Exercise session 7 – Prob. & Stat. 2 — Dec 1, 2022

⟹ Zápočtová písemka (credit exam) next week – Dec 8, 2022 ⟸

The Poisson process

Recall the definition of the Poisson process by means of exponential waiting times. And also the theorem speaking about distribution of N_t, more precisely $N_{t_{k+1}} - N_{t_k} \sim Pois(\lambda(t_{k+1} - t_k))$. You may also use the following two results, analogical to what we learned about the Bernoulli process:

- Given a Poisson process with intensity $\lambda > 0$ and $p \in (0, 1)$. We create a new Poisson process by keeping each arrival with probability p. More precisely: Having defined times T_k as we have in the lecture, at time T_k we toss a coin (with probability p) to decide if something actually happened at time T_k. Then we define new sequences L'_k, T'_k N'_t based on the new arrival times.

  Then the new process is a Poisson process with intensity λp.

- Similarly: having two Poisson processes, with intensity λ_1 and λ_2, their merge is a Poisson process with intensity $\lambda_1 + \lambda_2$.

1. Customers depart from a bookstore according to a Poisson process with rate λ per hour. Each customer buys a book with probability p, independent of everything else.
(a) Find the distribution of the time until the first sale of a book.
(b) Find the probability that no books are sold during a particular hour.
(c) Find the expected number of customers who buy a book during a particular hour.

2. An athletic facility has 5 tennis courts. Pairs of players arrive at the courts and use a court for an exponentially distributed time with mean 40 minutes. Suppose a pair of players arrives and finds all courts busy and k other pairs waiting in queue.
(a) What is the expected waiting time to get a court?
(b) What is the probability that they get to play within 2 hours?

3. By considering Poisson process, derive that for independent random variables $X_i \sim Pois(\lambda_i)$ we have

$$X_1 + \cdots + X_n \sim Pois(\lambda_1 + \cdots + \lambda_n).$$

4. * Consider a Poisson process. Given that a single arrival occurred in given interval $[0, t]$, show that the distribution of the arrival time is uniform over $[0, t]$.

5. Beginning at time $t = 0$, we start using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a type-A bulb and a type-B bulb. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else. Lighbulbs of type A have $X \sim Exp(1)$, of type B $X \sim Exp(3)$.
(a) Find the expected time until the first failure.
(b) Find the probability that there are no bulb failures before time t.
(c) Given that there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.
(d) Find the variance of the time until the first bulb failure.
(e) Find the probability that the 12th bulb failure is also the 4th type-A bulb failure.
(f) Up to and including the 12th bulb failure, what is the probability that a total of exactly 4 type-A bulbs have failed?
(g) Determine either the PDF or the transform associated with the time until the 12th bulb failure.
(h) Determine the probability that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

(i) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance of the total period of illumination provided by type-B bulbs while the process is in operation.

(j) Given that there are no failures until time t, find the expected value of the time until the first failure.

6. A pizza restaurant serves n different types of pizza, and is visited by a number K of customers in a day, where K is a Poisson random variable with mean λ. Each customer orders a single pizza, with all types of pizza being equally likely, independent of the number of other customers and the types of pizza they order. Find the expected number of different types of pizzas ordered by at least one customer.

Bernoulli process

7. (Sum of a geometric number of independent geometric random variables.)
Let $Y = X_1 + \cdots + X_N$, where the random variables X_i are geometric with parameter p and N is geometric with parameter q. Assume that the random variables $N, X_1, X_2, \ldots$ are independent. Show that Y is geometric with parameter pq. Hint: Interpret the various random variables in terms of a split Bernoulli process.

8. * (The bits in a uniform random variable form a Bernoulli process.)

Let $X_1, X_2, \ldots$ be a sequence of binary random variables taking values in the set $\{0, 1\}$. Let Y be a continuous random variable that takes values in the set $[0, 1]$. We relate X and Y by assuming that Y is the real number whose binary representation is $0.X_1X_2X_3\ldots$

More concretely

$$Y = \sum_{i \geq 1} 2^{-i} X_i.$$

(a) Suppose that the X_i form a Bernoulli process with parameter $p = 1/2$. Show that Y is uniformly distributed. [Hint: Consider the probability of the event $(i-1)/2^k < Y < i/2^k$, where i and k are positive integers.]

(b) Suppose that Y is uniformly distributed. Show that the $X_1, X_2, \ldots$ form a Bernoulli process with parameter $p = 1/2$.

Experiments

9. Choose one of the following ways to generate a Bernoulli proces: generate a sequence of independent Bernoulli trials $X_1, X_2, \ldots$ and compute the waiting times L_t and number of arrivals N_t. OR generate a sequence of independent geometric RVs $L_1, L_2, \ldots$ and from this deduce T_t, L_t and N_t. Verify that the computed variables have the distribution it should have (by computing its variance and mean, or by plotting the distribution of the sampled one and of a separately generated samples from the correct distribution).

10. Generate a Poisson process: generate a sequence of independent exponencial RVs $L_1, L_2, \ldots$ and from this deduce $N(t)$. Verify that it has Poisson distribution – as above, by sampling the varible many times, estimating the mean and variance and possibly by plotting the distribution. Or you may use Kolmogorov-Smirnov test (scipy.stats.kstest).