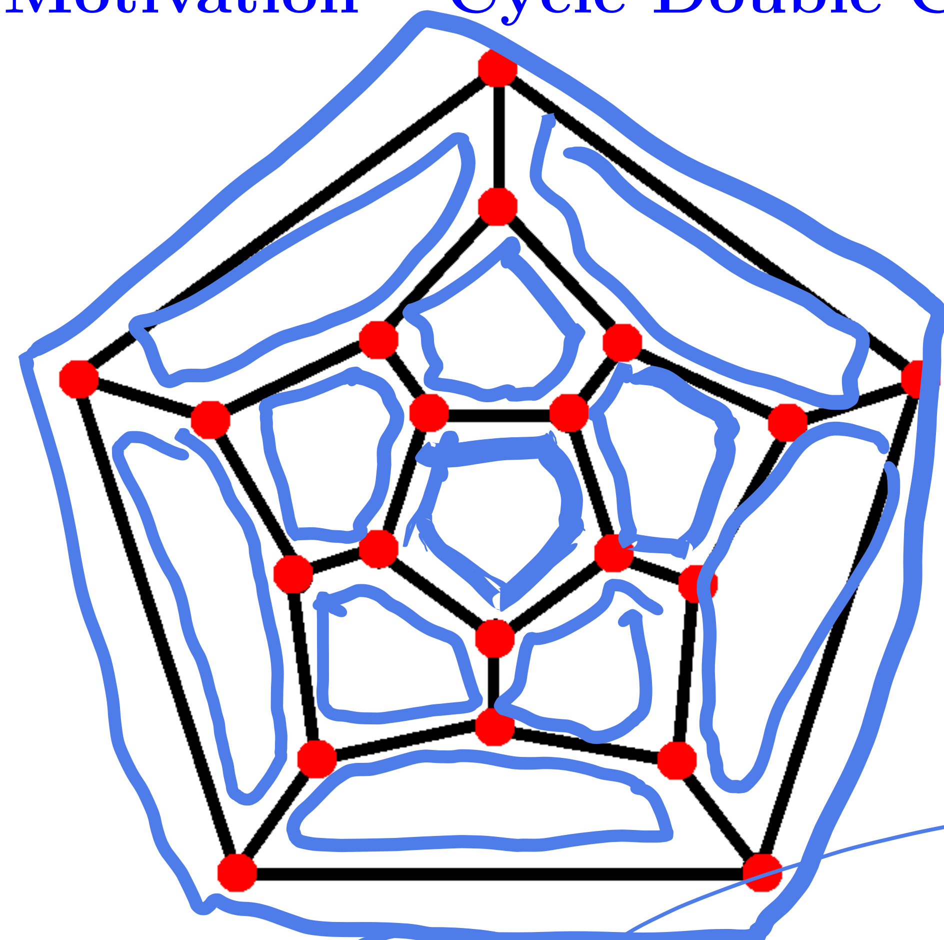
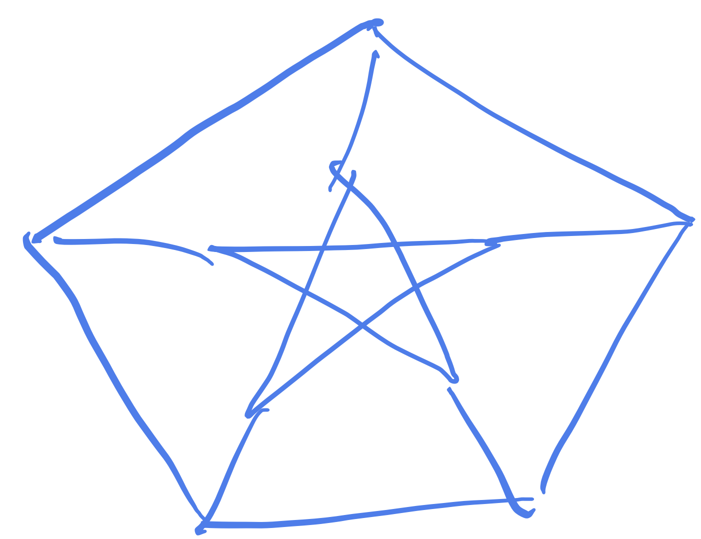
Motivation – Cycle Double Cover



for planar bridgeless graphs the face-boundaries are a collection of circuits that cover every edge exactly twice. What about nonplanar?

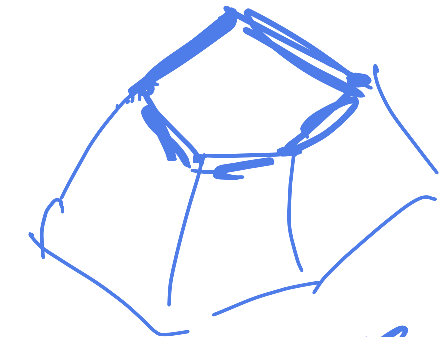


Petersen graph

Conjecture (Seymour, Szekeres)

Definitions

- circuit (kružnice) := 2-regular connected graph (subgraph of another graph)
- cycle (cyklus) = even graph = eulerian graph := edge disjoint union of circuits
- digraph := directed multigraph, loops allowed
- group := abelian group



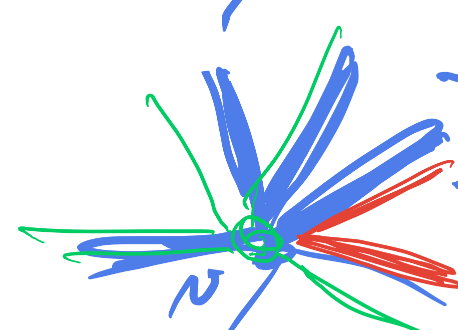
⟷ graph with all degrees even

$$\deg_{C_1} v + \deg_{C_2} v = \deg_{C_1 \triangle C_2} v + 2 \deg_{C_1 \cap C_2} v$$

subsets of $E(G)$

cycles are closed on $\triangle$

C_1, C_2 cycle $\Rightarrow$ $C_1 \triangle C_2$ is a cycle



Definition 1. *Let G be a digraph, Γ a group. A mapping $f : E(G) \to \Gamma$ is called a* flow *(or, more explicitly, a Γ-flow), if for every vertex $v \in V(G)$ the Kirchhoff law is valid:*

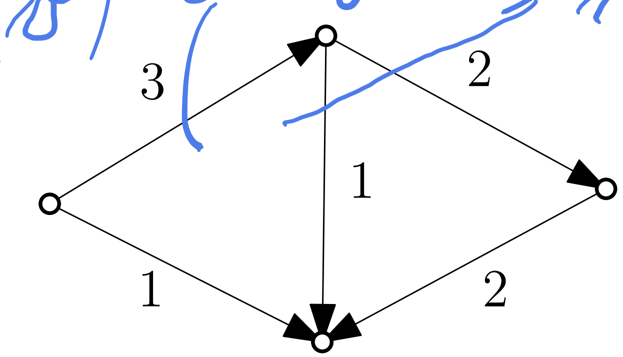
$$\sum_{e=(v,u)} f(e) = \sum_{e=(u,v)} f(e)\,.$$

$f^+(v)$ = *the left-hand side of the above equation, the amount of flow that leaves v,*
$f^-(v)$ = *the right-hand side of the above equation, the amount of flow that enters v.*

network flows

max-flow
= min cut

$f^-(v)$ v $1+2 = f^+(v)$



$\deg^+$ ... outdegree
$\deg^-$ ... in degree

- $f \equiv 0$ is a flow.
- if f, g are flows, then $f \pm g$ are also flows
- the set of all Γ-flows on a given digraph is again an (abelian) group.
- If Γ is a field, than the set of all Γ-flows is a vector space.
- related notion – flows in networks.
- $\mathbb{R}^d$-flow. The same definition. Esp. for $d = 3$ has a meaning in physics: momentum-preservation, Feynmann diagrams.

$\Gamma = \mathbb{Z}_2$

C is a cycle

$\Updownarrow$

f_C is a $\mathbb{Z}_2$ flow

$$f_C = \begin{cases} 1 & \text{on } e \in C \\ 0 & \text{on } e \notin C \end{cases}$$

Notation A, B are sets of vertices

$$f(A, B) = \sum f(e) : e \text{ starts in } A \text{ and ends in } B$$

$$f^+(A) = f(A, \bar{A})$$

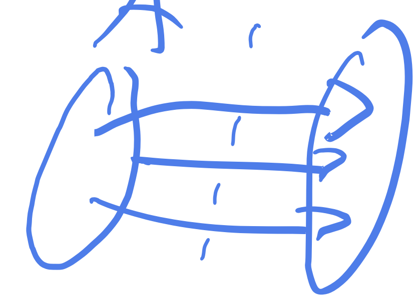
$$f^-(A) = f(\bar{A}, A)$$

(where $\bar{A} = \cancel{V}(G) \setminus A$).

E

$\bar{A} = A^c = E \setminus A$

A B



$f(A, B)$

$\delta^+(A) =$ set of edges leaving A

Observation 2. *Let G be a digraph, Γ a group, f a Γ-flow. Then for $A \subseteq V(G)$*

$$f^+(A) = f^-(A)\,.$$

Proof. Let us sum the Kirchhoff law for all $v \in A$. ∎

$f^+(v) = f^-(v)$ for $v \in A$

$\sum$

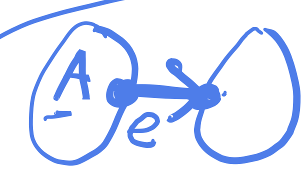
$$\underbrace{\sum_{v\in A} f^+(v)}_{= f^+(A) + f(A,A)} = \underbrace{\sum_{v\in A} f^-(v)}_{f^-(A) + f(A,A)}$$

Corollary 3 (a flow and small cuts)**.** *Let G be a digraph, Γ a group, f a Γ-flow.*

- *If e is a bridge then $f(e) = 0$.*
- *If e, e' form a 2-cut (and are oriented in the same direction) then $f(e) + f(e') = 0$.*

$\boxed{\delta^+(A) = \{e\}}$

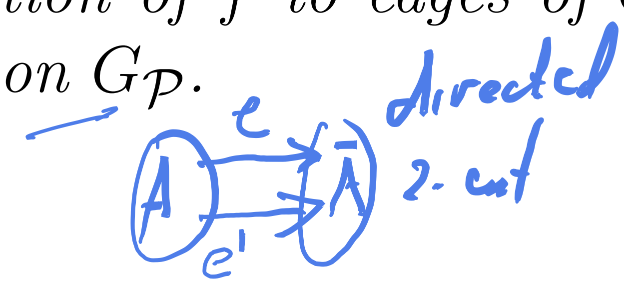
set of edges



$f^+(A) = f^-(A) = 0$

$f^+(A) = f(e)$

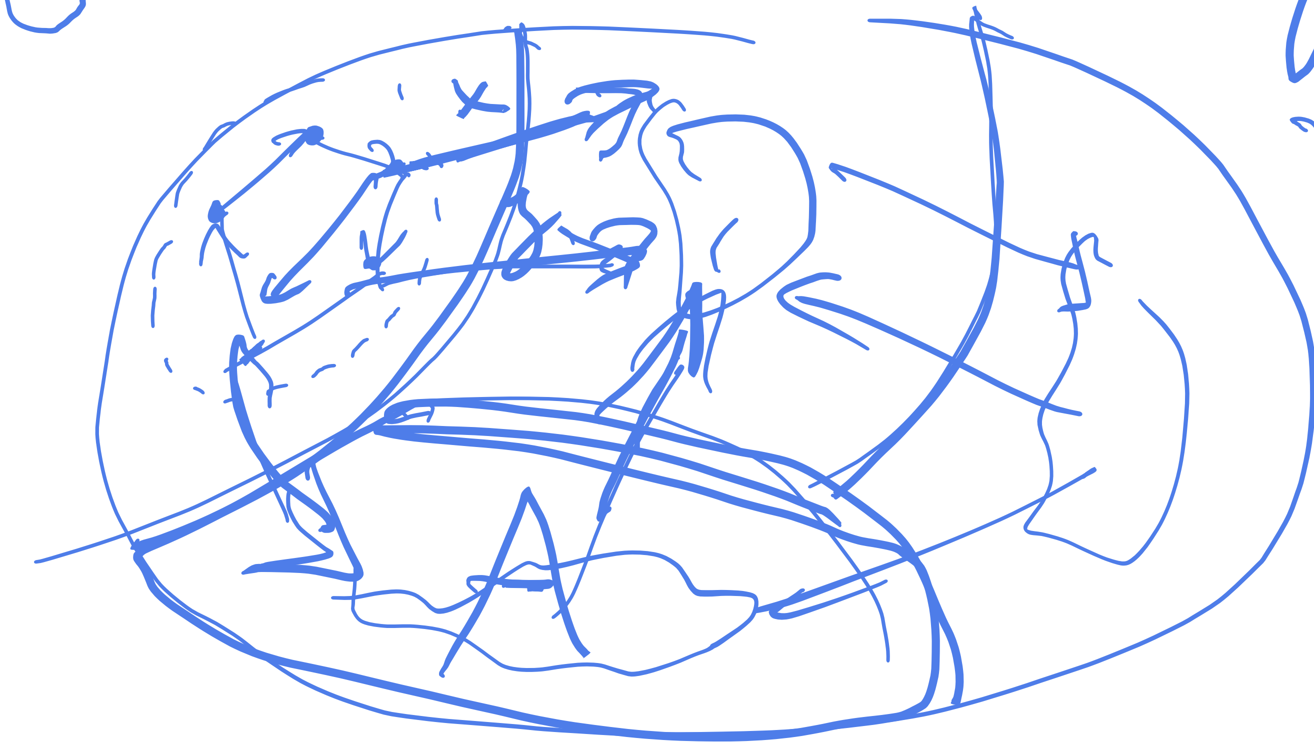
Corollary 4 (a flow and a partition)**.** *Let G be a digraph, Γ a group, f a Γ-flow. Consider any partition $\mathcal{P}$ of $V(G)$. Let $G_{\mathcal{P}}$ be the graph where each equivalence class is identified to a vertex and all edges are preserved and let $f_{\mathcal{P}}$ be the restriction of f to edges of $G_{\mathcal{P}}$. Then $f_{\mathcal{P}}$ is a Γ-flow on $G_{\mathcal{P}}$.*



