Exercise session 1 – Prob. & Stat. 2 — Oct 6, 2022

Definition of Markov chains

Recall that a sequence $(X_t)_{t=0}^{\infty}$ of random variables with range S is a (discrete time, time-homogeneous) *Markov chain* if for every $t \geq 0$ and every $a_0, \dots, a_{t+1} \in S$ we have

$$P(X_{t+1} = a_{t+1} \mid X_t = a_t \,\&\, \dots \,\&\, X_0 = a_0) = P(X_{t+1} = a_{t+1} \mid X_t = a_t) = p_{a_t, a_{t+1}},$$

for some collection of transition probabilities $p_{i,j}$.

1. A Markov chain with states $\{1, 2, 3\}$ has transition matrix

$$\begin{pmatrix} 0.4 & .4 & .2 \\ 0 & .8 & .2 \\ 0.9 & 0 & .1 \end{pmatrix}$$

Draw its transition graph.

2. Consider the Markov chain from the previous problem.
(a) Find $P(X_4 = 3 \mid X_3 = 2)$.
(b) Find $P(X_3 = 1 \mid X_2 = 3)$.
(c) Suppose $P(X_0 = 1) = 0.2$. Find $P(X_0 = 1 \,\&\, X_1 = 2)$.
(d) Suppose $P(X_0 = 1) = 0.2$. Find $P(X_0 = 1 \,\&\, X_1 = 2 \,\&\, X_2 = 3)$.
(e) Suppose that $X_0 = 1$. What is $P(X_3 = 1)$?

3. Show that any sequence of independent identically distributed random variables taking values in a countable set S is a Markov chain. What if the variables are independent, but each may have a different distribution?

4. Let us modify the example with broken/working machine from the class: If the machine is broken, the probability that it will be repaired in another day is still 0.9. However, if it is broken for the second day in a row, the probability that it will be repaired is only 0.5. If the machine is broken for three days in a row, it is broken forever.
(a) Can you represent this with a Markov chain?
(b) Suppose the probability that a working machine breaks increases to 0.1 after a year (starting from the last successful repair). Is this a Markov chain?

5. A mouse moves along a tiled corridor with $2m$ tiles, where $m > 1$. From each tile $i \neq 1, 2m$ it moves to either tile $i - 1$ for $i + 1$ with equal probability. From tile 1 or tile $2m$, it moves to tile 2 or $2m - 1$, respectively, with probability 1. Each time the mouse moves to a tile $i \leq m$ or $i > m$, an electronic device outputs a signal L or R, respectively.
(a) Is the position of the mouse a Markov chain?
(b) Can the generated sequence of signals L and R be described as a Markov chain with states L and R?

6. Consider the Markov chain that describes fly's movement (in the example from the lecture). Assume that the process starts at any of the four states, with equal probability. Let $Y_n = 1$ whenever the Markov chain is at state 0 or 1, and $Y_n = 2$ whenever the Markov chain is at state 2 or 3. Is the process Y_n a Markov chain?

7. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition graph.

(a) The largest number M_n shown up to the n-th roll.

(b) The number N_n of sixes in n rolls.

(c) At time r, the time A_r since the most recent six.

(d) At time r, the time B_r until the next six.

8. Suppose (X_t) is a Markov chain. Show that $(X_{2t})_{t=0}^{\infty}$ is also a Markov chain. What is its transition matrix?

Classification of states

Given a Markov chain (X_t) with set of states S, we say a state $s \in S$ is *recurrent* if $P(\exists t > 0 : X_t = s \mid X_0 = s) = 1$. We say s is transient otherwise.

(Observation: a state s is recurrent iff for every $s' \in A(s)$ we have $s \in A(s')$.)

Further, we say that s is *periodic* if there is an integer $\Delta > 1$ such that $P(X_t = s \mid X_0 = s) = 0$ unless t is divisible by Δ. A Markov chain is periodic if all of its states are.

9. Think about the difference between $P(\exists t > 0 : X_t = j \mid X_0 = i) > 0$ and $\exists t > 0 : P(X_t = j \mid X_0 = i) > 0$.

10. A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is an integer. When the spider and the fly land in the same position, the spider captures the fly. (They may "exchange position" though.)

(a) Construct a Markov chain that describes the relative location of the spider and fly.

(b) Identify the transient and recurrent states.

11. Consider a Markov chain with states 1, 2, ..., 9. and the following transition probabilities: $p_{1,2} = p_{1,7} = 1/2$, $p_{i,i+1} = 1$ for $i \neq 1, 6, 9$ and $p_{6,1} = p_{9,1} = 1$. Determine which states are recurrent/transient/periodic.