$, $i < j$. For the rest of this section we redefine the notion of path pattern as follows:

Definition 6.20. Call the expression $m = x_1 + \cdots + x_l$ the *path pattern* of type (x_1, x_l) if all three properties defined in the previous paragraph are satisfied. Define the parity of the pattern as the number of elements from the sum, that are equal to a.

Lemma 6.21. *The $B(a,b,c)$-LInHom problem is NP-complete, whenever there exists an integer m, such that the only path patterns with respect to (a,b,c) are of type (a,a) and the odd parity, and of type (b,b) and (c,c) (of an even parity due to the definition) and that for each allowed type and parity at least one path pattern exists.*

Proof. The reduction from the BW(2,1) problem is straightforward, and is done by the same method as in Lemma 6.14. Recall that the $G^{:m} \xrightarrow{I} B(a,b,c)$ defines a proper black and white coloring, where the odd parity along the pattern of (a,a) type forces distinct colors of vertices incident to the corresponding edge in G, while the patterns of an even parity connect vertices of the same color.

The opposite direction is even simpler, since each color class of vertex color of G use patterns with both ends colored by the fixed color. □

Proof of Theorem 6.19. Put $m = a$. The only possible path patterns are $m = a = b+b+\cdots+b = c+c+\cdots+c$, that are required for the application of Lemma 6.21. □

We mention here that a seemingly harder class of $B(a,b,c)$-partial covering problem surprisingly belong among polynomially solvable problems.

Proposition 6.22. *The* $B(1,2,4)$-LInHom *problem is solvable in a polynomial time.*

Proof. Let G be the instance of the $B(1,2,4)$-LInHom problem. Call $v \in V_B$ the vertex of degree three belonging to the cycle C_4, and call w the other vertex of degree three. We refer the edge (v,w) as the central edge of B.

The multigraph B is bipartite, hence we can assume that G is bipartite too, otherwise no homomorphism $G \to B$ exists. If G is a cycle, then a homomorphism exists, if and only if G is an even cycle. In the following, we assume G has at least one vertex of degree three. Find the bipartition of $V_G = A \cup B$, and denote A_3, B_3 the vertices from the set A of degree three, or from the set B respectively.

We show a test of the existence of a locally injective homomorphism $f : G \to W$ satisfying $f(A_3) = v, f(B_3) = w$. By the symmetry of sets A and B, we can perform the same test with sets A and B interchanged. If both tests fail, then $G \xrightarrow{I} H$ does not hold.

Let f be a mapping on vertices of degree three, which we want to extend to the entire graph G. Consider a maximal path of length l in G with both endpoints u, u' of degree three. According to the length l and mapping of its endvertices, it can be decided in constant time, whether there exists a locally injective homomorphism from the maximal path extending f, and having none, single or both initial edges mapped onto the central edge of B. Denote the set of all possibilities $J(l, f(u), f(u')) \subseteq \{0,1\}^2$. Note that nonsymmetric pairs $[0,1]$ and $[1,0]$ can occur in $J(l,v,w)$.

We build a graph G' by replacing each maximal path of length l connecting vertices u and u' by a single edge, and put $J_{[u,u']} = J(l, f(u), f(u'))$ whenever both u and u' are of degree three and $J_{[u,u']} = \{0,1\}^2$ otherwise.

In addition put $I_u = \{1\}$ for all vertices of G', and ask whether there is a proper subset of flags $S \subseteq F(G')$, satisfying oriented constraints given by sets I_u and $J_{[u,u']}$. Due to Lemma 6.5 this instance of the flag factor (FF) problem can be solved in polynomial time.

The existence of the set S is a necessary condition for $G \xrightarrow{I} B$, and we show that it is also a sufficient condition. By the definition of sets $J_{[u,u']}$ there always exists a path pattern of the corresponding maximal path connecting vertices u and u', mapping only those initial segments on the central edge, that are selected by the subset of flags S, and mapping the other initial segments into cycles in B. Therefore, the subset of flags S can be transformed in polynomial time into a locally injective homomorphism $G \xrightarrow{I} B(1,2,4)$. □

Chapter 7

Complexity of H-LSurHom problems

In the case of locally surjective homomorphisms it is possible to provide full classification of the computational complexity. In contrary to the previous results only few target graphs H allow a polynomial time algorithm for the H-LSurHom problem.

Theorem 7.1. *For a graph H, the H-LSurHom problem is solvable in polynomial time if and only if*

- *either H has no edge,*
- *or H is bipartite and has at least one component isomorphic to K_2.*

In all other cases the H-LSurHom problem is NP-complete.

Before proving the theorem we mention here that in there were studied also other variants of the H-LSurHom problem. Namely we may in addition require that the homomorphisms is globally surjective. For connected graphs it makes no difference according to Lemma 3.4, but for disconnected graphs the classification is be different.

Special H-Locally Surjective Homomorphism (H-LSurHom*) *Instance:* A graph G. *Question:* Does G allow a locally sujective homomorphisms $G \xrightarrow{S} H$ which is also globally surjective?

The computational complexity of the H-LSurHom* problem will be settled in Sect. 7.3.

Another direction on examining existence of a locally surjective homomorphisms was studied in [56]. The following problem was defined there.

k-LSurHom *Instance:* A graph G. *Question:* Does exist a graph H on k vertices such that $G \xrightarrow{S} H$?

It is of interest in social network theory where networks are modeled, in which individuals of the same social role relate to other individuals in the same way. The networks of individuals are represented by simple graphs.

Again our aim is to fully characterize the computational complexity of the k-LSurHom problem. Clearly the 1-LSurHom problem is solvable in linear time, since it is sufficient to check whether G has no edges ($H = K_1$) or whether all vertices in G have degree at least one (H consists of one vertex with a loop). The 2-LSurHom problem is proven to be NP-complete in [56]. We generalized this result as follows:

Proposition 7.2. *The k-LSurHom problem is polynomially solvable for $k = 1$ and it is NP-complete for all $k \geq 2$.*

We refer here also to [20] for the computational complexity characterization of the H-LSurHom problem in the case when the target graph H is allowed to contain loops at some vertices.

7.1 Auxiliary constructions

Lemma 7.3. *Let G and H be a graph such that $f : G \xrightarrow{S} H$. If x, y are vertices of H connected by a path P_H then for each u with $f(u) = x$ a vertex $v \in V_G$ and a path P_G connecting u and v exist, such that r restricted to P_G is an isomorphism between P_G and P_H.*

Proof. We prove the statement by induction on the length of the path P_H. If x and y are adjacent, then the vertex u has a neighbor v mapping onto y, by the definition of the locally surjective homomorphism.

Now assume that the path P_H is of length $k \geq 2$, and that the hypothesis is valid for all paths of length at most $k-1$. Denote by y' the predecessor of y in P_H and by P'_H the truncation of P_H by the last edge, i.e., the path of length $k-1$ connecting x and y'. By the induction hypothesis G contains a vertex v' and a path P'_G such that $P'_G \simeq P'_H$ under f. Then it is easy to find a neighbor v of v' satisfying $f(v) = y$ and tack it to P'_G to get the desired path P_G. □

In particular we may also in the above lemma conclude that $\mathrm{dist}_H(x, y) \leq \mathrm{dist}_G(u, v)$. We get immediately the following claims:

Observation 7.4. *Let $f : G \xrightarrow{S} H$ and u be a vertex in a graph G. If H is connected, then $f(\{v \mid \mathrm{dist}(u, v) \leq \mathrm{diam}(H)\} = V_H$.*

Proof. Let $f(u) = x$. Suppose y is an arbitrary vertex in H. Let P_H be a shortest path in H connecting x and y. By Lemma 7.3 a vertex $v \in V_G$ exists with $f(v) = y$ and a path P_G of length $\mathrm{dist}_H(x, y) \leq \mathrm{diam}(H)$ connecting u and v. Hence $y \in f(\{v \mid \mathrm{dist}(u, v) \leq \mathrm{diam}(H)\}$. □

Lemma 7.5. *Let $f : G \xrightarrow{S} H$, $u \in V_G$ be a vertex, $x = f(u)$, and z, y be some other vertices of H. If in G each path connecting u to a vertex of mapped to y contains a vertex mapped to z, then the vertex z is a cutvertex in H.*

Proof. Since vertices mapped onto x and y are connected by a path in G, there exists a path in H connecting x to y. Moreover if z were not a cutvertex, then we can find such a path avoiding z. But then by Lemma 7.3 we can find a path in G from u to some vertex $v : f(v) = y$ avoiding any vertex mapped onto z under f. □

Our constructions of the NP-hardness reduction involve at several places graph product. At this moment we show several useful properties of graphs constructed by use of product with respect to locally surjective homomorphisms.

Lemma 7.6. *Let H be a graph without isolated vertices. If a graph G allows a locally surjective homomorphism $f : G \xrightarrow{S} H$, then the mapping $g : V_{G\times H} \to V_H$ given by $g((u, v)) = f(u)$ is a locally surjective homomorphism $G \times H \xrightarrow{S} H$.*

Proof. Consider a vertex (u, v) of $G \times H$ with $g((u, v)) = f(u) = x$.

Suppose x' is a vertex in $g(N_{G\times H}(u, v))$. Then there exists a neighbor (u', v') of (u, v) such that $g(u', v') = x'$. By definition of g we have $x' = g((u', v')) = f(u')$, and (u, u') is an edge in G by the definition of $G \times H$. Because f is locally surjective, the vertex x' must be a neighbor of $x = f(u) = g((u, v))$.

Now suppose x' is a vertex in $N_H(g((u, v))) = N_H(f(u)) = N_H(x)$. Since f is a locally surjective homomorphism $G \xrightarrow{S} H$, vertex u must have a neighbor u' mapped onto x'. Because H has no isolated vertices, the vertex v in H has a neighbor v'. Then (u, v) and (u', v') are adjacent vertices in $G \times H$. Hence, x' appears as an image of a vertex in $g(N_{G\times H}(u, w))$. □

In general, we cannot conclude that a graph G satisfies $G \xrightarrow{S} H \times H'$ if G allows both locally surjective homomorphisms $G \xrightarrow{S} H$ and $G \xrightarrow{S} H'$. However, we have shown in Lemma 3.8 that this does hold when $H = K_2$. We will later use this result in our NP-completeness proof for nonsimple target graphs.

We finish this section by construction of a graph allowing more locally surjective homomorphisms to the same target graph.

