Note on the computational complexity of covering regular graphs

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Abstract. We show that the *NP*-completeness of the computational complexity of the problem which for a fixed k-regular graph H ($k \geq 3$) determines whether there exists a locally isomorphic graph hommomorphism from an input graph G to H.

Introduction

We define the covering projection $f : G \to H$ as a homomorphism which restricted to the neigborhood $N[u]$ of an arbitrary vertex $u \in V(G)$ is an isomorphism to $N[f(u)]$.

The covering projection satisfies the following conditions:

1. f is a homomorphism, i.e. $\forall (u, v) \in E(G) : (f(u), f(v)) \in E(H)$
2. f is locally injective, i.e. $\forall u, v \in V(G), dist(u, v) = 2 : f(u) \neq f(v)$
3. f is degree-preserving, i.e. $\forall u \in V(G) : deg_G(u) = deg_H(f(u))$

It is easy to check that the local injectivity (no pair of edges can be merged into a single edge) and degree preserving (the target doesn't have incident more edges than the source) are necessary and sufficient conditions for the local isomorphism and therefore above definitions are equivalent.

The computational point of view states a question whether for given graphs G and H there exists a (partial) covering projection from G to H. If both graphs are part of the input, then the problem is trivially *NP*-complete, since selecting $H = K_4$ we test the existence of a proper 4-coloring of a cubic graph G such that on the closed neighborhood of every vertex all four colors are used [3].

We use a similar approach as is used for testing the existence of a graph homomorphism (i.e. the H-coloring problem) and define a class of H-cover problems where each problem corresponds to a specific graph H:

Problem: *H-cover [1]*
Input: *A graph G*
Question: *Does there exists a covering projection mapping the graph G onto the graph H?*

Without lost of generality we suppose that the input graph G is connected, since each block of connectivity of G have to (partially) cover the graph H if and only if the entire graph G covers H.

Covers of regular graphs

In this section we consider a regular graph H as the underlying graph for the H-cover problem. It was generally excepted that for all k-regular graphs with $k \geq 3$ the H-cover problem is *NP*-complete.

Here we show proof of this conjecture.

One more definition is necessary for the *NP*-completeness reduction.

Definition *A graph H is* solid *[2] if for any vertex $u \in V(H)$ the graph H_u that arises by splitting the vertex u of degree d into leaves $u_1, .., u_d$ involves only partial covers $H_u \to H$ that became automorphisms of H after unifying all vertices u_i into the original vertex u.*

The multicover and solid graphs were used in a construction of an gadget for a polynomial reduction from the hypergraph colorability.

Theorem 1 *[4] The H-cover problem is NP-complete for solid graphs.*

In addition two large classes of graphs were showed to be classes of solid graphs.

Theorem 2 *[4] All k-edge colorable k-regular graphs and all $\lceil \frac{k+2}{2} \rceil$-edge-connected k-regular graphs are solid.*

The characterization is tight in the sense that in [2] there was given an example of a $\lceil \frac{k+1}{2} \rceil$-edge-connected k-regular graph that is not solid.

We extend the result of Theorems 1 and 2 into the class of all regular graphs.

Theorem 3 *The H-cover problem is NP-complete for all k-regular graphs H of $k \geq 3$.*

Proof: Without loss of generality we assume that H is connected and that H is not a solid graph, in particular that H is not bipartite, since bipartite k-regular graphs are due to König-Hall marriage theorem k-edge colorable and hence solid.

The Kronecker double cover $\tilde{H} = H \times K_2$ [1] is k-edge colorable k-regular connected graph and hence the $\tilde{H}$-cover problem is due to Theorems 1 and 2 *NP*-complete.

We show a reduction of the $\tilde{H}$-cover problem to the H-cover problem. Consider a graph G whose covering projection $G \to \tilde{H}$ is questioned. We claim that G covers $\tilde{H}$ if and only if G is bipartite and G covers H.

The only if statement is trivial since $\tilde{H}$ is bipartite and only bipartite graphs can cover a bipartite graph (this holds even for a general graph homomorphism). Moreover any covering projection $G \to \tilde{H}$ can be extended to H by a composition with a covering projection $\tilde{H} \to H$.

In the other direction assume that $f : G \to H$ is a covering projection and that G is bipartite, and its proper bicoloring using black and white color is given. For each vertex u of H denote by $(u, 0)$ and $(u, 1)$ its two copies in $u \times K_2 \subset H \times K_2 = \tilde{H}$. We define a mapping $\tilde{f} : G \to \tilde{H}$ by $f(v) = (u, 0)$ if $f(v) = u$ and v is white, and $f(v) = (u, 1)$ if $f(v) = u$ and v is black.

Since each vertex has all neighbors colored by the complementary color, the above mentioned mapping satisfies all properties of a covering projection. $\square$

All graphs H with at most one cycle in each component of connectivity are polynomial instances for the H-cover problem, hence we get that the computational complexity of the class of all regular graphs is fully clasified.

References

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[3] Kratochvíl, J. Regular codes in regular graphs are difficult. *Discrete Mathematics 133* (1994), 191–205.

[4] Kratochvíl, J., Proskurowski, A., and Telle, J. A. Covering regular graphs. *Journal of Combinatorial Theory B 71* (1997), 1–16.

[1] $V(\tilde{H}) = V(H) \times \{0, 1\}, E(\tilde{H}) = \{((u, 1), (v, 0)), ((u, 0), (v, 1))) : (u, v) \in E(H)\}$