$f^+(A) = f(e) + f(e')$

$f^-(A) = 0$

G



$\mathcal{P}$ ... partition

$\leadsto G/\mathcal{P}$

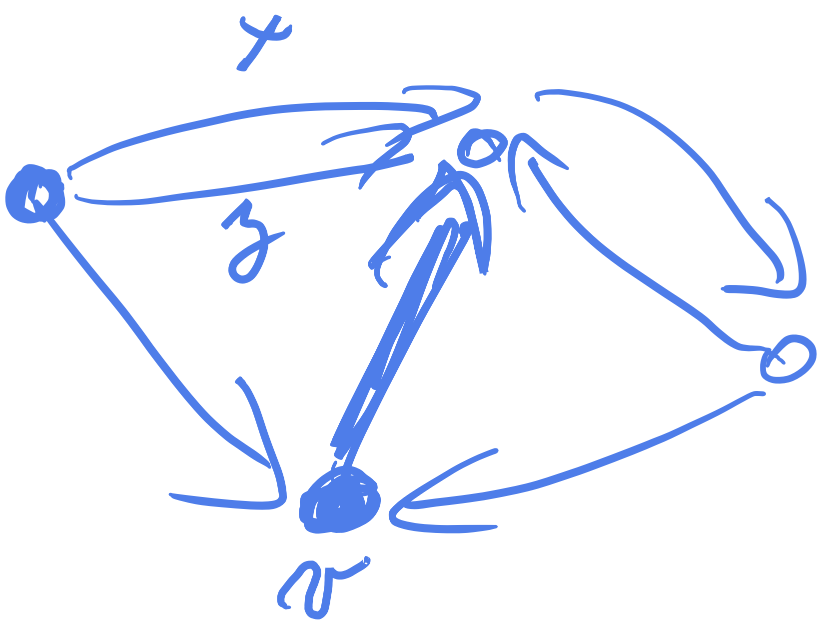
$f^+(A) = f_{\mathcal{P}}^+(v)$

$f^-(A) = f_{\mathcal{P}}^-(v)$

f ... flow on G

$\Downarrow$

$\boxed{f_{\mathcal{P}} \text{ on } G/\mathcal{P} \text{ is a flow}}$



Nowhere-zero flows

Definition 5. *Let G be a digraph, Γ a group, f a Γ-flow. We say that f is a* nowhere-zero Γ-flow, *if $f(e) \neq 0$ for all edges $e \in E(G)$. Frequently we will shorten nowhere-zero to NZ.*

- bridge $\Rightarrow$ no NZ flow.
- the opposite is also true
- dependence on the group Γ.

digraph

Theorem 6 (flow polynomial, Tutte 1954)**.** *For every graph G there is a polynomial $P_G(x)$ s.t. for every group Γ, the number of NZ Γ-flows on G is $P_G(|\Gamma|)$.*

$\Gamma_1 = \mathbb{Z}_4 \quad \Gamma_2 = \mathbb{Z}_2^2$

We will prove this by induction on $|E(G)|$.

- $P_G(x) = (x-1)P_{G-e}(x)$
- $P_G(x) = P_{G/e}(x) - P_{G-e}(x)$

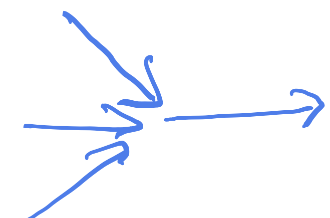
Corollary $|\Gamma_1| = |\Gamma_2|$

G has Γ_1-NZ-flow $\Leftrightarrow$ G has Γ_2 NZ-flow

$\Gamma = \mathbb{Z}_2$

G has a NZ $\mathbb{Z}_2$-flow

$\Updownarrow$



$1+1+1 = 1$

(in $\mathbb{Z}_2$)

all degrees are even

G is a cycle

$K_{3,3}$ NZ $\mathbb{Z}_3$-flow ?

✓

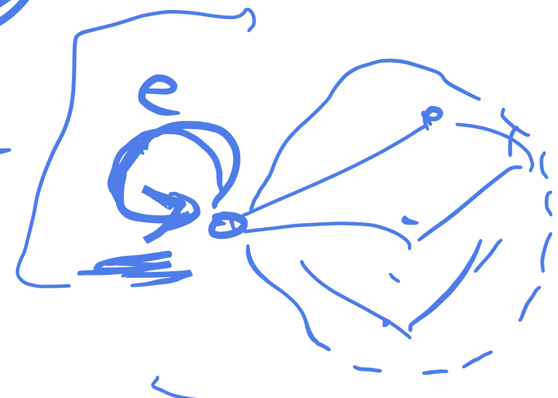
Proof of Thm 6

1) $E(G) = \emptyset$ # of there is one NZ-flow $(\emptyset)$

$$P_G(x) = 1$$

2) G has a bridge $P_G = 0$

3) G has a loop $\Longrightarrow P_G(x) = (x-1) \cdot P_{G-e}(x)$



G

G-e is smaller

$\Longrightarrow P_{G-e}$ exists

f flow on G-e $\rightsquigarrow$?? is a flow on G

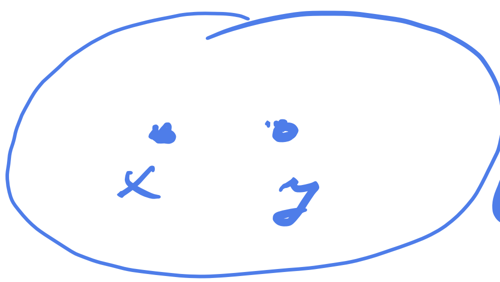
f $\exists$ $|\Gamma|-1$ f'

f' = f extended by a NZ value on e

G: $\boxed{N^z \text{ flow on } G}$

[illegible]: $f(A) = f(B) + \boxed{f(e)}$

$f(D) = f(C) + f(e)$



G − e

G/e

(e removed
other loops
kept)

$\boxed{N^z \text{ flow on } G/e}$

$$P_G \overset{?}{=} P_{G/e}$$

$f(A) + f(C) = f(B) + f(D)$

f is a flow on G/e :
Kirchhoff in G/e

$$\boxed{\sum_{e \in A} f(e) + \sum_{e \in C} f(e) = \sum_{e \in B} f(e) + \sum_{D} f(e)}$$

Tutte polynomial

Contraction/deletion invariant – a polynomial in two variables that counts NZ flows, colorings and many more graph invariants. The Tutte polynomial is usually denoted $T_G(x, y)$ and satisfies the relation $\boxed{T_G = T_{G-e} + T_{G/e}}$ if e is neither a loop, nor a bridge, with the base case $\boxed{T_G(x, y) = x^i y^j}$ for G with i bridges, j loops, and no other edges. One can use T_G to express the flow polynomial P_G as well as the *chromatic polynomial* $C(x)$ (the number of proper colorings using x colors).

f NZ flow on G/e

f can be extended to a unique flow on G

either f is NZ on G

or f is NZ on G-e

$P_{G/e} = P_G + P_{G-e}$

$P_G = P_{G/e} - P_{G-e}$ $\boxed{\text{e is not a loop/bridge}}$

Reversing orientations

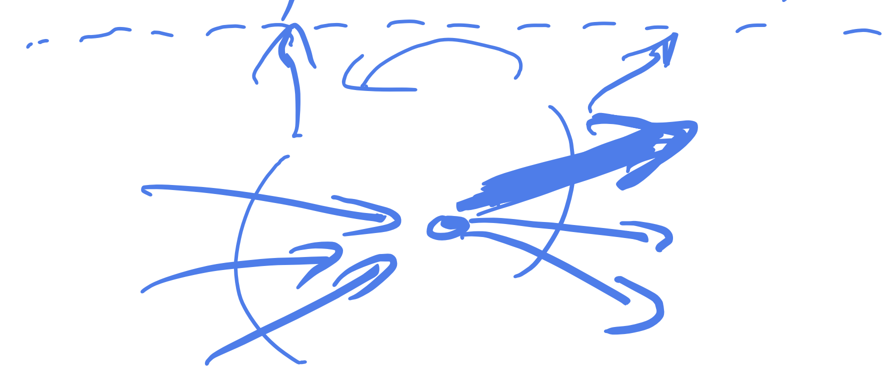
We need directed edges for the definition of flows. However, we will in fact study undirected graphs. To understand why, let us define a simple notation. Let G be a digraph, f a mapping $E(G) \to \Gamma$ and $F \subseteq E(G)$ any set of edges. We let G_F denote the digraph obtained from G after reorienting all edges in F. We define a mapping f_F as follows:

$$f_F(e) = \begin{cases} -f(e) & \text{if } e \in F \\ f(e) & \text{otherwise} \end{cases}$$

Observation 7. *Let f be a Γ-flow on a digraph G, let $F \subseteq E(G)$. Then f_F is a Γ-flow on G_F. Moreover, if f is NZ then f_F is also NZ.*

We can consider all pairs (G_F, f_F) to be different representations of “the same flow” and we pick the most convenient one.

$$\sum_{e\in\delta^-(v)} f(e) = \sum_{e\in\delta^+(v)} f(e)$$



Easy properties of flows

The following easy observation connects $\mathbb{Z}_2$-flows with cycles ($\neq$ circuits).

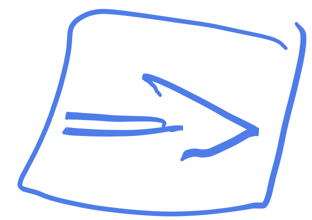
$\mathrm{Supp}(f) = \{e : f(e) \neq 0\}$

Observation 8 ($\mathbb{Z}_2$-flow)**.** *Let G be a graph and f any $\mathbb{Z}_2$-flow on G. Then the* support of f *(that is, the set of edges with nonzero value of f) is a cycle.*

In particular a graph has a NZ $\mathbb{Z}_2$-flow iff it is a cycle.

all edges $\mapsto 1$

Theorem 9 ($\mathbb{Z}_3$-flow of cubic graphs)**.** *Let G be a cubic (i.e., 3-regular) graph. Then G admits a NZ $\mathbb{Z}_3$-flow iff G is bipartite.*



$\{\cancel{0}, 1, 2\}$ in $\mathbb{Z}_3$

reverse all e s.t. $f(e) = 2$

Proof. If G is bipartite, we direct all edges from one part to the other and assign 1 to each edge, clearly this is the desired flow. On the other hand, . . . □

$\to \forall e\ f_F(e) = 1$, f_F is a flow in G_F

Theorem 10 ($\mathbb{Z}_2^2$-flow of cubic graphs)**.** *Let G be a cubic (i.e., 3-regular) graph. Then G admits a NZ $\mathbb{Z}_2^2$-flow iff G is edge 3-colorable.*

- As opposed to the previous two characterisations (being a cycle and being bipartite), the condition in this theorem is NP-complete to check.
- We will frequently meet graphs that are cubic and *fail to have edge 3-coloring* $\Rightarrow$ *snarks.*

Proof. • A NZ $\mathbb{Z}_2^2$-flow can use only values from $A = \{(0,1), (1,0), (1,1)\}$.

- As we are calculating modulo 2, we don't care about the orientation.
- It is easy to check that if three elements of A sum to zero, they must be in fact distinct.

∎

…

Corollary 11 (3-edge-coloring and bridges)**.** *Let G be a cubic graph with at least one bridge. Then*

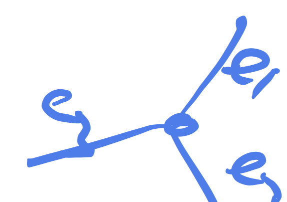
$\boxed{\Leftarrow}$

Is G bipartite?

— polynomial algo (linear time in $|V|+|E|$)

$\boxed{\Rightarrow}$:

If G has NZ $\mathbb{Z}_2^2$-flow then $\forall v$



$f(e_1), f(e_2), f(e_3)$ are all three elts of A

G is not edge-3-colorable.

In analogy with the chromatic number $\chi(G)$ we define the flow number of a graph G to be

$$\varphi(G) = \inf\{|\Gamma| : G \text{ has a NZ } \Gamma\text{-flow}\}\,;$$

$\inf(\emptyset) = \infty$

- $\varphi(G)$ is defined (as ∞) if G has no NZ flow.
- This happens iff G has a bridge.
- (In analogy: what graphs have no proper coloring?)
- Monotonicity: compare with χ.

(G has a bridge

easy $\Downarrow$ $\Uparrow$ true, not too hard

G has no NZ flow

G has a k-coloring

$\Downarrow$ obvious

G has a $(k+1)$-coloring

loops !

G has NZ $\mathbb{Z}_k$-flow $\overset{?}{\Longrightarrow}$ G has NZ $\mathbb{Z}_{k+1}$-flow

Definition 12. *Let G be a digraph, f a $\mathbb{Z}$-flow on G.*
f is a k-flow *if $|f(e)| < k$ $(\forall e)$.*
f is a nowhere-zero *k*-flow *if $0 < |f(e)| < k$ $(\forall e)$.*

k-NZF := nowhere-zero k-flow
Γ-NZF := nowhere-zero Γ-flow

Note: Many authors use k-flow to mean NZ k-flow.

Theorem 13 (Tutte)**.** *A graph has a k-NZF iff it has $\mathbb{Z}_k$-NZF.*

Motivated by this result we will sometimes use *k-flow* to mean Γ-flow for any Γ of size k.