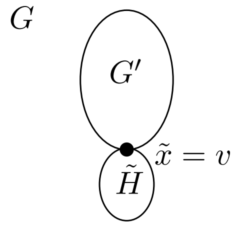


Figure 7.1: A construction of a glued subgraph.

Lemma 7.7. *Let H be a simple graph without isolated vertices. Then for any two vertices x and y a simple connected graph A exists that allows two locally surjective homomorphisms $f_1, f_2 : A \xrightarrow{S} H$, such that a vertex u exists in A with $f_1(u) = x$, and $f_2(u) = y$. Moreover, A can be constructed in time being polynomial with respect to the size of H.*

We have picked the name A to remind that for the special vertex there are two *alternatives*.

Proof. Start with the product graph $H \times H$. By Lemma 7.6 it is clear that the projections $r_1 : (z, z') \to z$ and $r_2 : (z, z') \to z'$ are locally surjective. Hence vertex $u = (x, y)$ satisfies the statement of the lemma. Since H is simple, $H \times H$ does not contain loops as well. Then we can take A as the component of $H \times H$ containing the vertex $u = (x, y)$. □

Definition 7.8. We say that a graph $\tilde{H}$ is *glued* in a graph G by a vertex $\tilde{x}$, if G can be obtained from $\tilde{H}$ and some other graph G' by identifying a vertex $v \in V_{G'}$ with the vertex $\tilde{x}$.

See Figure 7.1 for a more intuitive picture of such a glued graph.

For our NP-completeness proof we would like to construct instance graphs that contain vertices, for which we can deduce where are mapped into the target graph. For this purpose we utilize the maximum distance vertices.

A vertex $u \in V_G$ is called a *maximum distance vertex* if there exists a vertex $v \in V_G$ with $\text{dist}(u, v) = \text{diam}(G)$. We denote by D_G the set of all maximum distance vertices in G.

Lemma 7.9. *Let H be a simple connected graph and let x be a maximum distance vertex in H. Let further G be a graph such that $f : G \xrightarrow{S} H$, where $\tilde{H} \simeq H$ is glued in G by the vertex $\tilde{x}$, the isomorphic copy of x in $\tilde{H}$. Then f restricted to $V_{\tilde{H}}$ is an isomorphism between $\tilde{H}$ and H. In particular, f can be composed with an automorphism π of H to get an locally surjective homomorphism $\pi \circ f : G \xrightarrow{S} H$ such that $\pi \circ f(\tilde{x}) = x$.*

Proof. Choose a vertex $y \in V_H$ such that $\text{dist}(x, y) = \text{diam}(H)$. Then by Observation 7.4 all vertices of H must appear on vertices at distance at most $\text{diam}(H)$ from $\tilde{y}$. Since there are exactly $|V_H|$ many such vertices, namely only the vertices in $V_{\tilde{H}}$, the mapping f is a one-to-one mapping when restricted to $V_{\tilde{H}}$.

Every edge-preserving bijective mapping between two graphs with the same number of edges must be an isomorphism. Now let the automorphism $\pi : H \to H$ be defined by

$$\pi(y) = z \quad \Longleftrightarrow \quad f(\tilde{z}) = y,$$

where $\tilde{z}$ is the isomorphic copy of z in $\tilde{H}$. As locally surjective homomorphisms are preserved under composition the mapping $\pi \circ f$ is the desired locally surjective homomorphism $G \xrightarrow{S} H$ satisfying $\pi \circ f(\tilde{x}) = \pi(f(\tilde{x})) = x$. □

7.2 Connected target graphs

We assume that the instance graph G is simple, while the graph H may contain loops. In this section we consider the case where H is simple as well. Below we prove the conjecture of Kristiansen and Telle [41].

Proposition 7.10. *Let H be a connected graph. Then the H-LSurHom problem allows a polynomial time algorithm if $|V_H| \leq 2$ and it is* NP*-complete otherwise.*

Proof. First we show that H-LSurHom is polynomially solvable for $|V_H| \leq 2$.

- $|V_H| = 1$. Clearly, $G \xrightarrow{S} H$ if and only if G contains no edges.
- $|V_H| = 2$, i.e., $H \simeq K_2$. Then $G \xrightarrow{S} H$ if and only if G is a bipartite graph that does not contain any isolated vertices.

So let $|V_H| \geq 3$. Since we can guess a mapping $f : V_G \to V_H$ and check in polynomial time if f is a locally surjective homomorphism, the problem H-LSurHom belongs to the class NP. We prove NP-completeness by reduction from Hypergraph 2-Colorability (see Appendix). This is a well-known NP-complete problem (cf. [22]).

The instance of H2C problem consists of a set system $\mathcal{S} = \{S_1, \ldots, S_n\}$ over a set $Q = \{q_1, \ldots, q_m\}$. With such an instance we associate its incidence graph I, which is a bipartite graph on $Q \cup \mathcal{S}$, where (q, S) forms an edge if and only if $q \in S$.

Let $p = \min\{\deg_H(u) \mid u \in D_H\}$ and let v be a maximum distance vertex with $\deg_H(v) = p$. Denote the neighbors of v by $N_H(v) = \{w_1, \ldots, w_p\}$. Denote the *second common neighborhood* as $N_H^{(2)}(v) = \bigcap_{u \in N_H(v)} N_H(u) = \{v, v_2, \ldots, v_l\}$. Choose v such that l is minimal, i.e., there does not exist a vertex v' in D_H with $|N_H(v')| = p$ and $|N_H^{(2)}(v')| < l$. See Figure 7.2 for

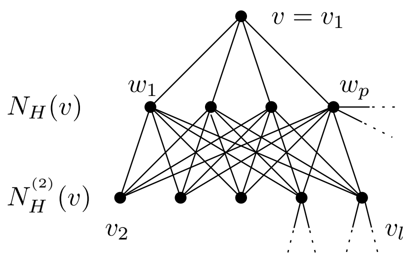


Figure 7.2: Neighborhood of a vertex v in H.

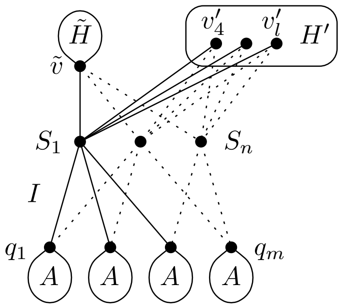


Figure 7.3: Construction of the graph G in Case 2.

a drawing of a possible situation. We distinguish four cases according to possible values of p and l.

Case 1: $p = 1, l = 1$. Then $H = K_2$, and we have already discussed this case above.

Case 2: $p = 1, l \geq 3$. We extend the incidence graph I as follows: According to Lemma 7.7 we construct a connected graph A for which allows two locally surjective homomorphisms mapping a particular vertex u to v_2 and v_3. We form an instance G as the union of the graph I and m disjoint copies of the graph H, where the vertex u of the i-th copy is identified with the vertex q_i of I. Finally we insert into G two extra copies $\tilde{H}, H'$ of the graph H, where $\tilde{v}$ is the isomorphic copy of v in $\tilde{H}$ and v'_k is the isomorphic copy of v_k in H' for $1 \leq k \leq l$. We add the following edges (cf. Figure 7.3):

- $(\tilde{v}, S_j)$ for all $S_j \in \mathcal{S}$,
- (v'_k, S_j) for all $S_j \in \mathcal{S}$ and all $4 \leq k \leq l$ (this set may be empty).

We show that the graph G formed in this way allows a locally surjective homomorphism $f : G \xrightarrow{S} H$ if and only if $(Q, \mathcal{S})$ is 2-colorable.

Assume first that such a f exists. Then according to Lemma 7.9 we assume that the vertex $\tilde{v}$ is mapped onto v and all vertices S_j are mapped to vertex w_1. Since their neighborhoods are saturated by common $l-3$ images on $v'_4, \ldots, v'_l$, at least two distinct vertices $v_a, v_b \in N_H^{(2)}(v) \setminus f(\{v'_4, \ldots, v'_l\})$ exist that are used on some neighbors of each S_j in the set $\mathcal{S}$.

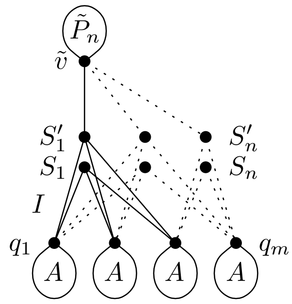


Figure 7.4: Construction of the graph G if $H = P_n$.

The partition $Q_1 = \{q_i \,|\, f(q_i) = v_a\}$ and $Q_2 = Q \setminus Q_1 \supseteq \{q_i \,|\, f(q_i) = v_b\}$ is the desired 2-coloring of $(Q, \mathcal{S})$.

In the opposite direction, any 2-coloring $Q_1 \cup Q_2$ can be transformed into an $f : G \xrightarrow{S} H$ by letting $f(q_i) = v_a$ if $q_i \in Q_a$ for $a = 1, 2$ and by further extension according to the two projections of the graph A and graph isomorphisms $\tilde{H} \to H$, $H' \to H$.

Case 3: $p = 1, l = 2$.

First assume that H is not isomorphic to a path P_n on $n \geq 3$ vertices. Let y be the first vertex on the unique path $P_H(v)$ from v in H that has degree $\deg_H(y) \geq 3$. Now we can use the same construction as in Case 2 after a couple of modifications: We replace each edge $(\tilde{v}, S_j)$ by a path from $\tilde{v}$ to S_j of the same length as $P_H(v)$. Furthermore, we make sure to add the right edges from each S_j to H' (instead of edges (v_k', S_j)) and to choose the right vertex $u \in H$.

If H is isomorphic to P_n for some $n \geq 3$, then we act as follows.

Assume that $n \neq 4$. We construct a graph G from I. First we insert n new vertices $S_1', \ldots, S_n'$ and a copy $\tilde{P}_n$ of the graph P_n. Then we identify each q_i with the vertex u of an extra copy of the graph H as in the previous case, but here A is constructed such that u can be assigned v or v_2.

These parts are linked as follows (cf. Figure 7.4):

- $(\tilde{v}, S_j') \in E_G$ for all $j \in \{1, \ldots, n\}$,
- $(q_i, S_j') \in E_G$ if and only if $(q_i, S_j) \in E_I$.

