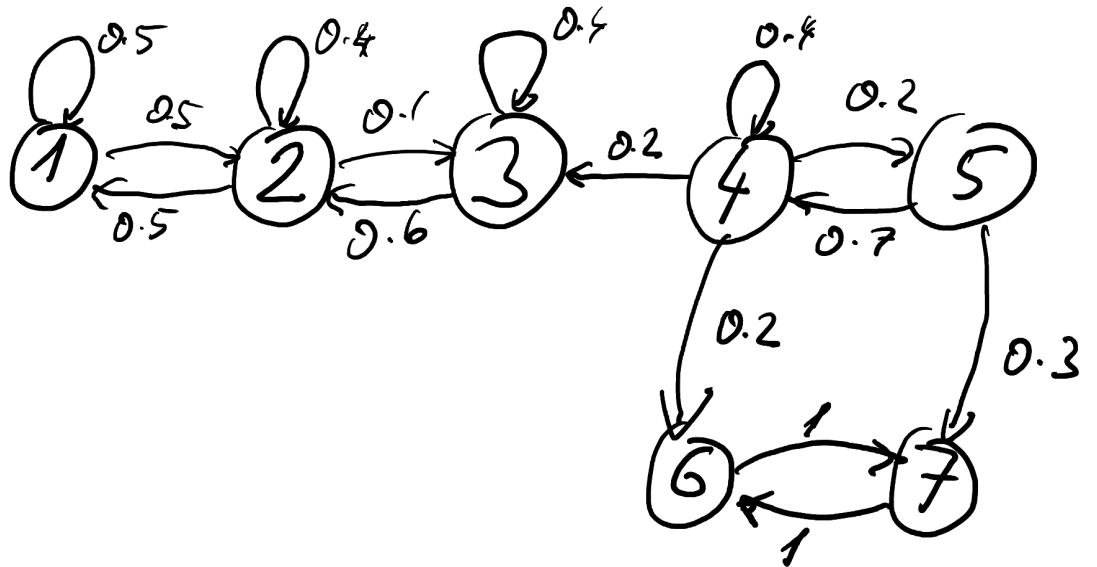# Exercise session 4 – Prob. & Stat. 2 — Oct 24, 2023

1. Assume the process starts in state 4.

(a) For each recurrent class find the probability, that we reach that class at some time.

(b) What is the expected time till we reach some recurrent state?

2. In a gambler's ruin problem the player is repeatedly betting \$1 in a fair bet, until she loses ℓ_1 dollars or gains ℓ_2 dollars.

(a) What is the probability of the first and the second way how the game can end? For a shorter solution: use the fact, that expected winning is zero at every time!

(b) What is the expected number of rounds before this happens? (Find a short solution for $\ell_1 = \ell_2 = n$ using the calculation from class last week.)

3. A modification of the previous problem: the probability of losing a dollar is 2/3, probability of gaining it 1/3. Suppose that you start with i dollars and finish when you have n dollars or you have lost everything. Let W_t be the amount you have after t rounds of play.

(a) Show that $\mathbb{E}(2^{W_{t+1}}) = \mathbb{E}(2^{W_t})$.

(b) Use part (a) to determine the probability of finishing with 0 dollars and the probability of getting to n dollars, when starting with i dollars at the beginning.

(c) Verify that the probabilities satisfy the equations for the absorption probabilities we covered in the lecture.

(d) Generalize to any probability of losing $p > 1/2$.
(Hint: consider $\mathbb{E}(c^{W_t})$ for a suitable c.)

4. Recall the Markov chain we used in the 3-SAT algorithm. (Random walk on 0, ...,n, with preference going right.) What is the expected time to reach 0?

Random Walks

Given an undirected graph $G = (V, E)$ we consider a Markov chain with states V and transition probabilities $p_{u,v} = \frac{1}{\deg(u)}$ for every edge $uv \in E$. (That is, the person at vertex u is equally likely to use any edge to leave u.)

5. Show that $\pi_v = \frac{\deg(v)}{2|E|}$ is stationary distribution (assuming the graph is finite).

6. The Markov chain is aperiodic iff the graph is not bipartite. (What are the implications for the convergence to stationary distribution?)

7. * A maximum over all vertices $v \in V$ of the expected time to visit all other vertices, when we start at v is called the *cover time* of G.
  (a) What do we know about cover time of a path?
  (b) Compute (or estimate) a cover time of a clique.
  (c) * Find a graph on n vertices with as large cover time as possible.

Applications ;–)

8. Define a particular way of shuffling cards as a Markov chain. (Suggestion: we take a couple of cards from the top of the stack at put them somewhere in the middle without changing their order.) Is the Markov chain aperiodic? Irreducible?

  Homework: Experimentally find the speed of convergence to the stationary distribution. More details will be provided in the Owl.

9. * There are $n+1$ sheep standing in a circle, let us label them $0, 1, \ldots, n$. A wolf eats sheep 0. Next, the wolf starts to walk randomly around the circle. At each position, he eats the sheep (if it is still there). Then he decides at random, which of the two directions to take next. (He does decide even on empty spots, where the sheep has been eaten already.)
  (a) What is the probability, that the wolf eats sheep n as the last one?
(Hint: use Markov chains on a path.)
  (b) What is the probability, that the wolf eats sheep i as the last one? (For any $i = 1, 2, \ldots, n$.)