Assume

$0 < |f(e)| < k$ $\Rightarrow$ Take $g(e) = f(e) \bmod k$

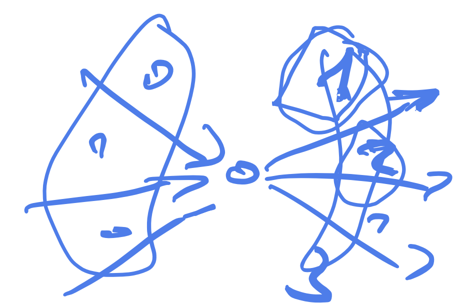
$f(e) \in \mathbb{Z}$

g is a NZF

$f(e) \in \mathbb{Z}_{\cancel{k}} \setminus \{0\}$

Observation: G has NZ $\mathbb{Z}_k$-flow $\Longleftrightarrow$ G has NZ Γ-flow ($|\Gamma| = k$)

($\mathbb{Z}_4$-NZF $\Longleftrightarrow$ $\mathbb{Z}_2^2$-NZF)



$\mathbb{Z}_q$

Corollary 14 (group-monotonicity). *Let Γ_1, Γ_2 be groups, with $|\Gamma_1| < |\Gamma_2|$. Then any graph with Γ_1-NZF has also a Γ_2-NZF.*

$$\begin{array}{ccc}
G \text{ has } \Gamma_1\text{-NZF} & & \Gamma_2\text{-NZF} \\
\Updownarrow & & \Updownarrow \\
G \text{ has } \mathbb{Z}_{|\Gamma_1|}\text{-NZF} & & \mathbb{Z}_{|\Gamma_2|}\text{-NZF} \\
\Updownarrow \text{ Thm 13} & & \Updownarrow \\
G \text{ has } |\Gamma_1|\text{-NZF} & \Longrightarrow & G \text{ has } |\Gamma_2|\text{-NZF}
\end{array}$$

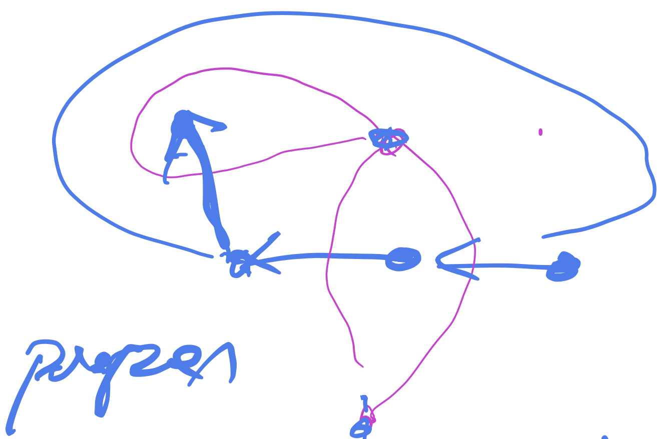
NZ flows in planar graphs

A general way to construct NZ flows originates from colorings and planar duality. We now present just a sample to show one of the early motivations for the study of NZ flows.

Let G be a planar digraph, consider a proper coloring of faces of G by elements of some group Γ – so that faces sharing an edge get distinct colors. Now for an edge e let $f(e)$ be the difference of the left face's value and the right face's value. It's easy to check that f is a NZ Γ-flow.

G^* ... dual graph

may have parallel edges, loops



If c is proper coloring

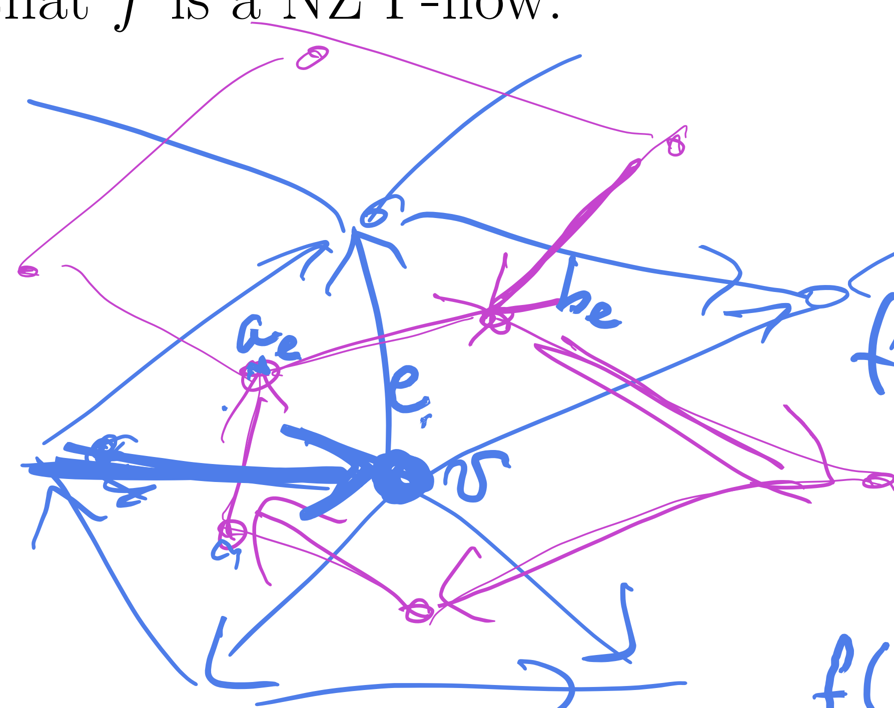
$c : V(G^*) = F(G) \longrightarrow \Gamma$

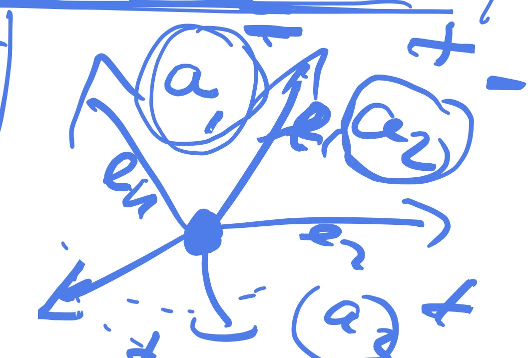
$\Downarrow$

f is NZ

$f(e) = \boxed{c(b_e) - c(a_e)}$

$\boxed{f \text{ is a flow}}$

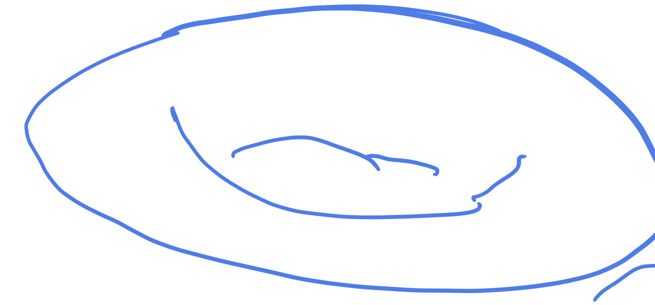




$f(e_1) + f(e_2) + \dots + f(e_k) = 0$

$(c(a_2) - c(a_1)) + (c(a_3) - c(a_2)) + \dots$

- It works for graphs drawn on arbitrary orientable surface.
- For planar graphs all NZ flows arise in this way,
- thus $\varphi(G) = \chi(G^*)$. (Proof later.)
- $\varphi(G) \leq 4$ whenever G is planar. & bridgeless
- OTOH $\varphi(\mathrm{Pt}) = 5$ (where Pt is the Petersen graph).
- It is open, whether $\varphi(G) > 5$ is possible.

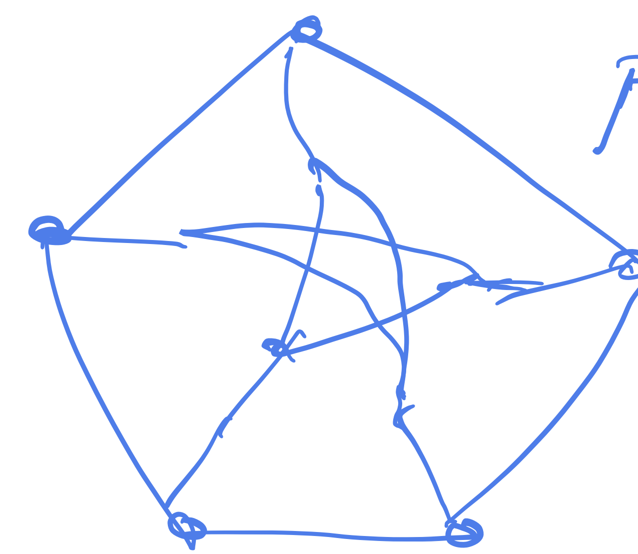


$\mathbb{Z}_k$-flows

(& # of flows in G = # color. of G^* / k)

($\varphi(G) \leq 6$ if G is bridgeless)

(G^* is planar & loop-less)



Pt

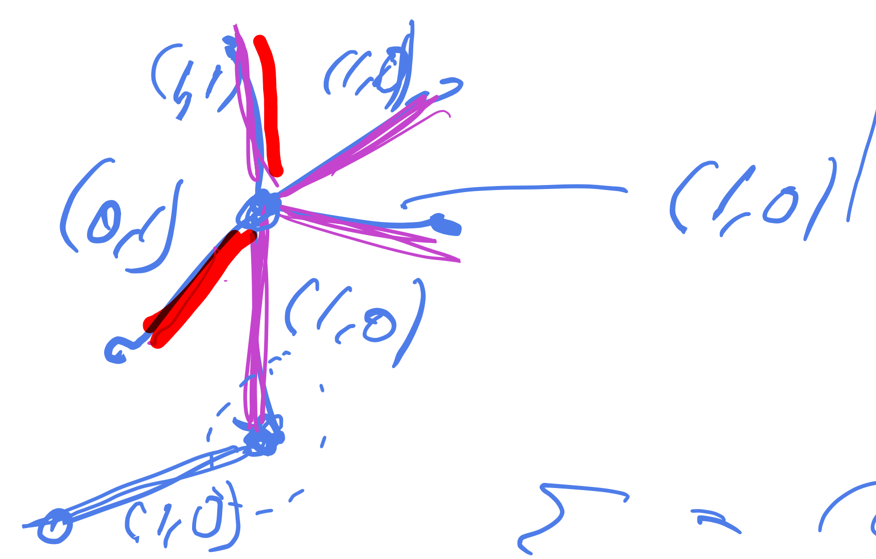
More basic properties

Theorem 15 (Jaeger). *The following are equivalent for any graph G*

1. G has a $\mathbb{Z}_2^2$*-NZF*

2. E(G) is a union of two cycles

Proof. Let f be a NZ $\mathbb{Z}_2^2$-flow on G, observe it only uses values $(0,1)$, $(1,0)$, $(1,1)$. …



$\Sigma = (0,0)$

$C_1 := \{e : \varphi(e)_1 = 1\}$

$C_2 := \{e : \varphi(e)_2 = 1\}$ cycles

$C_1 \cup C_2 = \{e : \varphi(e) \neq (0,0)\} = E(G)$

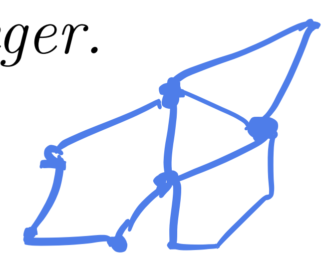
In the other direction: let $E(G) = E_1 \cup E_2$ and each E_i is a cycle. We take a $\mathbb{Z}_2$-flow f_i that is 1 precisely on E_i. Putting $f = (f_1, f_2)$ we get the desired flow.

An alternative proof: consider (integer) 2-flows g_i on E_i. Then $g = 2g_1 + g_2$ is a NZ 4-flow. □

Theorem 16 (Tutte). *Let $k \geq 2$ be an integer. A graph has a k-NZF iff it has* $\mathbb{Z}_k$*-NZF.*

2-elementary group

circuit



cycle = all degrees even

Proof. The forward implication is obvious. For the other one, let g be a $\mathbb{Z}_k$-NZF in a graph G. For any mapping $f : E(G) \to \mathbb{Z}$ we let $f(v)$ be the net flow out of a vertex v, that is $f(v) = \sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e)$. Recall that f is a flow iff $f(v) = 0$ for every vertex v. We won't achieve this directly, however, but by certain optimization.

∂f

• Start with $f(e) = g(e)$ the

$(\mathbb{Z}_k \leftrightarrow \{0, 1, \dots, k-1\})$

Let $f : E(G) \to \mathbb{Z}$ be such that

1. $f(e) \equiv g(e) \pmod{k}$ for each edge e,
2. $|f(e)| < k$ for each edge e, and
3. subject to the above, $\sum_{v\in V(G)} |f(v)|$ is as small as possible.

feasible solution

$|\partial f(v)|$

← optimize this

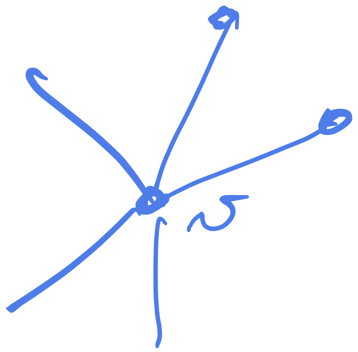
∃ feasible solution

(If the sum in part 3. is zero, then f is a flow and we are done.)

By possibly reorienting the edges of G we may assume that $f(e) > 0$ for each edge e.