Now the proof is similar to the proof of Case 2. If $f : G \xrightarrow{S} H$, then without loss of generality we may assume that $f(\tilde{v}) = v$. Then all S_j' are mapped onto w_1 since w_1 is the only neighbor of v. The images of all q_i hence belong to $N_H(w_1) = \{v, v_2\}$. Each S_j' requires the image v_2 to be present among its neighbors in Q. Moreover, if all neighbors of some S_j' in Q are assigned the vertex v_2, we get that S_j must be mapped to a neighbor of v_2 that is a leaf, which is only possible if $H = P_4$. We conclude that each S_j is mapped to w_1. Hence both v, v_2 appear on its neighborhood and the

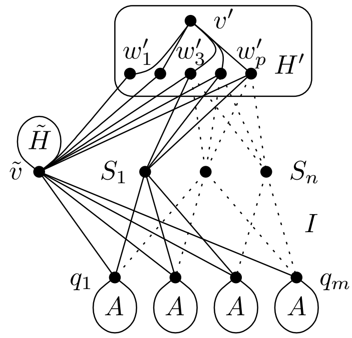


Figure 7.5: Construction of the graph G in Case 4.

partition $Q_1 = \{q_i \,|\, f(q_i) = v\}$ and $Q_2 = \{q_i \,|\, f(q_i) = v_2\}$ is a 2-coloring of $(Q, \mathcal{S})$.

In the opposite direction, a locally surjective homomorphism $f : G \to H$ can be constructed from a 2-coloring of $(Q, \mathcal{S})$ in a straightforward way as in the previous case.

If $n = 4$ we replace the edges between $\tilde{v}$ and each S_j by paths of length two, and we identify each q_i with the vertex $u \in V_H$ that can be assigned to both neighbors of v_2. After these modifications the proof is similar to the case $n \neq 4$. The case when H is isomorphic to the path P_n on $n \geq 3$ vertices was already shown to be NP-complete in [41].

Case 4: $p \geq 2$. As above we first build the graph A, which allows two locally surjective homomorphisms mapping a vertex u either to w_1 or to w_2.

The graph G consists of the graph I, where each q_i is unified with the vertex u of an extra copy of H. We further include two copies of H denoted by $\tilde{H}$ and H'. Finally we extend the set of edges by (cf. Figure 7.5):

- $(\tilde{v}, q_i)$ for all $q_i \in Q$,
- $(\tilde{v}, w'_k)$ for all $1 \leq k \leq p$,
- (S_j, w'_k) for all $3 \leq k \leq p$ (this set may be empty).

If a mapping $f : G \xrightarrow{S} H$ exists, then we assume that $f(\tilde{v}) = v$. For each S_j we have $N_G(S_j) \subseteq N_G(\tilde{v})$. So we know that S_j is assigned some vertex v_i for which $N_H(v_i) = N_H(v)$. Since v is a maximum distance vertex in H, S_j is mapped on a vertex from D_H as well. Because p is the smallest number of neighbors these vertices can have, $f(S_j)$ has degree at least p.

However, only $p-2$ possible images appear on $w'_3, \ldots, w'_p$. So two distinct vertices w_a and w_b are never used as images of $w'_3, \ldots, w'_p$. Then we define a 2-coloring of $(Q, \mathcal{S})$ by selecting $Q_1 = \{q_i \,|\, f(q_i) = w_a\}$ and $Q_2 = Q \setminus Q_1 \supseteq \{q_i \,|\, f(q_i) = w_b\}$.

Suppose a 2-coloring of $(Q, \mathcal{S})$ exist. Then a locally surjective homomorphism $f : G \to H$ can be derived from this 2-coloring as in the previous cases. □

Observe that all graphs G involved in our constructions had an isomorphic copy of the target graph glued in, and were connected, even if the incidence graph I was not connected.

7.3 Disconnected target graphs

Up to now we have only considered target graphs H that were connected. Due to this property we could easily derive that all vertices of H appear as the image of the vertex in the instance graph (cf. Lemma 3.4 (i)). We now focus our attention to the case of disconnected target graphs. Suppose H is a graph with set of components $\{H_1, \dots H_m\}$. We order the components such that the latter have a higher number of vertices. (Formally, for all $i \le j : |V_{H_i}| \le |V_{H_j}|$.)

Note that the identity mapping $\pi : V_{H_1} \to V_H$ is locally surjective, but Lemma 3.4 (i) is no longer valid here (take $G \simeq H_1$). Our argument guarantees that a locally surjective homomorphism is globally surjective only for connected target graphs. Within some social network models it is natural to demand that all images appear on the vertices of the instance graph. We show below that the computational complexity of the H-LSurHom problem for disconnected target graphs depends on whether such a property $f(V_G) = V_H$ is required or not.

Theorem 7.11. *Let H be a disconnected graph. Then the H-LSurHom* problem is polynomially solvable if all components have at most two vertices and it is* NP*-complete otherwise.*

Proof. Clearly the H-LSurHom* problem belongs to NP. For a connected graph H the statement immediately follows from Theorem 7.1.

Suppose H has $m \ge 2$ components ordered as shown above. If all components consist of only one vertex, then $G \xrightarrow{S} H$ if and only if G is a collection of at least m isolated vertices. For the other tractable case suppose H consists of k isolated vertices and $m-k$ isolated edges. Then $G \xrightarrow{S} H$ if and only if G contains at least k isolated vertices and at least $m-k$ bipartite components, each with at least one edge.

Now suppose $|V_{H_m}| \ge 3$. We prove NP-completeness by reduction from H_m-LSurHom. Without loss of generality we may assume that the instance graph G for the H_m-LSurHom problem is connected. Let G' be the graph with components $G, \tilde{H}_1, \dots, \tilde{H}_{m-1}$, where $\tilde{H}_i$ is isomorphic to H_i for $1 \le i \le m-1$. It is straightforward to see that $G' \xrightarrow{S} H$ if $G \xrightarrow{S} H_m$.

For the backward implication assume that $G' \xrightarrow{S} H$. Observe that both G' and H have the same number of components, so each component of

H provides images for exactly one component of G'. It is impossible to make a locally surjective homomorphism from $\tilde{H}_i$ to H_j when $|V_{\tilde{H}_i}| < |V_{H_j}|$. Hence the component G can only be mapped onto one of the components of maximum size.

If the image of the component G is H_m, then we are finished. Suppose the component G is mapped onto some H_i with $i \neq m$. Then $\tilde{H}_i$ maps to some other component H_j such that $|V_{\tilde{H}_i}| = |V_{H_j}|$ and by Corollary 3.6 we get $\tilde{H}_i \simeq H_j$. Hence, $G \xrightarrow{S} H_j$. If $j \neq m$ we repeat the argument, and after at most m iterations we find a desired locally surjective homomorphism $G \xrightarrow{S} H_m$. □

Now we show that without the condition of global surjectivity "$f(V_G) = V_H$", some polynomially solvable H-LSurHom problems exist for target graphs H with large components.

Take any graph H with bipartite components (of arbitrary size) but assure that at least one of these components is isomorphic to K_2. For simplicity, assume that H has no isolated vertices. Clearly, any graph G satisfies $G \xrightarrow{S} H$ if and only if G is bipartite without isolated vertices. This is because a nonbipartite graph G allow no homomorphism to any of the bipartite components of H, while a possible bipartition of G provides a natural homomorphism to K_2, which is in fact locally surjective.

The above observation together with the following proposition conclude the proof of Theorem 7.1.

Proposition 7.12 ([20]). *The H-LSurHom problem is* NP*-complete if and only if either H is nonbipartite or H is bipartite, contains at least one component that is not an isolated vertex, and does not have a component isomorphic to K_2.*

Proof. Suppose H is a graph with $m \geq 2$ components $\{H_1, \ldots, H_m\}$ ordered in a non-decreasing sequence.

First assume that H contains at least one nonbipartite component. Again we prove NP-completeness by reduction from H2C. Given an instance $(Q, \mathcal{S})$ we act as follows.

Choose the target graph $L = H_j$ to be the first nonbipartite component in the order, i.e., all components H_i with $i < j$ are bipartite. With respect to L we extend the incidence graph I corresponding to an instance $(Q, \mathcal{S})$ to an appropriate graph G as in the proof of Proposition 7.10. Note that G contains an isomorphic copy $\tilde{L}$ glued in G by a vertex $\tilde{v}$ that is the isomorphic copy of a vertex $v \in D_H$.

Starting with graph G we construct a new graph G^*. Let u, v be two vertices in L with distance $\mathrm{dist}_L(u, v) = \mathrm{diam}(L)$.

Construct a graph F_0 that consists of two isomorphic copies L_0^1 and L_0^2 of $L = H_j$ glued together by the vertex $u_0^{1,2} = u_0^1 = u_0^2$, the isomorphic copy

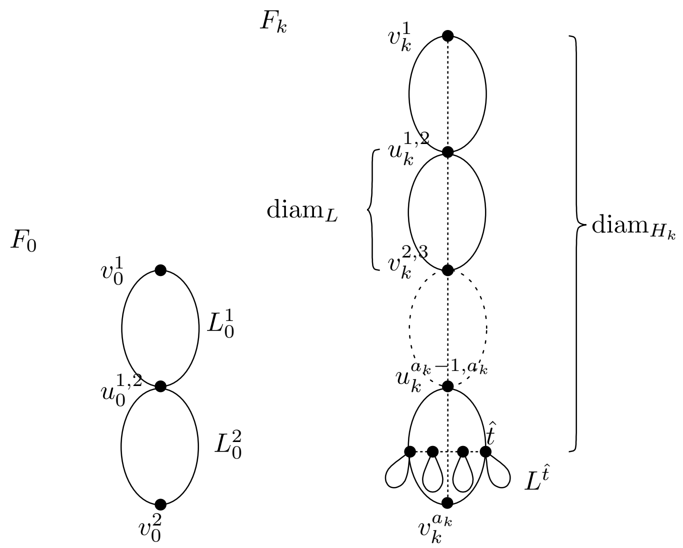


Figure 7.6: The graphs F_0 and F_k.

of u in *both* copies. (See Figure 7.6 for a picture of such a graph.) Glue F_0 to G by $\tilde{v}$ in such a way that $\tilde{v}$ is identified with the vertex $v_0^2 \in V_{F_0}$.

For each nonbipartite component H_k with $\mathrm{diam}(H_k) > \mathrm{diam}(L) = \mathrm{diam}(H_j)$ we construct an appropriate subgraph F_k as follows. Let a_k be the smallest even integer such that $\mathrm{diam}(H_k) < a_k \cdot \mathrm{diam}(L)$. The graph F_k contains a_k isomorphic copies of $L = H_j$ glued in a "chain": Each odd numbered copy L_k^i is linked with the successive copy L_k^{i+1} by the common vertex $u_k^{i,i+1} = u_k^i = u_k^{i+1}$, while each even numbered copy shares with its successor the vertex $v_k^{i,i+1}$.

We finalize F_k as follows. Let $d_k = \mathrm{diam}(L)$. Since we have added enough copies, the set $D_k = \{\hat{t} \mid \mathrm{dist}_{F_k}(\hat{t}, v_k^1) = d_k\}$ is non-empty. Let $\hat{t} \in D_k$ be the isomorphic copy of vertex t in $L = H_j$. We glue an isomorphic copy $L^{\hat{t}}$ of L to F_k in such a way that also in the new copy, $\hat{t}$ is identified with vertex t. We do this for all $\hat{t}$ in D_k. See Figure 7.6 for a picture of a graph F_k.

Finally we make the graph F_k connected to G by identifying vertices $\tilde{v} = v_k^{a_k}$. By repeating the above process for all components H_k with $\mathrm{diam}(H_k) > \mathrm{diam}(L)$ we have obtained graph G^*. See Figure 7.7 for a picture of this graph.

Claim 7.13. *The graph* G^* *satisfies* $G^* \xrightarrow{S} H$ *if and only if* $G^* \xrightarrow{S} L$.

We show this as follows. As $L \subset H$ we see that $G^* \xrightarrow{S} L$ implies $G^* \xrightarrow{S} H$. We prove the reverse statement by contradiction. Suppose G^* allows a locally surjective homomorphism to $f : G \xrightarrow{S} H$ but none such homomorphism to L only. Then $G^* \xrightarrow{S} H_k$ for a certain component H_k in H with $k \neq j$.

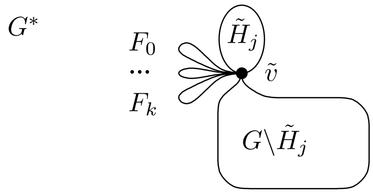


Figure 7.7: The graph G^*.

Because G contains a nonbipartite subgraph $\tilde{H}_j \simeq H_j$, G is nonbipartite and H_k cannot be bipartite too. Hence $|V_{H_k}| \geq |V_{H_j}| = |V_L|$.

First suppose H_k has diameter $\text{diam}(H_k) \leq \text{diam}(L)$. Let Z_0 be the set containing all vertices $z \in V_{G^*}$ with distance $\text{dist}_{G^*}(v_0^1, z) \leq \text{diam}(H_k)$. By Observation 7.4 we deduce that $f(Z_0) = V_{H_k}$.

By construction of G^*, the distance $\text{dist}_{G^*}(v_0^1, z)$ between v_0^1 and any vertex z not in L_0^1 is greater than $\text{diam}(L) \geq \text{diam}(H_k)$. Then Z_0 is a subset of $V_{L_0^1}$. Together with $f(Z_0) = V_{H_k}$ this implies that all vertices of H_k appear as an image of a vertex in $V_{L_0^1}$. This is only possible if H_k has no more vertices than L. Hence $|V_{H_k}| = |V_L|$ (and $\text{diam}(H_k) = \text{diam}(L) = \text{diam}(H_j)$).

If the vertex sets of H_k and L have the same cardinality, each vertex of H_k appears exactly once as the image of a vertex in L_0^1. If $f(u_0^{1,2})$ has a neighbor mapped onto y only appearing as an image of a neighbor of $u_0^{1,2}$ outside L_0^1, then $f(u_0^{1,2})$ must appear at least twice in L_0^1. Hence, $L_0^1 \xrightarrow{S} H_k$. By Corollary 3.6 we deduce that $H_k \simeq L_0^1 \simeq L \simeq H_j$. This would imply that $G^* \xrightarrow{S} L$ as well, a contradiction.

So we know that $\text{diam}(H_k) > \text{diam}(L)$ must hold. In that case G^* has a corresponding subgraph F_k. Let Z_k be the set containing all vertices $z \in V_{G^*}$ with distance $\text{dist}_{G^*}(v_k^1, z) \leq \text{diam}(H_k)$. Again we use Observation 7.4 to deduce that $f(Z_k) = V_{H_k}$. Then, by construction of G^*, a vertex $\hat{t} \in D_k$ exists that is mapped on a maximum distance vertex $x \in D_{H_k}$.

Note that $\hat{t}$ is a cutvertex in G^*. Because H_k has a strictly greater diameter than L, not all vertices of H_k appear as the image of a vertex in $L^{\hat{t}} \simeq L$. Then applying Lemma 7.5 yields that $f(\hat{t}) = x$ is a cutvertex of H_k, contradicting the fact that maximum distance vertices in a graph cannot be cutvertices.

Hence $G^* \xrightarrow{S} H$ if and only if $G^* \xrightarrow{S} L$. To obtain G^* we glued a number of graphs to G by $\tilde{v}$ that clearly allow a locally surjective homomorphism to L. Analogously to the proof of Theorem 7.10 we can show that $G^* \xrightarrow{S} L$ if and only if $(Q, \mathcal{S})$ is 2-colorable. Hence, $G^* \xrightarrow{S} H$ if and only if $(Q, \mathcal{S})$ is 2-colorable.

If H only contains bipartite components, then we choose for L the smallest component that is not an isolated vertex. (Recall that in this case H does not contain any K_2 nor any isolated vertex incident with a loop.) Our construction is exactly the same, only the reasoning differs at one point: Instead of showing that G^* cannot map onto a nonbipartite component H_k (see the first paragraph after the claim) we exclude — due to trivial reasons — the case when $G \xrightarrow{S} H_k$ for H_k being an isolated vertex without a loop.

Hence we conclude that also in this case the H-LSurHom problem is NP-complete. □

Chapter 8

Complexity of matrix comparison

8.1 Matrix comparison via local injectivity

In this section we consider the problem of deciding whether for given degree matrices M and N the comparison $M \overset{I}{\leqslant} N$ holds.

Observe that according to the definition of the quasiorder $(\mathcal{M}, \overset{I}{\leqslant})$, there is no obvious bound on the sizes of graphs G and H with M and N as degree matrices that should justify the comparison $M \overset{I}{\leqslant} N$.

The main result of this section is the following theorem:

Theorem 8.1. *Let M, N be degree matrices of order k and l. If $M \overset{I}{\leqslant} N$, then there exist a graph G of size $(klm^*)^{O(k^2l^2)}$ and a graph H of size $(klm^*n^*)^{O(k^2l^2)}$ such that $G \xrightarrow{I} H$, $M \in \mathcal{M}_G$ and $N \in \mathcal{M}_H$.*

Proof. Throughout this proof we assume that indices i, j, r, s used later always belong to feasible intervals $1 \leq i, r \leq k$ and $1 \leq j, s \leq l$.

The main idea of the construction is as follows. Assume that $M \overset{I}{\leqslant} N$ holds. Then we deduce from Proposition 3.7 that there exist a graph H and its subgraph $G \subseteq H$ witnessing $M \overset{I}{\leqslant} N$. Let $\{U_1, \ldots, U_k\}$ be the partition of G providing M and $\{V_1, \ldots, V_l\}$ be the one for H and N.

We further partition $V_G \subseteq V_H$ as follows. For each pair of indices r and s we define the set

$$W_{r,s} = \{v \mid v \in U_r \cap V_s\},$$

and for some vertex $w \in W_{r,s}$ we can write a vector describing the distribution of neighbors of w in the classes $W_{1,1}, \ldots, W_{k,l}$.

We first show that for given M and N the set T containing all such vectors is finite. Then, with help of T, we design a set of equations that allows a solution if and only if the desired graphs G and H exist. As the size of T is bounded, we can establish the desired bounds on the size of G and H.

Let $\mathbf{p}^{r,s}$ be a vector of length kl whose entries are positive integers and are indexed by pairs ij. If the vector $\mathbf{p}^{r,s}$ further satisfies

$$\sum_{j=1}^{l} p_{i,j}^{r,s} = m_{r,i} \qquad \text{for all } 1 \le i \le k, \tag{8.1}$$

$$\sum_{i=1}^{k} p_{i,j}^{r,s} \le n_{s,j} \qquad \text{for all } 1 \le j \le l, \tag{8.2}$$

then we call $\mathbf{p}^{r,s}$ an *injective distribution row for indices r and s.* Note that for given matrices M and N and any feasible choice of r, s the number of all different injective distribution rows for r and s is finite. We denote the set of all injective distribution rows for indices r and s by

$$T(r,s) = \{\mathbf{p}^{r,s(1)}, \ldots, \mathbf{p}^{r,s(t(r,s))}\}.$$

Due to (8.1), the number of distribution rows for every $\mathbf{p}^{r,s}$ is bounded by $t(r,s) \le \binom{m^*+l-1}{m^*}^k = O((m^*+1)^{kl})$. The total number of distribution rows is then

$$t_0 = \sum_{r,s} t(r,s) = O(kl(m^*+1)^{kl}).$$

Now consider a set of t_0 variables $w^{r,s(t)}$ for all feasible r, s and all $1 \le t \le t(r,s)$. We claim that the existence of a nontrivial *nonnegative* solution of the following homogeneous system of k^2l^2 equations in t_0 variables:

$$\sum_{t=1}^{t(r,s)} p_{i,j}^{r,s(t)} w^{rs(t)} = \sum_{t'=1}^{t(i,j)} p_{r,s}^{ij(t')} w^{ij(t')} \qquad 1 \le i, r \le k,\ 1 \le j, s \le l \tag{8.3}$$

is a necessary and sufficient condition for the existence of finite graphs G and H witnessing $M \overset{I}{\leqslant} N$.

Necessity: For given G and H we assume without loss of generality that $G \subseteq H$. Firstly determine the sets $W_{r,s}$, and for each vertex $u \in W_{r,s} \subseteq V_G$ compute the distribution vector of its neighbors $\mathbf{p}(u) = (|N(u) \cap W_{1,1}|, \ldots, |N(u) \cap W_{k,l}|)$. Then the vector $\mathbf{w}$ with entries $w^{r,s(t)} = |\{u \mid \mathbf{p}(u) = \mathbf{p}^{r,s(t)}\}|$ is a nontrivial solution of (8.3), since in each equation both sides are equal to the number of edges connecting sets $W_{r,s}$ and $W_{i,j}$.

Sufficiency: Assume that the system (8.3) has a nontrivial nonnegative solution. By appropriate scaling we obtain a nonnegative integer solution $\mathbf{w} = (w^{1,1(1)}, \ldots, w^{k,l(t(k,l))})$ with each $w^{r,r(t)}$ is even.