• Let f be the optimum

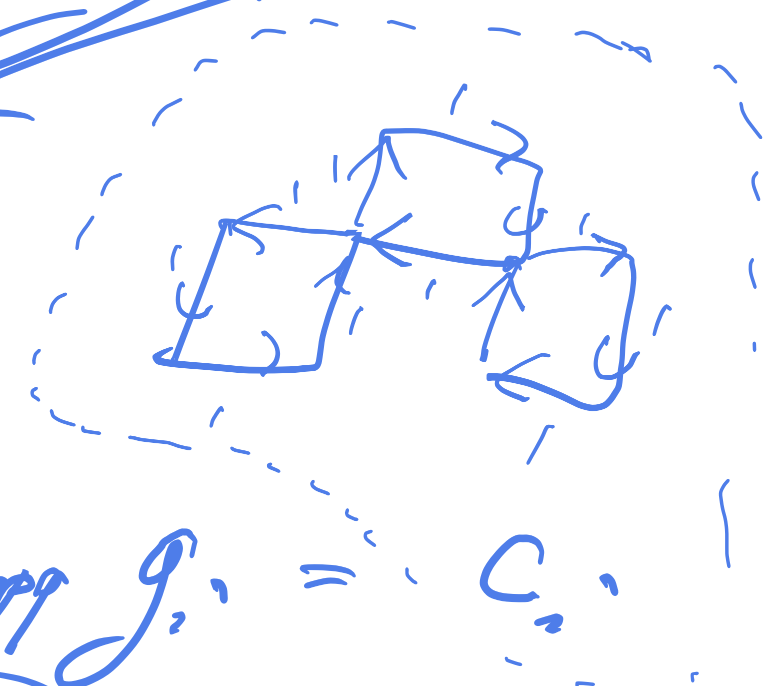
$$\partial f(v) = \sum_{e \text{ out}} f(e) - \sum_{e \text{ in}} f(e)$$



$$\sum_{e \ni v} \varphi(e) = (0,0) \leadsto \sum \varphi(e)_1 = 0 \text{ in } \mathbb{Z}_2$$

$\deg_{C_1} v$ = # edges around v s.t. $\varphi(e)_1 = 1$

= this is even



C_1, C_2

g_1 is 2-flow on G s.t. $\underbrace{\text{supp}\, g_1}_{\{e \mid g_1(e) \neq 0\}} = C_1$.

$\{0, \pm 1\}$ in $\mathbb{Z}$

We know:
$\forall e$ $g_1(e) \neq 0$
or $g_2(e) \neq 0$

$$g = \underbrace{2g_1}_{\{0,\pm 2\}} + \underbrace{g_2}_{\{0,\pm 1\}}$$

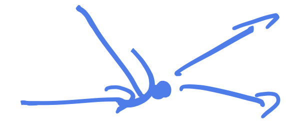
- $f(v) \equiv 0 \pmod{k} \qquad \forall v$
- $V^+ := \{v : f(v) > 0\}$
- $V^0 := \{v : f(v) = 0\}$
- $V^- := \{v : f(v) < 0\}$
- If $V^0 = V$ we are done.
- Otherwise, observe that both V^+ and V^- are nonempty and pick $a \in V^+$, $b \in V^-$.
- Either there is a directed $a - b$ path or there is a set A containing a but not b such that no directed edge leaves A.
- The second possibility immediately yields a contradiction:

$$\boxed{\sum_{v \in A} f(v)} = -\sum_{e \in \delta^-(A)} \overbrace{f(e)}^{\to 0} < 0$$

$a \in A := \{v \in V(G) : \exists \text{ dir. path } a \to v\}$

$f(a) > 0$

$\exists\, b' \in V^- \cap A$

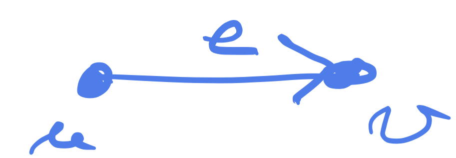


$f(v) = \sum_{e \in \delta^+} f(e) - \sum_{e \in \delta^-} f(e)$

g is a flow

$\overset{\text{mod } k}{\equiv} \sum_{\delta^+} g(e) - \sum_{\delta^-} g(e) = 0$

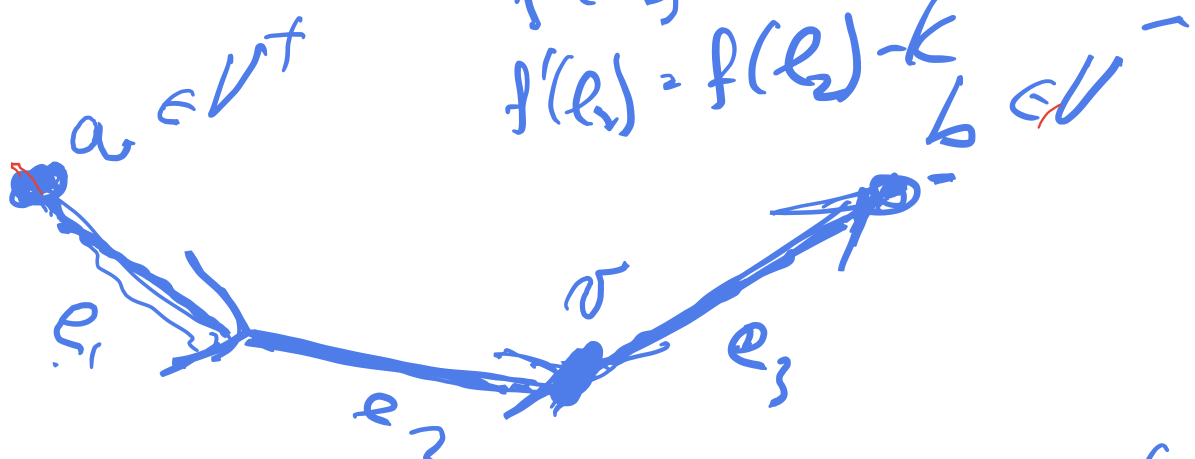
$\sum_{v \in V} f(v) = 0$



$\exists$ dir. path $a \to b'$

- So there is a directed $a - b$ path P with $a \in V^+$, $b \in V^-$.
- We define a mapping f' by letting $f'(e) = f(e) - k$ for $e \in E(P)$, and $f'(e) = f(e)$ otherwise.

$\square$



for v inside P & for v out of P

$f'(v) = f(v)$

$f'(a) = f(a) - k$

$f(a) > 0$

$\Rightarrow f(a) \geq k \Rightarrow |f'(a)| < |f(a)|$

$f'(b) = f(b) + k$

≤ 0

$f(b) \leq -k$

$|f'(b)| < |f(b)|$

$f(e) \in \{1, \ldots, k-1\}$

$f'(e) \in \{-k+1, -k+2, \ldots, -1\}$

$f(v) = f(e_3) - f(e_2)$

f' is better than f

$f'(e_i) = f(e_i) - k$

for e_i on P

$f' \equiv f \mod k$

$\| \; g$

- The *existence* of a k-NZF and $\mathbb{Z}_k$-NZF are equivalent,
- but the *numbers* of them not (in general)
- However, the number of k-NZF's of a given graph is also a polynomial in k.
- (Proof using Ehrhart method).

$\{\pm 1, \pm 2, \ldots, \pm(k-1)\}$-flows

$P \subseteq \mathbb{R}^{E(G)}$

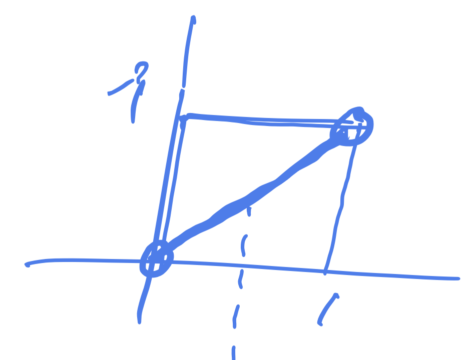
$P = \{ f : \boxed{f \text{ is a flow}}, \ \forall e : 0 \le f(e) \le 1 \}$

$\forall v \quad \sum \ldots = \sum \ldots$

(polytope)

kP° ..... flows with values in $(0,k)$

$\underbrace{|kP^{\circ} \cap \mathbb{Z}^{E(G)}|}_{\substack{k\text{-NZF} \\ \text{with pos. values}}} \to$ flows is a polynomial in k



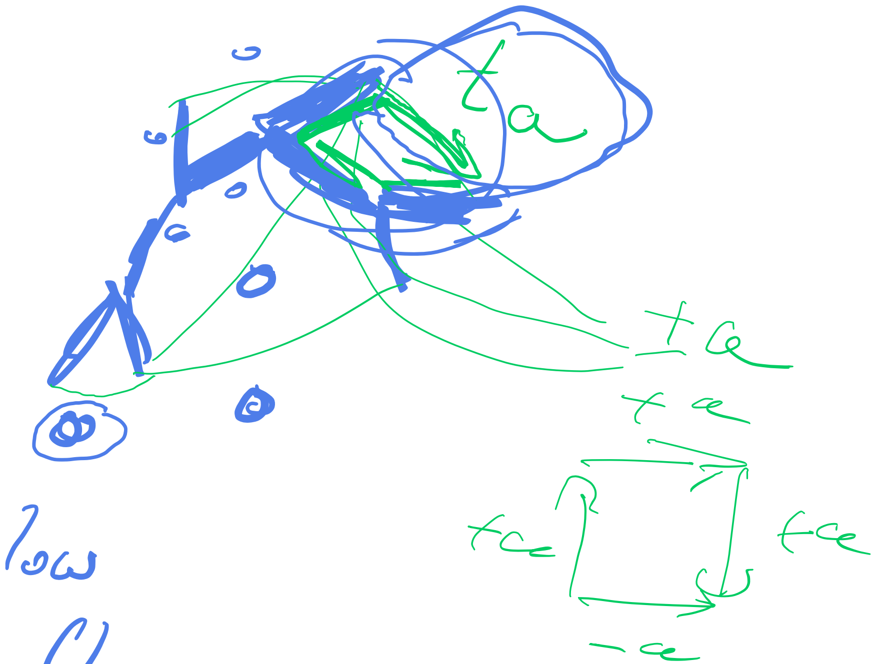
P° ... interior of P

... $0 < f(e) < 1$

Flows and spanning trees – sum

Let T be a spanning tree of G. Now for every edge $t \in E(G) \setminus E(T)$ and every $a \in \Gamma$ we let $\varphi_{t,a}$ be the (unique) flow in G such that

- $\varphi_{t,a}(t) = a$
- $\varphi_{t,a}(e) = 0$ for $e \neq t$ and $e \in E(G) \setminus E(T)$



... *elementary flow with respect to T.*

a Γ-flow

$\vec{0}$: zero-flow

- $\mathcal{F}_\Gamma(G) :=$ the vector space of all flows
- (we need Γ to be a field).

vector space over F-field

- For any fixed spanning tree T the elementary flows $\{\varphi_{t,1} : t \in E(G) \setminus E(T)\}$ form a basis of $\mathcal{F}_\Gamma(G)$.
- Any mapping $\varphi : E(G) \setminus E(T) \to \Gamma$ can be uniquely extended to a Γ-flow on G.
- No control over the edges of T, thus we can't use this easily to construct a NZ flow.

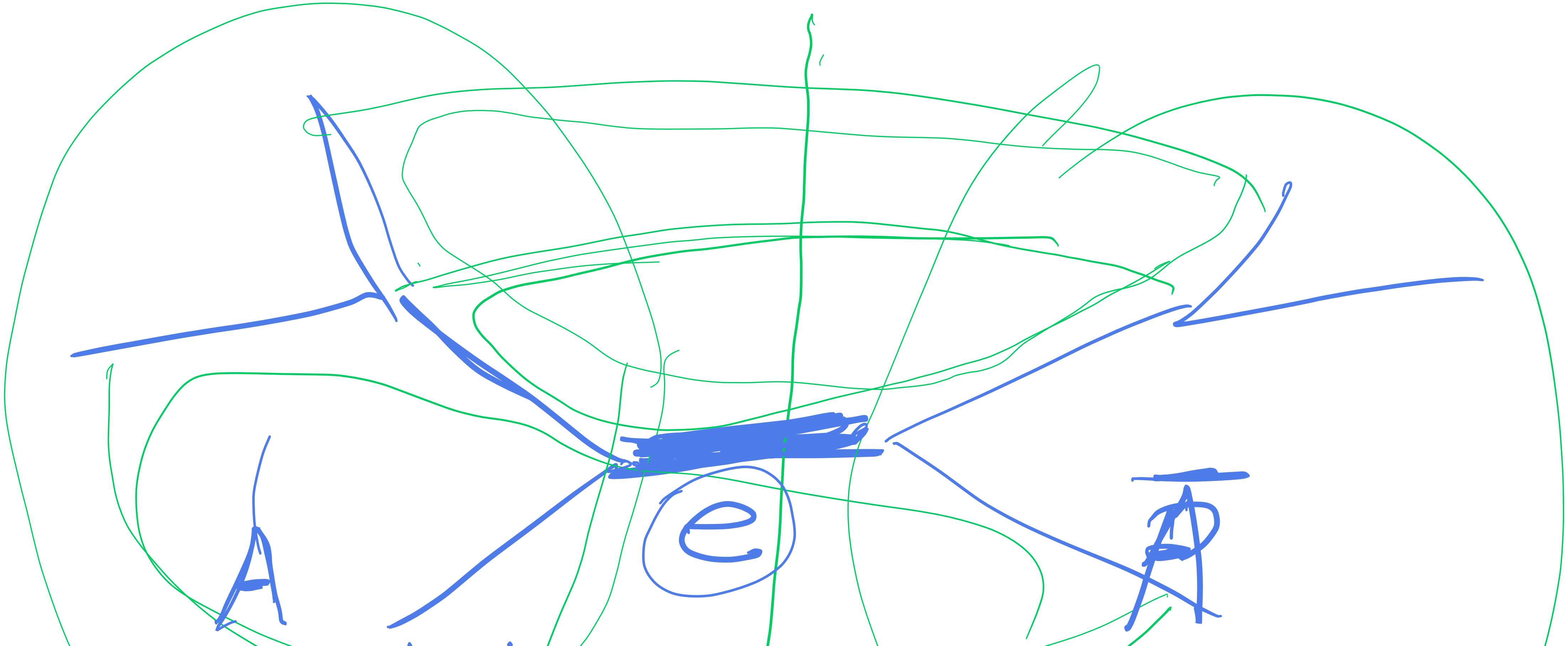
$f \in \mathcal{F}(G)$ what?

$f = \sum_{t \in E(G) \setminus E(T)} \varphi_{t,1} \, f(t)$

$t_0 \in E(G) \setminus E(T)$

$f(t_0) = 1 . f(t_0)$

only $t = t_0$ has NZ value of $\varphi_{t,1}(t_0)$



$\sum h(A,\bar{A}) - h(\bar{A},A) = 0$

$f(e) = g(e)$ $\forall e \in E(T)$ & f, g are flows

$h = f - g$: flow s.t. $\forall e \in E(T)$ $h(e) = 0$

Flows and spanning trees – product

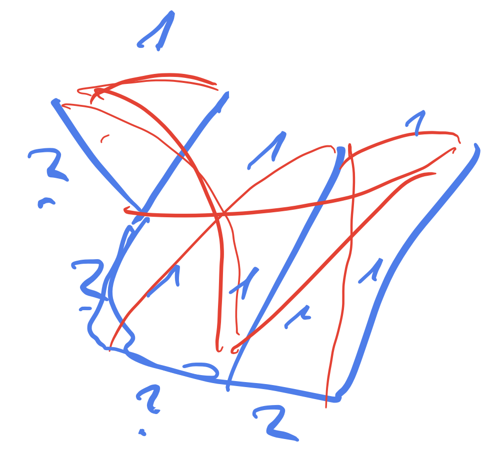
Theorem 17. *Any 4-edge connected graph admits a $\mathbb{Z}_2^2$-NZF.*

Proof. If G is 4-edge connected, then there are two disjoint spanning trees, T_1 and T_2 (proof later).

Let f_i be the $\mathbb{Z}_2$-flow on G that equals 1 on all edges not in T_i. (Such flow exists — see above.)

Now put $f = (f_1, f_2)$. This is indeed a $\mathbb{Z}_2^2$-flow, and if $f(e) = 0 = (0,0)$ for some edge e then e lies in both T_1 and T_2, a contradiction. $\square$

$G - X$ is connected if $X \subseteq E(G)$, $|X| < 4$



T_1

T_2

$f = (f_1, f_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$

	$E(T_1)$	$E(T_2)$	rest	
f_1	?	$+1$	$+1$	$\mathbb{Z}_2$
f_2	$+1$	?	$+1$	$\mathbb{Z}_2$
	$\neq (0,0)$	$\neq (0,0)$	$\neq (0,0)$	

Theorem 18 (Jaeger). *Any bridgeless graph admits a $\mathbb{Z}_2^3$-NZF.*

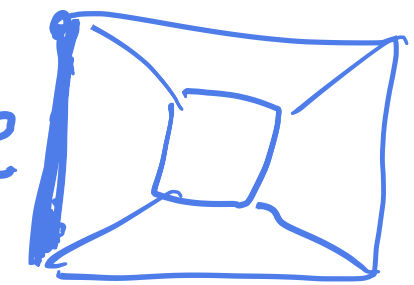
($\iff \mathbb{Z}_8$-NZF $\iff$ 8-NZF)

Proof. Suppose first that G is 3-edge connected, we will use spanning trees similarly as in the construction of a NZ 4-flow.

We let G' be the (multi)graph obtained from G by adding to each edge a new one, parallel to it.

G' is 6-edge connected ...

e



(no 2 EDST exist)

8 vert.

7 edges

7+7 > 12 edges)

$\exists$ 3 EDST T_1, T_2, T_3

101 010

	$E(T_1)$	$E(T_2)$	$E(T_3)$	Rest
f_1	?	+1	+1	+1
f_2	+1	?	+1	+1
f_3	+1	+1	?	+1

e_1 e_2

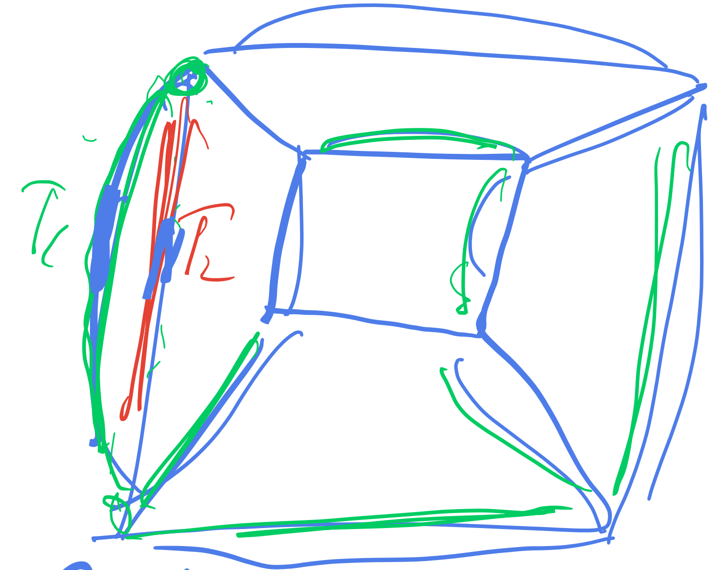
e_1 e_2

WARN: NZF on G not G'

$f(e) := f'(e_1) + f'(e_2)$

$f' = (f_1, f_2, f_3)$ is a NZF in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in G'

T_1', T_2', T_3' --- EDST in G'



$\underline{T_1, T_2, T_3}$ trees in G

s.t. $\forall e$ of G is in ≤ 2 of them

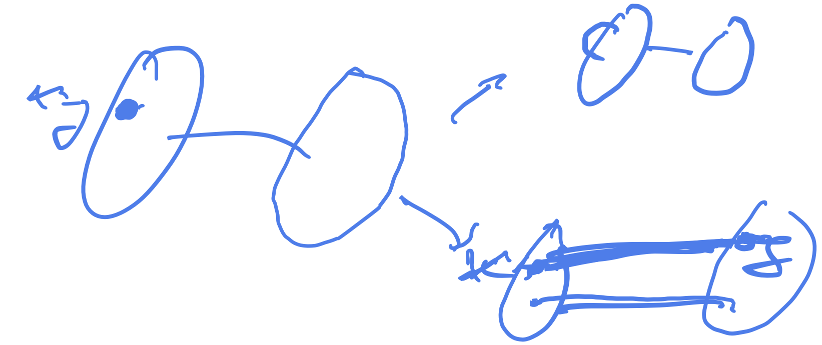
$$\left.\begin{aligned} f_i(e) &= +1 \text{ on } E(G) - E(T_i) \\ &= ? \text{ on } E(T_i) \end{aligned}\right\} \mathbb{Z}_3\text{-flow}$$

$f = (f_1, f_2, f_3)$ --- $\mathbb{Z}_3^3$-flow $\longrightarrow$ a NZF in G

if $\underline{f(e) = (0,0,0)}$ then $e \in E(T_1) \cap E(T_2) \cap E(T_3)$ ↯

So the theorem holds for all 3-edge-connected graphs. To prove it for all bridgeless graphs, suppose there is a counterexample and choose one with minimal number of edges, let it be denoted G. …

□



if G is 3-e-c --- we know G has NZF in $\mathbb{Z}_2^3$.

if G has a bridge ---- nothing claimed

?

$G' = G/e'$ — smaller & bridgeless

$\Longrightarrow$ G' has $\mathbb{Z}_2^3$-NZF f

use f for G, extended by

$f(e') = f(e)$ This is a flow

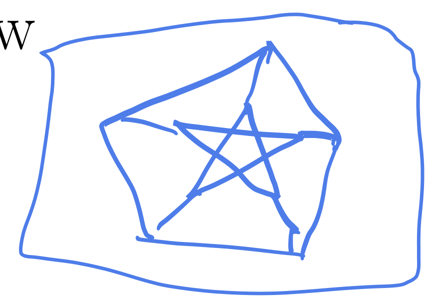
Small flows – for bridgeless graphs

(4-e.c. ⇒ 4-flow)

3-flow

- 1-flow: impossible
- 2-flow: exists precisely in cycles
- 3-flow: for cubic graphs exists precisely in bipartite graphs
- 3-flow should exist in every 4-edge-connected graph by a **conjecture of Tutte**, 1966. It exists in every 6-edge-connected graph. It suffices to prove it for 5-edge-connected graph.
- 4-flow for a cubic graph is the same as 3-edge-colorability. By a **conjecture of Tutte**, every bridgeless graph that does not have Petersen graph as a minor admits a 4-flow. Proved for cubic graphs by Robinson, Seymour and Thomas (unpublished) by reducing to four-color theorem.
- 4-edge-connected graph has a 4-flow

(0,1), (1,0), (1,1)

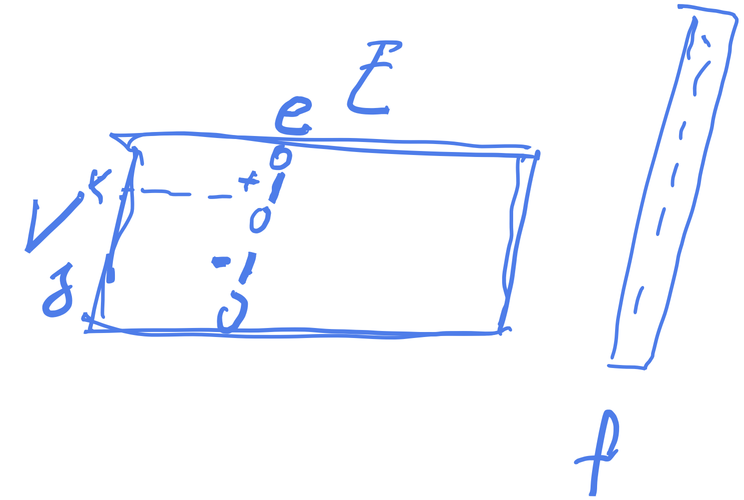


- **Conj. [Tutte 1954]** 5-flow exists in every bridgeless graph
- 6-flow exists in every graph [Seymour 1981]
- 8-flow exists in every graph [Jaeger]
- In particular $\varphi(G) \leq 6$ for each bridgeless graph G.

Flows and tensions: a linear-algebra point of view

Let $G = (V, E)$ be a digraph. The *incidence matrix* of G is $B = (B_{v,e})_{v \in V, e \in E}$ defined by $B_{v,e} = +1$ if e starts at v, $B_{v,e} = -1$ if e ends at v, and $B_{v,e} = 0$ otherwise.

- Γ^E – set of mappings / vector space
- for every $f \in \Gamma^E$ the product $\boxed{Bf}$ has the v-coordinate equal to $f^+(v) - f^-(v)$. $=: f(v)$
- flows $= \ker B$



$e = x \longrightarrow y$

Tensions

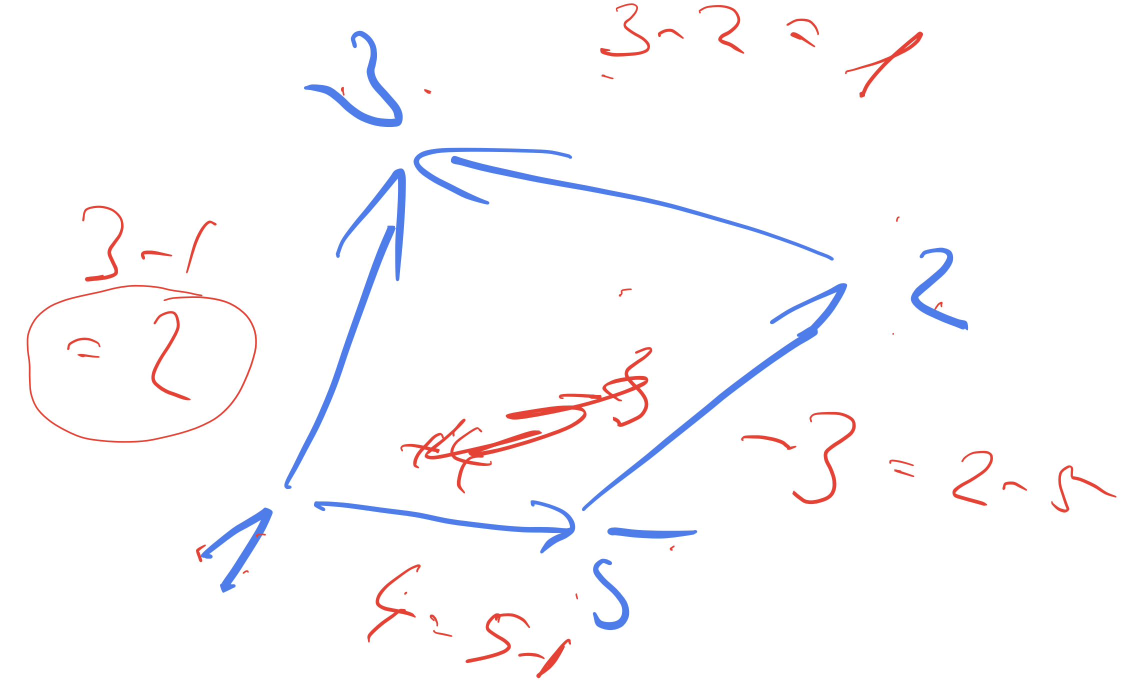
Let $t : E \to \Gamma$ be a mapping. We say that t is a *tension* whenever

$$\sum_{e \in C^+} t(e) = \sum_{e \in C^-} t(e) \qquad (1)$$

holds for every circuit C with C^+ and C^- being the edges oriented forward and backward, respectively, along C.