We first build a multigraph G_0 upon t_0 sets of vertices $W^{1,1(1)}, \ldots, W^{k,l(t(k,l))}$, where $|W^{r,s(t)}| = w^{r,s(t)}$ (some sets may be empty) as follows: Denote $W^{r,s} = W^{r,s(1)} \cup \cdots \cup W^{r,s(t(r,s))}$.

Our choice of even values $w^{r,r(t)}$ allows us to build an arbitrary $p_{r,r}^{r,r(t)}$-regular multigraph on each set $W^{r,r(t)}$.

As $\mathbf{w}$ satisfies (8.3), we can easily build a bipartite multigraph between any pair of different sets $W^{r,s}$ and $W^{i,j}$ such that the number of edges between them is equal to $\sum_{t=1}^{t(r,s)} p_{i,j}^{r,s(t)} w^{r,s(t)} = \sum_{t'=1}^{t(i,j)} p_{r,s}^{ij(t')} w^{i,j(t')}$.

For any vertex u in $W^{r,s(t)}$ with more than $p_{i,j}^{r,s(t)}$ neighbors in $W^{i,j}$ there exists a vertex u^* in some $W^{i,j(t^*)}$ with less than $p_{i,j}^{r,s(t^*)}$ neighbors, and vice versa. Now we remove an edge between u and some neighbor $v \in W^{i,j}$ and add the edge (u', v). We repeat this procedure until all vertices of $W^{r,s}$ have the right number of neighbors in $W^{i,j}$. Then we do the same for vertices in $W^{i,j}$.

This way we have constructed a bipartite multigraph between $W^{r,s}$ and $W^{i,j}$ such that each vertex of each $W^{r,s(t)}$ is incident with exactly $p_{i,j}^{r,s(t)}$ edges and each vertex of each $W^{i,j(t')}$ is incident with exactly $p_{r,s}^{i,j(t')}$ edges.

It may happen in some instances that multiple edges are unavoidable. In that case let $d \leq m^*$ be the maximal edge multiplicity in G_0. We obtain the graph G by taking d copies of the multigraph G_0 and replace each collection of d parallel edges of multiplicity $d' \leq d$ by a simple d'-regular bipartite graph.

Due to the construction, it is straightforward to check that vertices from sets that share the same index r form the r-th block of the equitable partition of G and that $M \in \mathcal{M}_G$.

For the construction of H we first distribute the vertices of G into sets $V_1', \ldots, V_l'$, where

$$V_s' = \bigcup_{r=1}^{k} \bigcup_{t=1}^{t(r,s)} W^{r,s(t)}.$$

Since N is a degree matrix, the following homogeneous system whose equations represent the number of edges between two different blocks in N has nontrivial solutions:

$$n_{s,j} v_s \;=\; n_{j,s} v_j \qquad 1 \leq j, s \leq l \tag{8.4}$$

Then we form sets $V_1, \ldots, V_l$ by further inserting new vertices into $V_1', \ldots, V_l'$ until for each $s, j : |V_s| n_{s,j} = |V_j| n_{j,s}$ and $|V_s| > 0$ is even.

Next we build a multigraph H_0 by constructing an $(n_{s,j}, n_{j,s})$-regular bipartite multigraph between any two sets V_s and V_j, and an $n_{j,j}$-regular multigraph on each V_j. In case multiple edges cannot be avoided we take sufficient copies of H_0 and make the appropriate reparations. So we perform these steps in the same way as before, however without removing any edges between vertices in (any copy of) G.

Clearly, G is a subgraph of the resulting graph H and H has N as its degree matrix.

To conclude the proof of the theorem we discuss the size of G and H. Note that all coefficients $p_{i,j}^{r,s(t)}$ of system (8.3) are at most m^*. Then, by

Lemma 1.1, we find a nontrivial nonnegative integer solution $\mathbf{w}$ whose entry sizes are bounded by $O(k^2l^2\log(klm^*))$.

We can use the entries of $2\mathbf{w}^*$ for the sizes of the blocks in the multigraph G_0. Since we take at most m^* copies of G_0 to obtain our final graph G, we find that $\langle G\rangle = (klm^*)^{O(k^2l^2)}$.

Analogously, the size of each entry of a solution $\mathbf{v}$ of system 8.4 is bounded by $O(l^2\log(ln^*))$. Since multigraph H_0 must contain graph G, we use the entries of $\langle G\rangle$ for the block sizes of H_0. We need at most n^* copies of H_0 for graph H. Hence, each block size $|V_i|$ can be chosen within the upper bound $\langle G\rangle \cdot (ln^*)^{O(l^2)}$ implying that $\langle H\rangle = (klm^*n^*)^{O(k^2l^2)}$. □

We can now settle the first computational complexity result for the following matrix comparison problem:

Matrix Injectivity (MI)
Instance: Degree matrices M and N.
Question: Does $M \overset{I}{\leqslant} N$ hold?

Corollary 8.2. *The problem* MI *belongs to the class* NP *of nondeterministically polynomial problems.*

Proof. The proof of Theorem 8.1 showed that $M \overset{I}{\leqslant} N$ if and only if system (8.3) has a nontrivial nonnegative solution. Then by Lemma 1.1 there exists a nontrivial nonnegative integral solution with at most k^2l^2+1 nonzero entries, which are each bounded in size by $O(k^2l^2\log(klm^*))$.

So we only have to consider vectors of this form. The nondeterministical proof word would hence consist of the k^2l^2+1 nonzero entries of the vector $\mathbf{w}$ together with the corresponding injective distribution rows. The size of this proof is $O(k^4l^4\log(klm^*))$, which is polynomial in the size of matrices M and N.

It can be tested in linear time (with respect to the length of the proof word) whether all distribution rows are valid for (8.1, 8.2). The test whether the vector $\mathbf{w}$ satisfies (8.3) can also be performed in polynomial time. □

8.1.1 An application on universal covers

As we have discussed in the introduction, the matrix order $(\mathcal{M}, \overset{I}{\leqslant})$ was considered as a nontrivial necessary condition for the decision problem whether $G \xrightarrow{I} H$. As the size of M and N should vary from being independent in the size of the given graphs to be of approximately the same size of G, H, even the exponential time-complexity of the MI problem might be plausible as a precomputation for some instances.

We apply Theorem 8.1 to disprove the following interesting conjecture on the equivalence between comparison of degree matrices in $\overset{I}{\leqslant}$ and inclusion of universal covers.

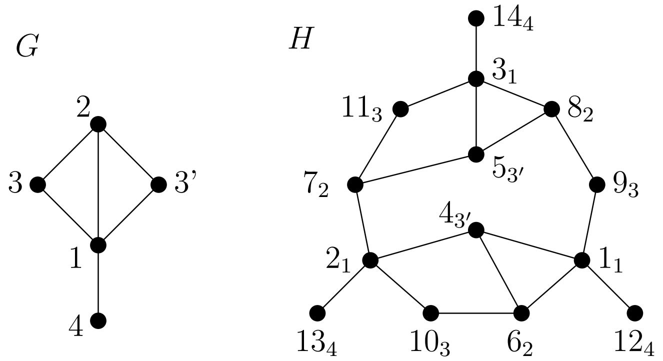


Figure 8.1: Graphs G and H, vertices of H are labeled by $u_{f(u)}$ for a $f : H \xrightarrow{S} G$.

Conjecture 8.3. *For any two degree matrices M and N the following equivalence holds:*

$$M \overset{I}{\leqslant} N \iff T_M \subseteq T_N.$$

We note here that the affirmative answer for the only if implication was already shown in Observation 4.10. The following example acts both as an example for the application of Theorem 8.1, and as an counterexample of Conjecture 8.3.

Corollary 8.4. *There exist matrices M and N such that $T_M \subseteq T_N$, but $M \overset{I}{\leqslant} N$ does not hold.*

Proof. We first construct graphs G and H such that $H \xrightarrow{S} G$. Denote $M = \mathrm{drm}(G)$ and $N = \mathrm{drm}(H)$. Then according to Observation 4.10 we get that $T_M \subseteq T_N$. We will now show that the MI problem for matrices M and N has a negative answer.

The graphs G and H together with a mapping $f : H \xrightarrow{S} G$ are depicted in Fig. 8.1. The graph G has 4 classes in its degree refinement and H has 14 classes. Then N is the adjacency matrix of H and the degree refinement matrix of G is

$$M = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In order to obtain a contradiction suppose $T_M \xrightarrow{I} T_N$ holds. By Proposition 3.7 there exists a graph $M \in \mathcal{M}_G$ and a graph $H' \xrightarrow{B} N$ such that $G' \subseteq H'$. Let $\{U_1, \ldots, U_k\}$ be the equitable partition of G' and $\{V_1, \ldots, V_l\}$ the one for H'. We define the sets $W_{r,s}$ as in proof of Theorem 8.1.

As we have seen in the proof of Theorem 8.1 the pair (G', H') corresponds with a nontrivial solution of (8.3). Below we will show, however, that (8.3) only allows the trivial solution. For simplicity reasons we will first restrict the length of the injective distribution rows.

A vertex in class U_1 has four neighbors in G'. A vertex in class V_4 has three neighbors in H'. This means that a vertex of U_1 can never be in V_4, i.e., $W_{1,4}$ is empty. Hence the set $T(1,4)$ is empty. By the same argument we find that the sets $T(r,s)$ with $(r,s) = (1,5),\ldots,(1,14),(2,9),\ldots,(2,14),(3,12),\ldots,(3,14)$ are empty.

A vertex in U_2 has a neighbor of degree four in G'. A vertex in V_1 does not have a neighbor of degree four in H'. Hence the set $T(2,1)$ is empty. By the same argument we exclude pairs $(2,2),(2,3),(3,1),(3.2),(3,3),(4,1),(4,2),(4,3)$.

Any vertex in U_4 has degree one in G'. Suppose $u \in U_4$ belongs to V_4. So it does not have degree one in H'. Let $v \in U_1$ be the (only) neighbor of u in G'. Then v has degree four in G' and must belong to $V_1 \cup V_2 \cup V_3$. The other three neighbors of v all have degree greater than one in G'. However, one of these three remaining neighbors of v must have degree one in H'. Hence, the set $T(4,4)$ is empty. In the same way we may exclude pairs $(4,4),\ldots,(4,11)$.