- "does not depend on orientation" (if I change sign of $t(e)$ appropriately)
- For any $F \subseteq E$, t is a tension in G whenever t_F is a tension in G_F.
- an easy way to get tensions: *potential difference.*
- For $p : V \to \Gamma$ we define its difference $\delta p : E \to \Gamma$ by letting $(\delta p)(u, v) = p(v) - p(u)$.
- δp is a tension for every p.

& there are no other examples



$+5 - 3 + 1 = 2$

- $\langle f, g\rangle := \sum_{e\in E} f(e)g(e)$.
- not in general an inner product! (Why?)
- Still, many notions from linear algebra generalize easily. In particular, t is orthogonal to all elements of a vector space, whenever it is orthogonal to all elements of a generating set.
- Equation (1) $\iff$ $\langle t, c\rangle = 0$ for a particularly simple flow c: one that is zero outside of a circuit C and has values ± 1 on C.
- We already know that circuits generate $\mathcal{F}$ the space of all Γ-flows on G. (Why?)
- Thus, t is an element of $\mathcal{F}^{\perp}$.

"lin. in f"
"lin. in g"
$\langle f, f\rangle \geq 0$ & $\langle f, f\rangle \neq 0$ unless $f = 0$?
false

$\langle t, x\rangle = 0$ $\forall x \in V = \langle G\rangle$
$\Updownarrow$
$\langle t, x\rangle = 0$ $\forall x \in G$

$\varphi_{t,l}$ ... elem. flows

(1) $\iff$ $\forall$ circ. c $\langle t, c\rangle = 0$

$\Rightarrow$ $\forall$ flow φ $\langle t, \varphi\rangle = 0$

Theorem 19. *Let $\mathcal{F}$, $\mathcal{T}$ be the vector spaces (or modules) of all flows and all tensions, respectively, defined on a digraph G. Then*

$$\mathcal{F}^\perp = \mathcal{T} \qquad \text{and} \qquad \mathcal{T}^\perp = \mathcal{F}\,.$$

Moreover, $\mathcal{F} = \ker B$ and $\mathcal{T}$ is the row space of B, the icidence matrix of G.

Consequently

> for every tension t there is a potential p such that $t = \delta p$.

$p : V \to \Gamma$

$p^T B$

?

Indeed, δp can be expressed as $B^T p$. And, obviously, $B^T p$ is a general form of a linear combination of rows of B.

the def-

$X^\perp = \{ f : \forall x \in X \ \langle f, x \rangle = 0 \}$

we know: $\forall \varphi$ flow $\forall t$ tension $\langle t, \varphi \rangle = 0$

$\Downarrow$

$\mathcal{F}^\perp \supseteq \mathcal{T}$, $\mathcal{T}^\perp \supseteq \mathcal{F}$

$x \in \mathcal{F}^\perp$

$\Downarrow$

$\forall c$ $\langle x, c \rangle = 0$

$\Downarrow$

x is a tension

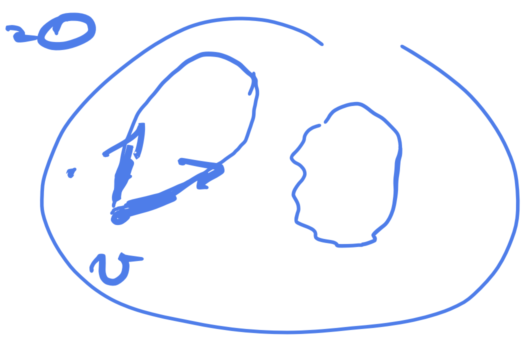
$y \in \mathcal{T}^\perp$

$\Downarrow$

$\forall v$

$\Downarrow$

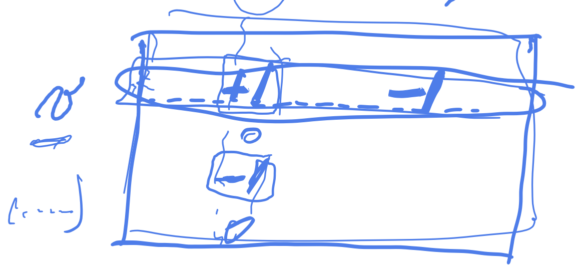
$\langle y, \delta \rangle = 0$



$y^+(v) - y^-(v)$

$\longrightarrow y \in \mathcal{F}$

e out of v, in v



t_v

row space $= \langle \text{rows} \rangle \in \mathcal{T}$

$\mathcal{F} = \{\varphi \in \Gamma^E : \varphi \text{ is a flow}\} = \{\varphi : \langle \varphi, \tau_v \rangle = 0 \ \forall v\}$ $\quad \Gamma = \mathbb{Z}_p$

$\sum_e \varphi(e)\, \tau_v(e)$

$\mathcal{T} = \{\tau : \tau \text{ is a tension}\}$

$\hat{=} \quad \tau_v \in \mathcal{T}, \quad \varphi_C \in \mathcal{F}$

$= \{\tau : \langle \tau, \varphi_C \rangle = 0 \ \forall \text{ cycle } C\}$

± 1 around C

$\mathcal{T} \supseteq \text{span}\{\tau_v : v \in V\}$

φ

$\mathcal{F}^\perp = \mathcal{T} \quad \& \quad \mathcal{T}^\perp = \mathcal{F}$

$\mathcal{F}^\perp = \{\tau : \langle \tau, \varphi \rangle = 0 \ \forall \varphi \in \mathcal{F}\} \subseteq \mathcal{T} \subseteq \mathcal{F}^{\perp\perp}$

$\mathcal{T}^\perp = \{\varphi : \langle \varphi, \tau \rangle = 0 \ \forall \tau \in \mathcal{T}\} \subseteq \mathcal{F} \subseteq \mathcal{T}^{\perp}$

$\mathcal{F} = \text{span}\{\varphi_C\} \Rightarrow \forall \tau \forall \varphi \quad \langle \tau, \varphi \rangle = 0$

$\tau_v = \delta p_v$

$p_v(u) = 1 \quad u \neq v$

$0 \quad u = v$

$\boxed{p(v) - p(u)}$

$u \longrightarrow v$

Claim

$$\mathcal{J} \supseteq \mathrm{span}(\{\tau_v\})$$

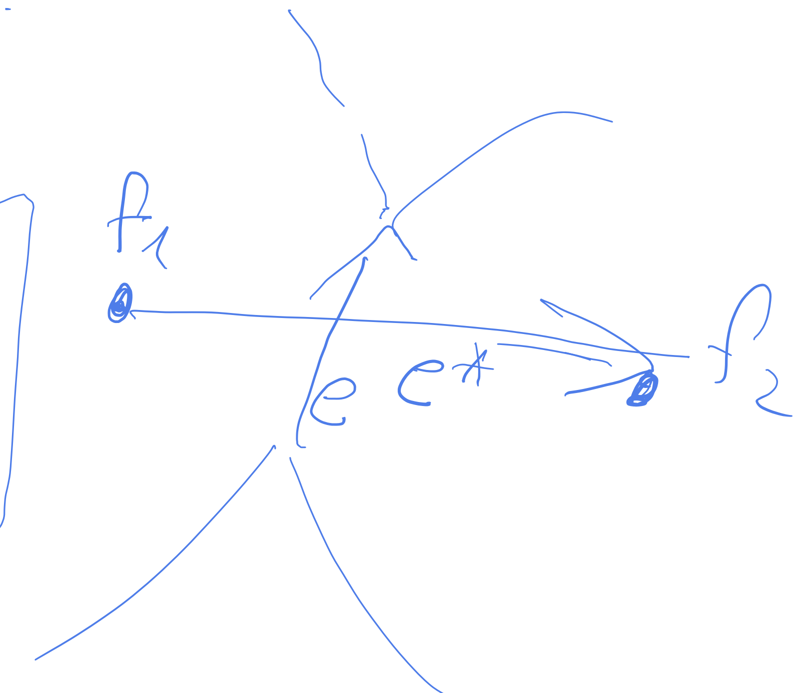
$$\mathcal{F}^{\perp}$$

$$\mathcal{J} \subseteq \mathrm{span}(\{\tau_v\}) \quad ?$$

$$\forall \tau \in \mathcal{J} \quad \exists \lambda_v : \quad \sum \lambda_v \tau_v = \tau$$

Flows and tensions for a planar graph

- G: a plane digraph (planar digraph with a fixed drawing in the plane).
- *dual graph* G^* is a digraph with vertices being the faces of G and with edges corresponding to edges of G: the edge e^* connects the face on the left of e to the face on the right of e.
- $(G^*)^* = ?$



Theorem 20. *The following is equivalent for a plane graph G:*

1. *G has a NZ k-flow* ?
2. *G^* has a NZ k-tension*
3. *G has a proper face-coloring by k colors.*

$\boxed{2 \Rightarrow 3}$ $\forall t\ \exists p$:

$t = \delta p$

$\boxed{2 \Leftarrow\Rightarrow 3}$

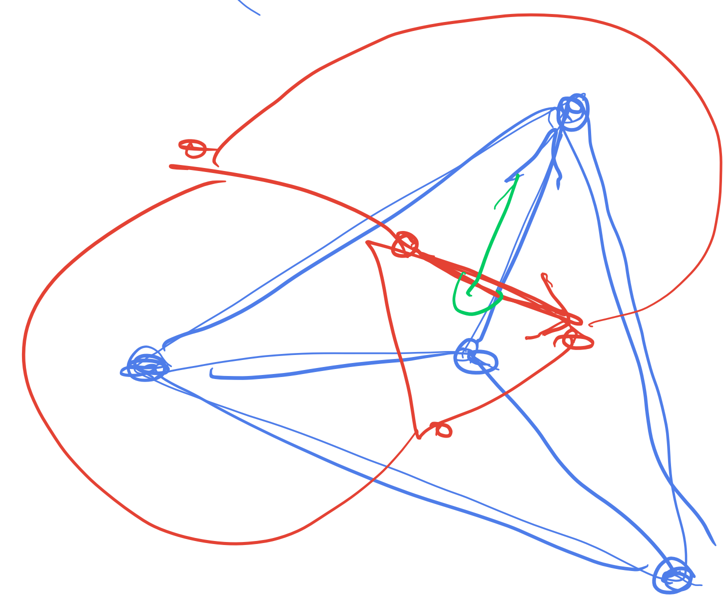
$\exists$ proper vertex-col. of G^*

$p : V(G^*) \to \mathbb{Z}_k$

$\delta p(uv) = p(v) - p(u) \neq 0$

$\forall u \to v \quad p(v) \neq p(u)$

G^*



G

$(G^*)^* = \cancel{G?}\ G^{op}$

$1 \Longleftrightarrow 2$

$1 \Rightarrow 2$

G has φ

Mapping $t(e^*) = \varphi(e)$

$\forall$ cycle C .... that is a facial walk

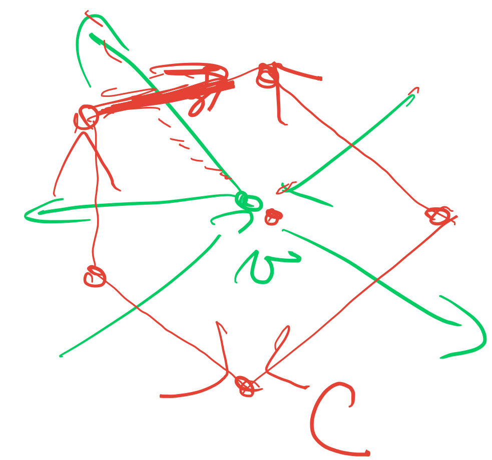
I have $\sum_{e^*} t(e) = \sum_{C} t(e)$

$2 \Rightarrow 1$

G^* has a tension t

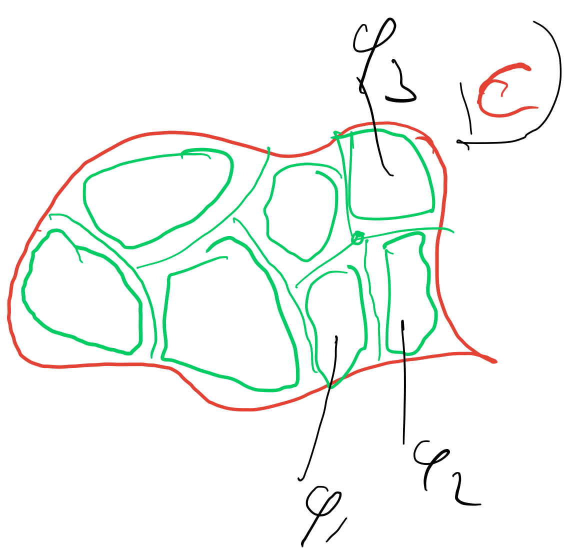
$\varphi(e) = t(e^*)$ ... φ is a flow

$\forall v$ $\varphi^+(v) = \varphi^-(v)$



this translates to

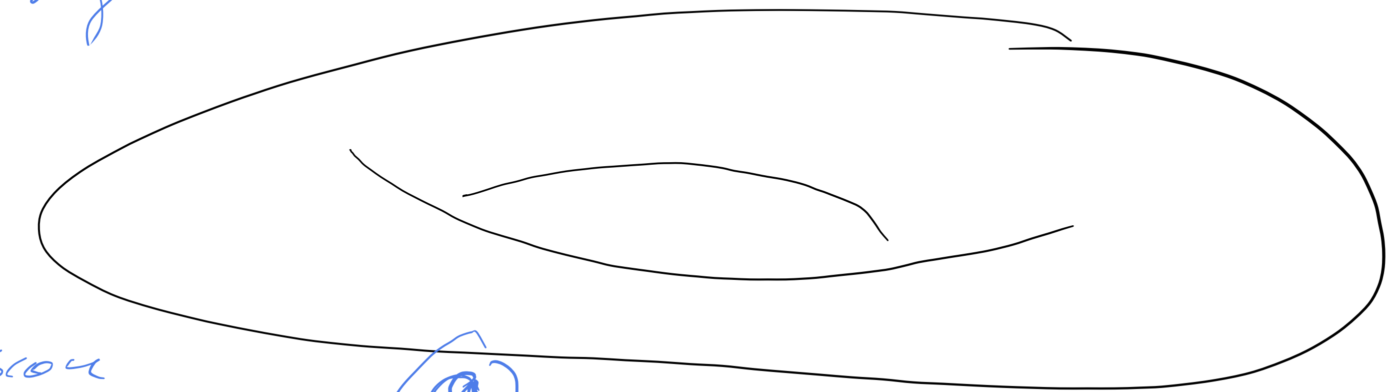
$\sum_{C^+} \varphi = \sum_{C^-} \varphi$



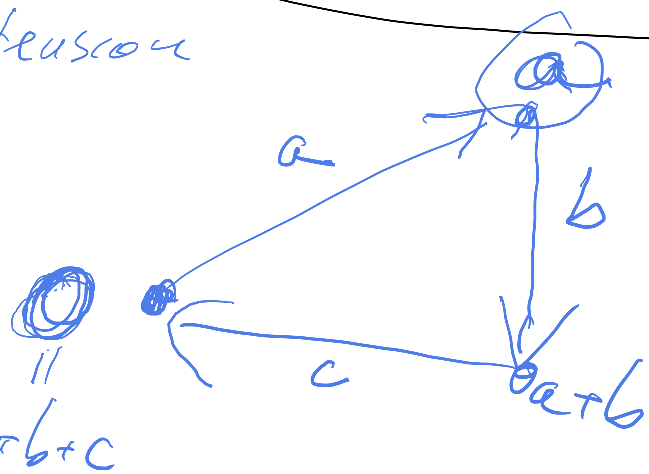
$\varphi_C = \sum_{i=1}^{k} \varphi_i$ $\langle t, \varphi_i \rangle$ $\forall i$ (Because φ_i is facial) $\Rightarrow$ $\langle t, \varphi_C \rangle = 0$

φ_i .... elem. cycles corr. to facial walk for faces inside C

coloring



tension



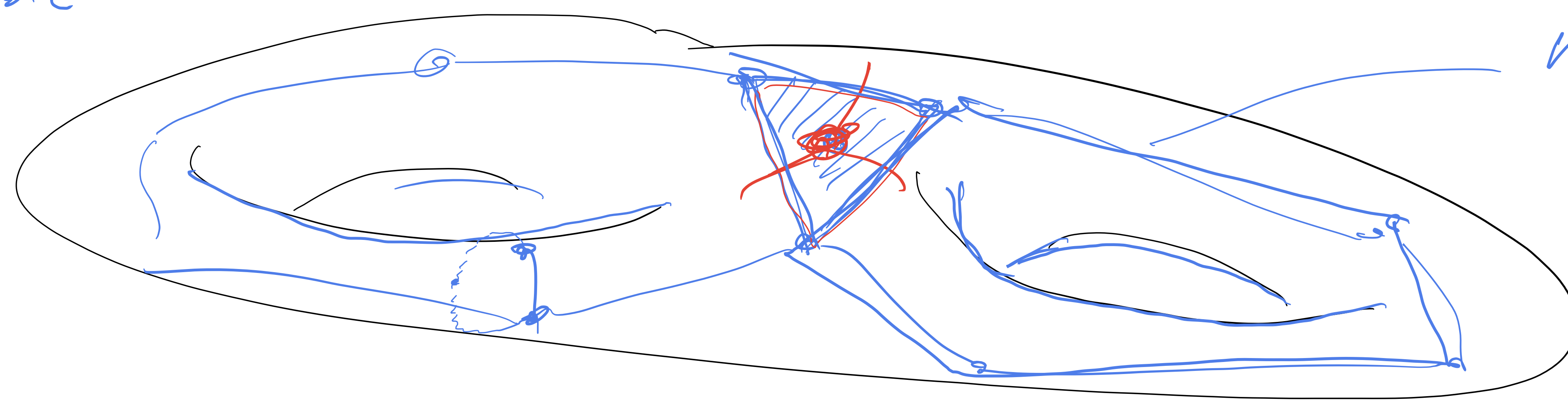
$a+b+c=0$

t is a local tension

$\forall$ contractible cycle C

$$\sum_{C^+} t(e) = \sum_{C^-} t(e)$$

flow in the dual



non contractible curves

Further abstraction possible

- A mapping $f : E(G) \to E(H)$ is Γ-*flow continuous* iff for every Γ-flow on H, $\varphi \circ f$ is a Γ-flow on G.
- A mapping $f : E(G) \to E(H)$ is Γ-*tension continuous* iff for every Γ-tension on H, $\varphi \circ f$ is a Γ-tension on G.
- Motivation: can help to solve flow-related problems:

Theorem 21. *If $g : V(G) \to V(H)$ is a graph homomorphism that the induced mapping on edges $((u, v) \mapsto (g(u), g(v)))$ is Γ-tension continuous for every Γ*

Def. $E(G) \xrightarrow{f} E(H) \xrightarrow{\varphi} \Gamma$

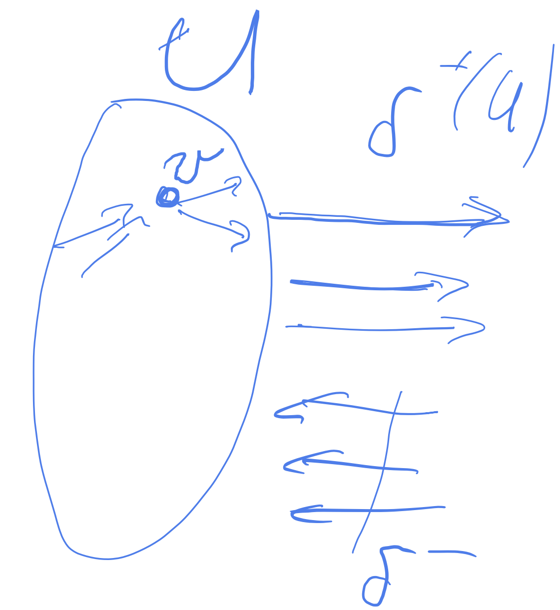
IF SUCH MAPPING EXIST AND H has some NZ Γ-flow THE G —/—

Conj: $\forall$ bridgeless G has $\mathbb{Z}_2$-flow cont. mapping to Petersen $\longrightarrow$ CDC Conj.

An equivalent formulation of NZ flows

Theorem 22 (Hoffman's Circulation Theorem)**.** *Let G be a digraph, let $0 < a \le b$ be integers. Then the following are equivalent.*

1. *There is a $\mathbb{Z}$-flow f on G such that $a \le f(e) \le b$ for each edge e of G.*
2. *There is a $\mathbb{R}$-flow f on G such that $a \le f(e) \le b$ for each edge e of G.*
3. *For each $U \subset V(G)$ we have $\frac{a}{b} \le \frac{|\delta^+(U)|}{|\delta^-(U)|} \le \frac{b}{a}$.*



Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): take any set U. As "the net flow over each cut is zero", we have

$$a \cdot |\delta^+(U)| \le \sum_{e\in\delta^+(U)} f(e) = \sum_{e\in\delta^-(U)} f(e) \,. \le b \cdot |\delta^-(U)|$$

$$b \cdots \ge \qquad\qquad \ge a\,|\delta^-(U)|$$

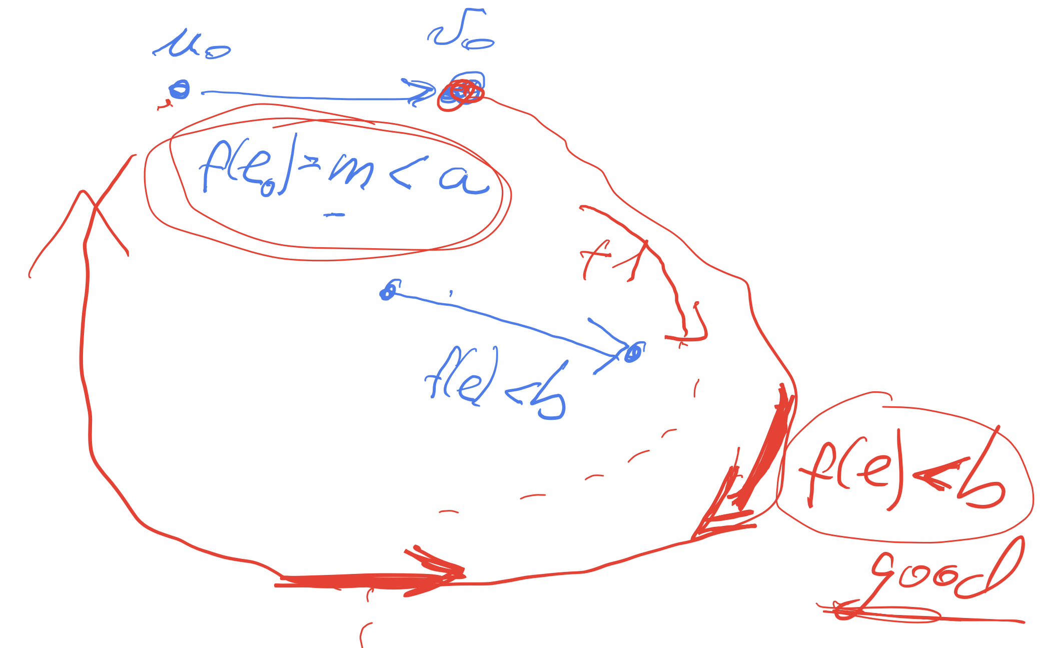
(3) $\Rightarrow$ (1): We call a $\mathbb{Z}$-flow reasonable if $0 \le f(e) \le b$ for each edge e. Find reasonable flow that is optimal in the following sense:

have: $\forall e \quad 0 \le f(e) \le b$

need: $a \le f(e) \le b$

1. • $m := \min\{f(e) : e \in E(G)\}$ is as large as possible;
2. • among flows with the same m we choose the one with as few edges attaining $f(e) = m$ as possible.

We claim that the optimal reasonable flow does in fact satisfy $f(e) \ge a$ for every edge, which would prove (1). For contradiction, suppose there is an edge $e_0 = u_0v_0$ for which $f(e_0) = m < a$.

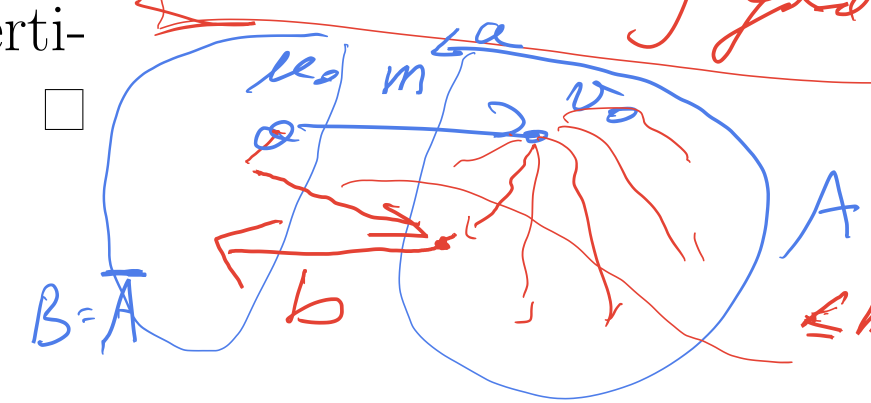


Good edges e:

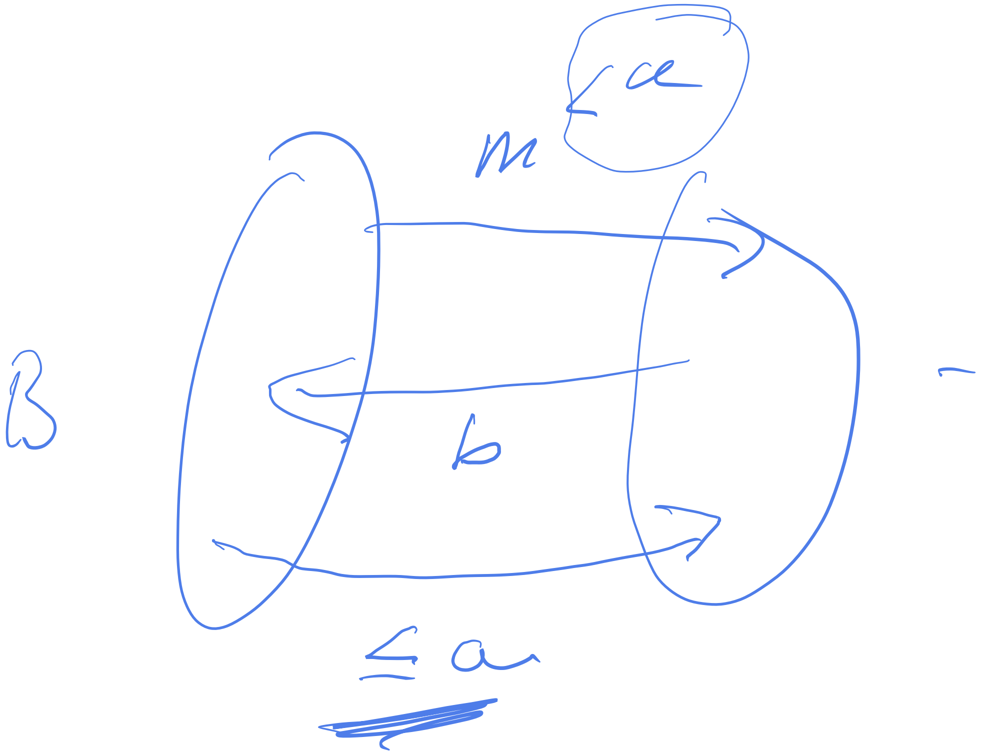
- $f(e) < b$ and we use e forward,
- $f(e) > m + 1$ and we use e backward. ?