Every vertex in $W_{2,4}$ needs a neighbor in $W_{3,1}$ or $W_{3,2}$. These sets are empty, since both $T(3,1)$ and $T(3,2)$ are empty. Hence $T(2,4)$ is empty, and consequently, by a similar argument, $T(3,6)$ is empty. Furthermore, $T(2,4) = \emptyset$ implies that a vertex in $W_{1,2}$ does not have neighbor in $W_{3,7}$. Since every vertex in $W_{3,7}$ must have a neighbor in $W_{1,2}$, the latter implies $T(3,7) = \emptyset$, and consequently $T(2,5) = \emptyset$ and $T(3,8) = \emptyset$.

Only the pairs $(3,4)$ and $(3,5)$ allow two distribution rows, the other pairs all allow one. So we have reduced the total number of feasible distribution rows to $4 \cdot 14 - 20 - 9 - 8 - 5 + 2 = 16$, see Table 8.1.

The equation (8.3) for $p,q = 1,1$ and $i,j = 2,6$ gives $w^{1,1} = w^{2,6}$. Analogously, $w^{1,1} = w^{3,4(1)}$ while $w^{2,6} = w^{3,4(1)} + w^{3,4(2)}$. Hence $w^{3,4(2)} = 0$. Further $w^{3,4(2)} = w^{1,2} = w^{3,10} = w^{2,6}$, and $w^{1,2} = w^{2,7} = w^{3,11} = w^{1,3}$. Consequently, $w^{1,1} = w^{1,2} = w^{1,3} = 0$.

It can be further shown that (8.3) allows only trivial solution via values of $w^{r,s}$. However, at this moment we can already claim that no witnesses G,H for $M \overset{I}{\leqslant} N$ exist, since it is impossible to map vertices from the first class of degree partition of G on any vertex of H. □

8.2 Matrix comparison via local surjectivity

In Section 4.4 it has been shown that the $\overset{S}{\leqslant}$ relation is a quasiorder on degree matrices. We are now interested inthe following matrix comparison problem:

i	1	1	1	2	2	2	3	3	3	3	3	4	4	4
j	1	2	3	6	7	8	4	5	9	10	11	12	13	14
$p^{1,1}$				1			1		1			1		
$p^{1,2}$					1		1			1			1	
$p^{1,3}$						1		1			1			1
$p^{2,6}$	1						1			1				
$p^{2,7}$		1						1			1			
$p^{2,8}$			1					1	1					
$p^{3,4(1)}$	1			1										
$p^{3,4(2)}$		1		1										
$p^{3,5(1)}$			1		1									
$p^{3,5(2)}$			1			1								
$p^{3,9}$	1					1								
$p^{3,10}$		1		1										
$p^{3,11}$			1		1									
$p^{4,12}$	1													
$p^{4,13}$		1												
$p^{4,14}$			1											

Table 8.1: The distribution rows for M (only nonzero entries are shown)

MATRIX SURJECTIVITY (MS)
Instance: Degree matrices M and N.
Question: Does $M \overset{S}{\leqslant} N$ hold?

Definition 8.5. Let G be a graph and let M be a degree matrix of order k. We write $G \xrightarrow{S} M$ if there is a partition of V_G into sets $\mathcal{B} = \{B_1, \ldots, B_k\}$ that for every i and $u \in B_i$ satisfies:

$$\forall j : |N(u) \cap B_j| \begin{cases} = 0 & \text{if } m_{i,j} = 0 \\ \geq m_{i,j} & \text{if } m_{i,j} > 0 \end{cases} \tag{8.5}$$

Observe that $G \xrightarrow{S} A_H$ if and only if there exists a locally surjective homomorphism from G to H, in which case we write $G \xrightarrow{S} H$.

8.2.1 Graph reconstruction

Lemma 8.6. *Let N be a degree matrix of order two with zeros on the diagonal. Let G be a graph with $G \xrightarrow{S} N$. Then for any graph H with $N \in \mathcal{M}_H$ there exists a graph G^* such that $G^* \xrightarrow{B} G$ and $G^* \xrightarrow{S} H$.*

The lemma statement can be depicted by the following commutative

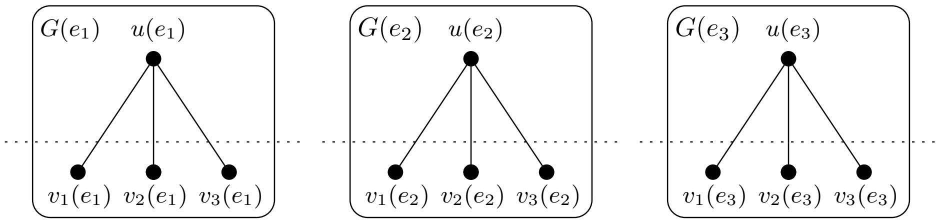
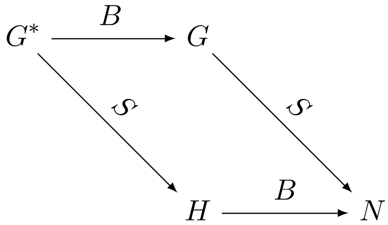


Figure 8.2: Before swapping edges for V_1.

diagram. Here the arrow $H \xrightarrow{B} N$ express the fact $N \in \mathcal{M}_H$.



Proof. Since $G \xrightarrow{S} N$ we have a partition (V_1, V_2) of V_G satisfying equation (8.5). Let H be a graph with $N \in \mathcal{M}_H$ witnessed by a partition (W_1, W_2) of V_H.

First take the graph G' as the disjoint union of $|E_H| = |W_1|n_{1,2} = |W_2|n_{2,1}$ copies of the graph G. The copy of the graph G (the block V_1, etc.) corresponding to the edge $e \in E_H$ will be denoted by $G(e)$ ($V_1(e)$ etc.).

We define homomorphism $f : G' \to H$ such that for every edge $e = (x, y) \in E_H$ with $x \in W_1$ and $y \in W_2$ all vertices in $V_1(e)$ are mapped on x and similarly $f(V_2(e)) = y$.

To make this homomorphism f locally surjective we perform appropriate edge swappings. For any vertex x in W_1 we act as follows. Let $N_H(x) = \{y_1, \dots, y_{n_{1,2}}\}$. We denote the corresponding edges in H by $e_h = (x, y_h)$. Consider the $n_{1,2}$ copies $G(e_h)$ in which $f(V_1(e_h)) = x$ and $f(V_2(e_h)) = y_h$. Below we explain how we swap suitable edges such that every vertex mapped onto x will in the resulting graph obtained from G' have at least one neighbor mapped onto y_h for $1 \le h \le n_{1,2}$ in such a way that M is still a degree matrix of he modified graph.

As $G \xrightarrow{S} N$ by our assumptions, any u in V_1 has at least $n_{1,2}$ neighbors in V_2. Out of $N_G(u)$ we choose $n_{1,2}$ different neighbors $v_1, v_2, \dots, v_{n_{1,2}}$. We denote the isomorphic copy of u in $V_1(e)$ by $u(e)$ and for $1 \le i \le n_{1,2}$ we denote the isomorphic copy of v_i in $V_2(e)$ by $v_i(e)$. Now we swap as follows (see also Figure 8.2 and Figure 8.3):

- Delete edges $(u(e_h), v_i(e_h))$ for all $1 \le i, h \le n_{1,2}$.
- Add edges $(u(e_h), v_{2h+i-2}(e_{h+i-1}))$ modulo $n_{1,2}$ for all $1 \le i, h \le n_{1,2}$.

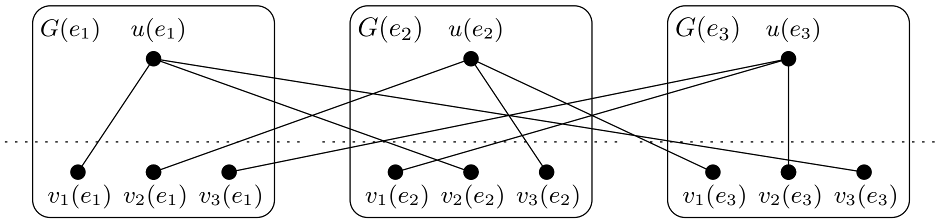


Figure 8.3: After swapping edges for V_1.

It is clear that after performing appropriate edge swappings for all x in W_1 the resulting graph G'' still allows a locally bijective homomorphism to G via the projection $\pi : u(e) \to u$ and that the homomorphism f is surjective on the neighborhood of every vertex in any $V_1(e_h)$.

We now make sure that also the neighbors of any vertex in $V_2(e_h)$ will receive their desired images (again, without changing the degree matrix of the resulting graph).

From our previous edge swappings it is clear that G'' is the disjoint union of $|W_1|$ disjoint bipartite graphs $F_i(A_i, B_i)$ that are all isomorphic to each other such that all vertices in a set A_i are mapped onto x_i.

For any vertex $y \in W_2$ we act as follows. Consider y in V_H. Let $N_H(y) = \{x_1, \ldots, x_{n_{2,1}}\}$. We denote the corresponding edges in H by $e_j = (x_j, y)$. For simplicity we assume that the graphs F_i are numbered in such a way that $V_2(e_j)$ is in graph F_j for $j = 1 \leq j \leq n_{2,1}$.

Recall that $f(V_2(e_j)) = y$ and that all neighbors of vertices from $V_2(e_j)$ are mapped onto x_j. Because $G \xrightarrow{S} N$, the number of neighbors of any vertex in any $V_2(e_j)$ is at least $n_{2,1}$. From our previous edge swappings it is clear that any isomorphic copy $v(e_i)$ of any vertex $v \in V_2$ with $p = |N_G(v)|$ neighbors $u_1, \ldots, u_p$ is adjacent to copies $u_1(e_{j_1}), u_2(e_{j_2}), \ldots, u_p(e_{j_p})$ for some $j_1, j_2, \ldots, j_p$ (with possibly $j_s = j_t$ for some $1 \leq s, t \leq p$). Then it is clear that just as before we can choose $n_{2,1}$ neighbors of v and perform appropriate edge swappings in G'' such that

- for $1 \leq i \leq n_{2,1}$ the neighbors of any vertex in B_i are mapped into the desired vertices in H;
- for $1 \leq i \leq n_{2,1}$ the neighbors of any vertex in A_i still maintain their right images;
- the resulting graph maps locally bijectively to G via the projection π.

See Figure 8.4 and Figure 8.5 for an example of two subgraphs F_1 and F_2 in which the copies $v(e_1)$ and $v(e_2)$ of a vertex v with $|N_G(v)| = 4$ and $n_{2,1} = 2$ are displayed together with their neighbors. We swap in the same way as for V_1. (Instead of the edge swappings as in Figure 8.5 we could, of course, also have chosen for other swappings.)

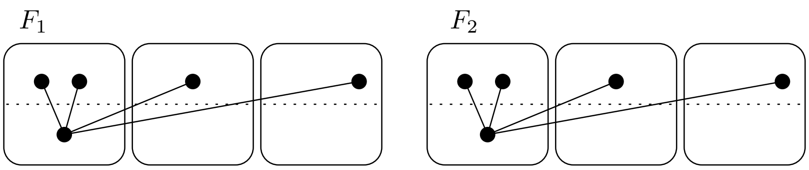


Figure 8.4: Before swapping edges for V_2.

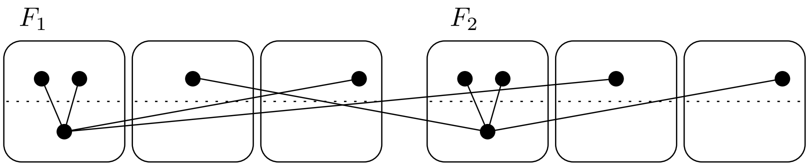


Figure 8.5: After swapping edges for V_2.

After performing appropriate edge swappings for all $y \in W_2$ we obtain our desired graph G^*. □

The case of matrices of order one cannot be treated directly as in the above case. The reason is that the construction heavily depends on the bipartition of the graph H, which cannot be assumed in this new setting. We show that the construction of Kronecker double cover allows us to transform this case to bipartite graphs.

Lemma 8.7. *Let N be a degree matrix of order one. Let G be a graph with $G \xrightarrow{S} N$. Then for any graph H with $H \xrightarrow{B} N$ there exists a graph G^* such that $G^* \xrightarrow{B} G$ and $G^* \xrightarrow{S} H$.*

Proof. For the proof of the lemma take the Kronecker double cover $G' = G \times K_2$, $H' = H \times K_2$ and

$$N' = \begin{pmatrix} 0 & n_{1,1} \\ n_{1,1} & 0 \end{pmatrix}.$$

Apply Lemma 8.6 for N', G' and H'. By Lemma 3.8 the resulting graph G^* satisfies $G^* \xrightarrow{B} G' \xrightarrow{B} G$ and $G^* \xrightarrow{S} H' \xrightarrow{B} H$, which proves the statement. □

Theorem 8.8. *Let M and N be degree matrices of size k and l respectively. The following statements are equivalent.*

(i) $M \overset{S}{\leqslant} N$.

(ii) There exists a graph G such that $M \in \mathcal{M}_G$ and $G \xrightarrow{S} N$.

(iii) For all $H : N \in \mathcal{M}_H$ there exists a graph G such that $M \in \mathcal{M}_G$ and $G \xrightarrow{S} H$.

Proof. $(iii) \Rightarrow (i)$ This is true by the definition of the quasiorder $\overset{S}{\leqslant}$ on degree matrices.

$(i) \Rightarrow (ii)$ The composition of $G \xrightarrow{S} H$ and $N \in \mathcal{M}_H$ gives $G \xrightarrow{S} N$.

$(ii) \Rightarrow (iii)$ This is the core implication of the proof. Since $G \xrightarrow{S} N$ we have a partition $\{V_1, \ldots, V_l\}$ of V_G satisfying equation (8.5) and since $N \in \mathcal{M}_H$ we obtain a partition $\{W_1, \ldots, W_l\}$ of V_H.

Let $V_{F_N} = \{1, 2, \ldots, l\}$, where vertex i corresponds to the i-th row & column of N. Let i, j be a pair of distinct adjacent vertices in F_N, $i < j$. Let $H^{(i,j)}$ be the subgraph of H induced by $W_i \cup W_j$. Consider the subgraph $G^{(i,j)}$ of G induced by $V_i \cup V_j$. Construct a graph $G^{(i,j)*}$ as in the proof of Lemma 8.6. I.e., take $|E_{H^{(i,j)}}|$ copies $G^{(i,j)}(e)$ of $G^{(i,j)}$ that correspond to edges $e \in E_{H^{(i,j)}}$, and perform edge swappings in such a way that we have $f^{(i,j)} : G^{(i,j)*} \xrightarrow{S} H^{(i,j)}$ and any edge $(u(e_1), v(e_2))$ between two copies $G^{(i,j)}(e_1)$ and $G^{(i,j)}(e_2)$ (with possibly $e_1 = e_2$) corresponds to an edge (u, v) in $G^{(i,j)}$. Let $V_{G^{(i,j)*}} = V_i^{(i,j)*} \cup V_j^{(i,j)*}$, where all vertices in $V_i^{(i,j)*}$ are mapped inside the block W_i and all vertices in $V_j^{(i,j)*}$ inside W_j.

Observe that by the construction each $G^*_{i,j}$ the preimage of every vertex from V_i have the same size $\forall u \in W_i : |(f^{(i,j)})^{-1}(u)| = \frac{|V_i^{(i,j)*}|}{|W_i|}$, and vice versa for vertices from the block W_j.

According to the proof Lemma 8.7 construct similarly for each loop (i, i) in F_N a graph $G^{(i,i)*}$ such that $G^{(i,i)*} \xrightarrow{S} G^{(i,i)}$ and allows $f^{(i,i)} : G^{(i,i)*} \xrightarrow{S} H^{(i,i)}$. As above each vertex has the same number of preimages in $f^{(i,i)}$, i.e. $|(f^{(i,i)})^{-1}(u)| = \frac{|V_i^{(i,i)*}|}{|W_i|}$ holds for each vertex u from W_i.

At this moment we have have constructed graphs that will provide connections between blocks V_i^* and V_j^* of the final graph G^*. The graph G^* will be formed by a series of unification from sufficiently copies of graphs $G^{(i,j)*}$, where (i, j) are taken over all arcs and loops in F_n s.t. $i \leq j$. The number of copies $x^{(i,j)}$ of the subgraph $G^{(i,j)*}$ has to assure that it will be possible to merge vertices coming from different parts $G^{(i,j)*}, G^{(i,j')*}, G^{(i,j'')*}, \ldots, G^{(k,i)*}, \ldots$ In particular it is enough to assure

$$x^{(i,j)} \cdot |V_i^{(i,j)*}| = x^{(i,j')} \cdot |V_i^{(i,j')*}| = \cdots = x^{(k,i)} \cdot |V_i^{(k,i)*}| = \ldots \tag{8.6}$$

Then every vertex $u \in W_i$ has the same number of preimages in all sets $V_i^{(i,j)*}, V_i^{(i,j')*}, \ldots$. Now the unification could be made by collapsing of a matching between $x^{(i,j)}$ copies of $(f^{(i,j)})^{-1}(u)$ and $x^{(i,j')}$ copies of $(f^{(i,j')})^{-1}(u)$, etc., for every $u \in W_i$.

The only thing which remains to verify is whether we can find nontrivial integers $x^{(i,j)}$ that satisfy the system of equations (8.6) taken for all $i = 1, \ldots, l$.

If we fix some $x^{(i,j)} > 0$, the size of sets W_i^* and W_j^* are uniquely determined. Then also are determined the values of all $x^{(j,k)}$ for all arcs (j,k) in F_N, and these values remain positive.

We express the size of some W_i^* as follows:

$$|W_i^*| = x^{(i,j)} \cdot |W_i^{(i,j)*}| = x^{(i,j)} \cdot |W_i| \cdot |E_{H^{(i,j)}}| = x^{(i,j)} \cdot |W_i| \cdot |V_i| \cdot n_{i,j} \quad (8.7)$$

W.l.o.g. assume that F_N contains a cycle $(1,\ldots,c)$. Then the size of W_c^* can be expressed in two ways as

$$\begin{aligned} |W_c^*| &= |W_1^*| \cdot \frac{|W_c|}{|W_1|} \cdot \frac{|V_c|}{|V_1|} \cdot \frac{n_{c,1}}{n_{1,c}} \\ |W_c^*| &= |W_1^*| \cdot \frac{|W_c|}{|W_1|} \cdot \frac{|V_c|}{|V_1|} \cdot \prod_{j=1}^{c-1} \frac{n_{j+1,j}}{n_{j,j+1}} \end{aligned}$$

Here in the first case we have considered only the arc $(1,c)$ while in the other we have iterated (8.7) along the path $1,2,\ldots,c$.

As F_N satisfies the cycle product identity due to Theorem 2.3, the two expressions above cause no conflict. Hence, values of $\mathbf{x}$ can be derived from a single entry $x^{(i,j)}$ of each connected component of F_N, regardless which paths were used during the computation. Since all coefficients in the system of linear equations determining $\mathbf{x}$ are integers, a nontrivial integer solution exists as well. □

8.2.2 Computational complexity

We are now ready to show decidability of the Matrix Surjectivity problem, i.e. deciding if $M \overset{S}{\leqslant} N$ for two degree matrices M and N. We use case (*ii*) of Theorem 8.8 and show that the existence of a suitable G can be nondeterministically verified in polynomial time. We mimic the method of Section 8.1, so here we briefly describe differences.

Theorem 8.9. *Let M, N be degree matrices of order k and l. If $M \overset{S}{\leqslant} N$, then there exists a graph G of size $(klm^*)^{O(k^2l^2)}$ such that $G \xrightarrow{S} N$ and $M \in \mathcal{M}_G$.*

Proof. We first explore properties of such a hypothetical graph G. As $M \in \mathcal{M}_G$ we have a partition $\{U_1,\ldots,U_k\}$ of $V_G$and since $G \xrightarrow{S} N$ we get a partition $\{V_1,\ldots,V_l\}$ of V_G satisfying equation (8.5). We combine these partitions s.t. for each pair of indices r and s we define the set $W_{r,s} = \{v \mid v \in U_r \cap V_s\}$.

For vertices from $W_{r,s}$ we determine all feasible vectors describing the distribution of neighbors in the classes $W_{1,1},\ldots,W_{k,l}$. These are vectors of length kl whose entries are positive integers and are indexed by pairs ij. If such a vector $\mathbf{q}^{r,s}$ further satisfies

$$\begin{array}{rcrcll} & & \sum_{j=1}^{l} q_{i,j}^{r,s} & = & m_{r,i} & \text{for all } 1 \leq i \leq k, \\ n_{s,j} > 0 & \Rightarrow & \sum_{i=1}^{k} q_{i,j}^{r,s} & \geq & n_{s,j} & \text{for all } 1 \leq j \leq l. \\ n_{s,j} = 0 & \Rightarrow & \sum_{i=1}^{k} q_{i,j}^{r,s} & = & 0 & \text{for all } 1 \leq j \leq l. \end{array}$$

then we call $\mathbf{q}^{r,s}$ a *surjective distribution row for indices r and s.* The set of all different surjective distribution rows for r and s is finite and we write it as $\{\mathbf{q}^{r,s(1)}, \ldots, \mathbf{q}^{r,s(t(r,s))}\}$.

Now consider a set of t_0 variables $w^{r,s(t)}$ for all feasible r, s and all $1 \leq t \leq t(r,s)$. The existence of a nontrivial *nonnegative* solution of the following homogeneous system of k^2l^2 equations in t_0 variables:

$$\sum_{t=1}^{t(r,s)} q_{i,j}^{r,s(t)} w^{r,s(t)} = \sum_{t'=1}^{t(i,j)} q_{r,s}^{i,j(t')} w^{i,j(t')} \qquad 1 \leq i, r \leq k, \ 1 \leq j, s \leq l \tag{8.8}$$

is a necessary and sufficient condition for the existence of the desired graph G and can be proved exactly as in Theorem8.1. If the positive case Lemma 1.1 assures that the system (8.8) has a nontrivial nonnegative integral solution with at most $k^2l^2 + 1$ nonzero entries, which are each bounded in size by $O(k^2l^2 \log(klm^*))$. □

Corollary 8.10. *The problem* MS *belongs to the class* NP*.*

Proof. We construct the nondeterministical proof from the vector $\mathbf{w}$ and the corresponding surjective distribution rows as for the MI problem.

Both tests, namely whether distribution rows are valid and whether $\mathbf{w}$ satisfies (8.8), are doable in polynomial time with respect to the size of M and N. □

Appendix A

Related decision problems

A.1 Tractable problems

In this Appendix we briefly review polynomially solvable problems that were used as subroutines for algorithms presented in this thesis.

> 2-Satisfiability (2-Sat)
> *Instance:* A formula Φ in conunctive normal form where each clause consists of two literals.
> *Question:* Does Φ allow an satisfying assignment?

The 2-Sat problem is solvable in linear time see [22, problem L01] or [14].

> Subtree Isomorphism
> *Instance:* Trees T and T'.
> *Question:* Is T a subtree of T'?

Solvable in time $O(\frac{k^{1.5}}{\log k}n)$ where $k = |V_T|$, $n = |V_{T'}|$ [59], see also [22, problem GT48] and [55].

As a special case of problem we get the tree isomorphism problem, which is well known to be solvable in linear time [34].

A.2 NP-complete problems

In this Appendix we list NP-complete problems that were used in hardness proofs in polynomial reductions.

> Satisfiability (Sat)
> *Instance:* A formula Φ in conunctive normal form.
> *Question:* Does Φ allow an satisfying assignment?

The Sat problem remains NP-complete even if all clauses contain at most three literals (3-Sat) see [22, problems LO1 and LO2].

Not All Equal 3-Satisfiability (NAE-3-Sat)
Instance: A formula Φ in conunctive normal form.
Question: Does Φ allow an satisfying assignment such that each clause contains negatively valued literal?

This problem can be reduced to the case when all literal are positive occurences, i.e. no clause contains a negation. Another view on this problem can be done via hypergraph coloring:

Hypergraph 2-Colorability (H2C)
Instance: A set $Q = \{q_1, \ldots, q_m\}$ and a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ with $S_j \subseteq Q$ for $1 \leq j \leq n$.
Question: Is there a 2-coloring of $(Q, \mathcal{S})$, i.e., a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$?

Both the two above problems are NP-complete [22, problem LO3].

H-Coloring
Instance: A graph G.
Question: Does there exist a graph homomorphism $G \to H$?

Solvable in polynomial time if H is bipartite and is NP-complete in all other cases [29]. Includes graph k-Coloring as a special case when $H = K_k$.

BW(k, j)-Coloring (BW(k, j))
Instance: A $(k + j)$-regular graph G.
Question: Does there exist a coloring of $V(G)$ with black and white colors s.t. each vertex is adjacent to exactly k vertices of its own color?

When k or j is zero or both are one, the problem is trivially solvable, but all other cases are NP-complete: for BW$(2, 1)$ see [35] and for the case of an even $k \geq 2$ and an arbitrary $j \geq 1$ see [37]. The remaining cases of an odd k can be treated similarly.

Edge Precoloring Extension (EPE)
Instance: A cubic bipartite graph G, where some edges are precolored with three colors.
Question: Can the precoloring be extended to all edges?

Shown to be NP-complete via reduction to NAE-3-Sat [17]. Remains NP-complete if only two colors are used in the precoloring. It become polynomially solvable if only one color is used [40].

Index

List of Symbols and Abbreviations

A	auxiliary graph allowing two homomorphisms $A \xrightarrow{S} H$, page 78
A_G	adjacency matrix of the graph G, page 13
$B(a_1, a_2, a_2)$	barbell graph consisting of a path of length a_1 joining cycles of length a_2 and a_3, page 57
C_n	cycle of length n, page 14
D_G	set of all maximum distance vertices in G, page 78
E_G	set of edges in the graph G, page 10
$F(a_1, \ldots, a_k)$	flower graph with cycles of length $a_1, \ldots, a_k$, page 57
F_G	multiset of flags of G, page 59
$\overline{G}$	complement the graph G, page 13
$G^{:t}$	graph G where each edge was subdivided $t-1$ times, page 34
K_n	complete graph on n vertices, page 13
$K_{n,n'}$	complete bipartite graph on blocks of sizes n and n', page 13
m^*	2 + the maximum absolute value in the matrix M, page 18
M^T	transpose of the matrix M, page 18
$N_H^{(2)}(u)$	the set of vertices that share a neighbor with u, page 79
$N_G(u)$	(open) neighborhood of the vertex u, page 11
$N_G[u]$	closed neighborhood of the vertex u, page 11
$O(g)$	asymptotic upper bound function, page 16
$o(g)$	asymptotically negligible function, page 16

P_n	path of length $n-1$, page 14
S_n	star on $n+1$ vertices, page 13
T_G	universal cover of the graph G, page 36
V_G	vertex set of the graph G, page 10
$\mathbf{p}^{r,s}$	injective distribution row for indices r and s, page 89
$\mathbf{q}^{r,s}$	surjective distribution row for indices r and s, page 100
$\mathcal{B}$	equitable partition of a graph, page 20
$\mathcal{E}_G$	edge space of the graph G, page 18
$\mathcal{G}^c$	class of all finite connected nonisomorphic graphs, page 16
$\mathcal{M}_G$	set of all degree matrices of the graph G, page 20
$\mathcal{M}^c$	set of all degree matrices of finite connected graphs, page 20
$\mathcal{P}(X)$	set of all subsets of the set X, page 11
$\mathcal{S}$	Instance of the H2C problem, page 79
$\mathcal{S}_G$	cycle space of the graph G, page 18
$\mathcal{Q}_G$	the set of equitable partitions of G, page 43
$\mathbb{Z}_p$	ring of residues modulo p, page 18
$\mathrm{Aut}(G)$	automorphism group of the graph G, page 16
$\deg_G(u)$	degree of the vertex u, page 11
$\mathrm{diam}(G)$	diameter of the graph G, page 14
$\dim(\mathcal{S})$	dimension of the space $\mathcal{S}$, page 18
$\mathrm{dist}_G(u,v)$	distance between vertices u and v, page 14
$\mathrm{drm}(G)$	degree refinement matrix of the graph G, page 27
$\mathrm{girth}(G)$	length of the shortest cycle in the graph G, page 13
$\mathrm{indeg}_{\vec{G}}(u)$	indegree of the vertex u, page 11
$\ker(M)$	kernel of the matrix M, page 18
$\mathrm{outdeg}_{\vec{G}}(u)$	outdegree of the vertex u, page 11
$\mathrm{rank}(M)$	rank of the matrix M, page 18

NP	nondeterministically polynomial problems, page 17
P	polynomially solvable problems, page 17
$\mathrm{BW}(k, j)$	Black and white coloring problem with parameters (k, j), page 102
DMA	Degree Matrix Association problem, page 27
DMD	Degree Matrix Determination problem, page 22
EPE	Edge Precoloring Extension problem, page 102
FF	Flag factor problem, page 59
H-Coloring	H-Coloring problem, page 102
H-LBiHom	H-Locally Bijective Homomorphism problem, page 50
H-LInHom	H-Locally Injective Homomorphism problem, page 50
H-LSurHom	H-Locally Surjective Homomorphism problem, page 50
H-LSurHom*	Special H-Locally Surjective Homomorphism problem, page 75
H2C	Hypergraph 2-Colorability problem, page 102
MI	Matrix Injectivity problem, page 91
MS	Matrix Surjectivity problem, page 94
Sat	Satisfiability problem, page 101
2-Sat	2-Satisfiability problem, page 101
NAE-3-Sat	Not All Equal 3-Satisfiability problem, page 102
Subtree Isomorphism	Subtree Isomorphism, page 101
$\Delta(G)$	the maximum degree in the graph G, page 13
$\delta(G)$	the minimum degree in the graph G, page 13
$\Omega(g)$	asymptotic upper bound function, page 16
$\omega(g)$	asymptotically dominant function, page 16
π_i	projection to the i-th coordinate, page 16
$\Theta(a_1, \ldots, a_k)$	theta graph with paths of length $a_1, \ldots, a_k$, page 57
$\Theta(g)$	asymptotic tight bound function, page 16

$G \times H$	product of graphs G and H, page 16
$G \times K_2$	Kronecker double cover, page 33
$\mathcal{B} \preceq \mathcal{C}$	equitable partition $\mathcal{B}$ is coarser than $\mathcal{C}$, page 43
$G \simeq H$	isomorphic graphs G and H, page 15
$L \propto L'$	polynomial time reduction from the problem L to L', page 17
$G \to H$	homomorphism from the graph G to H, page 14
$G \xrightarrow{B} H$	locally bijective homomorphism from the graph G to H, page 30
$G \xrightarrow{I} H$	locally injective homomorphism from the graph G to H, page 30
$G \xrightarrow{S} H$	locally surjective homomorphism from the graph G to H, page 30
$[n]$	set $\{1, 2, ..., n\}$, page 18
$\langle G \rangle$	computational size of the graph G, page 17
$\langle M \rangle$	computational size of the matrix M, page 17

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