Path or cycle of good edges $\Rightarrow$ ✓

Either a ~~u_0~~v_0-path of good edges OR a cut certifying it. . . . $v_0 u_0$ □



$A = \{v : \exists\, v_0 v \text{ path of good edges}\}$ $\ni v_0$, $\not\ni u_0$



$\delta^+(B) \cdot a$

$$\sum_{\delta^+(B)} f(e) = \sum_{\delta^-(B)} f(e)$$

$$a|\delta^+(B)| \geq \quad = b|\delta^-(B)|$$

$$\frac{\delta^+}{\delta^-} > \frac{b}{a}$$

$\le$

$\lightning$

Circular flows

Definition 23. *Let G be a digraph, f a $\mathbb{R}$-flow, $r \in \mathbb{Q}$. We say that f is* nowhere-zero circular r-flow, *if*

$$\boxed{f(e) \in [1, r-1]}$$

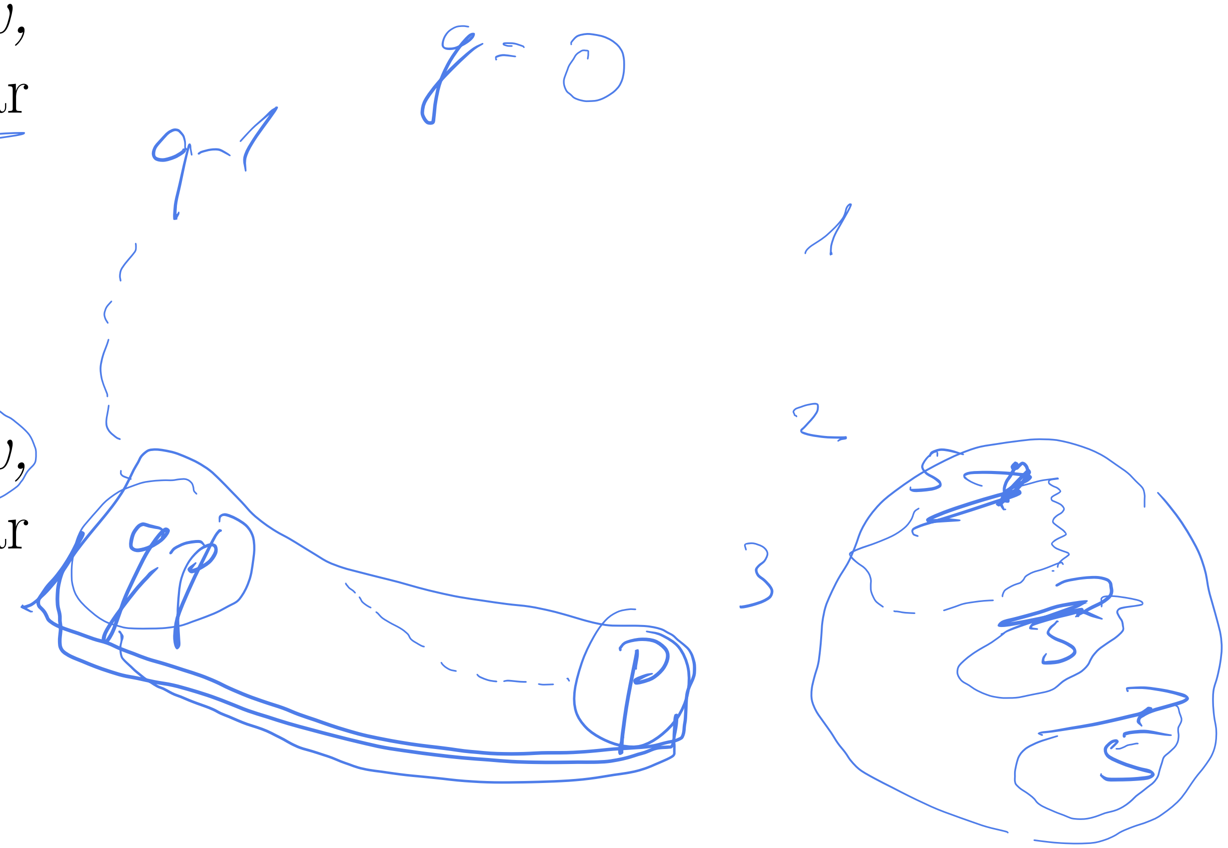
for all edges $e \in E(G)$.

Definition 24. *Let G be a digraph, f a $\mathbb{Z}_q$-flow, $p, q \in \mathbb{N}$. We say that f is* nowhere-zero circular p/q-flow, *if*

(modular)

$$f(e) \in \{p, p+1, \ldots, q-p\}$$

for all edges $e \in E(G)$.



- Definition 23 and 24 are equivalent (for $r = p/q$).
- A variant of the circulation lemma for real a, b also true (use just (2) and (3)).
- It follows that k-flow implies existence of k'-flow for all $k' > k$.

(Same proof?)

$\boxed{[1, 5-\varepsilon]}$ $[1, 5)$

Q: [illegible] circ. 5.5-flow?

$(6-\varepsilon)$-flow?

Snarks

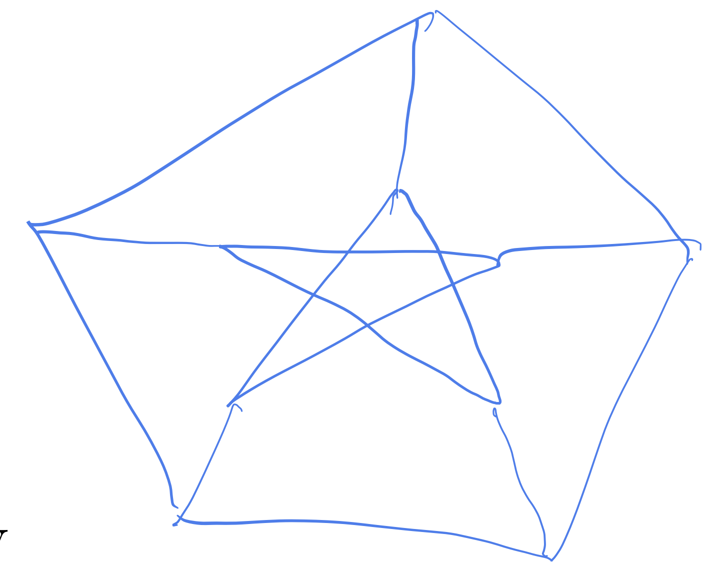
A graph is called a *snark*, if it is

- cubic, 3-reg.
- bridgeless and
- not 3-edge-colorable.

Equivalently, it has no 4-NZF.

Some authors require a higher edge-connectivity (we may insist on the graph to be cyclically 6-edge-connected), but we won't do it here.

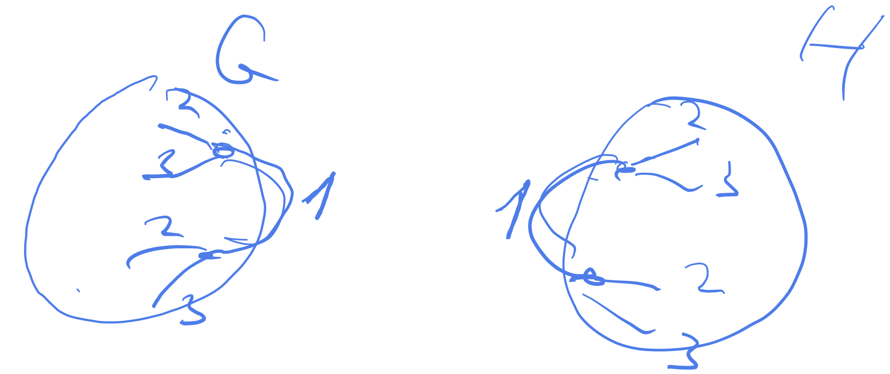
Snarks are canonical counterexamples.



Petersen

Snarks with a 2-cut

Start with graphs G and H each with a specified edge. To form the graph $G{=}H$ we cut the specified edges in G and H and glue the "half-edges" to connect G and H.

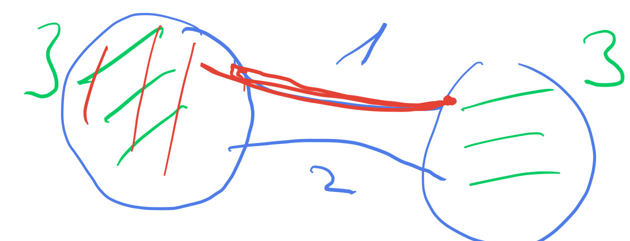


has 3-e.c. $\Longleftarrow$ G & H have 3-e.c.

$$G{=}H \text{ is a snark} \iff G \text{ or } H \text{ is a snark.}$$

Equivalently: when we "add anything to an edge of a snark", we get again a snark and all snarks with a 2-cut are obtained this way.

Any edge 3-coloring of $G{=}H$ gives the same color to the two edges of the 2-cut. Consequently, we may use the coloring of $G{=}H$ to get colorings of G and of H. OTOH …



not possible!

color class is a perf. matchg.

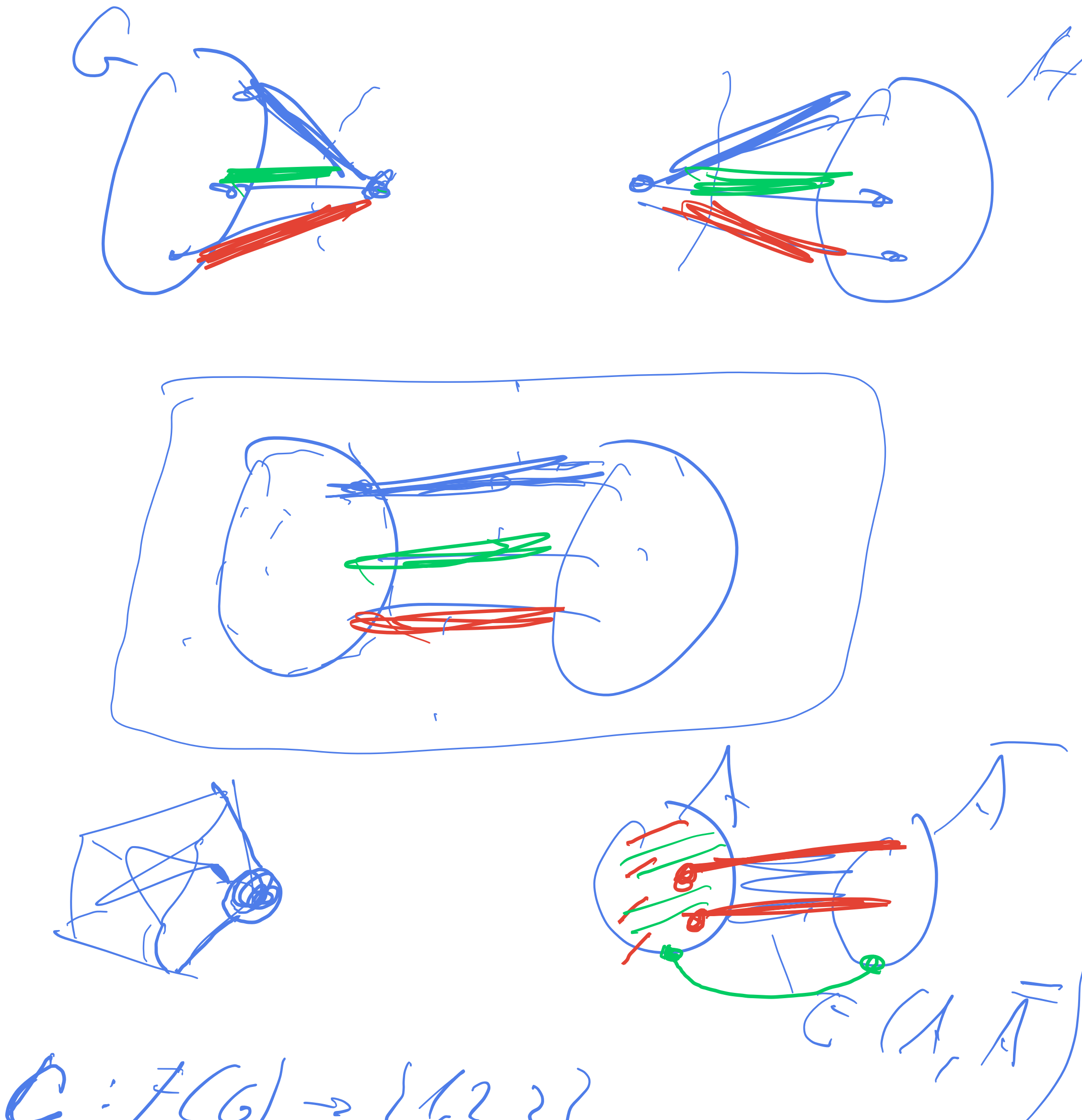
Snarks with a 3-cut

Now we start with cubic graphs G, H each with one specified vertex. We split these specified vertices in three vertices of degree 1, and identify the three pendant of G with those of H. (There are 3! ways to do so.) We use $G \equiv H$ to denote the resulting graph.

is 3-e.c. G & H are 3-e.c.

$$\boxed{G \equiv H \text{ is a snark} \iff G \text{ or } H \text{ is a snark.}}$$

Equivalently: when we "add anything to a vertex of a snark", we get again a snark, and all snarks with a 3-cut are obtained this way.



Parity lemma

$\forall G$ cubic

$\forall A \subseteq V(G)$ $\forall$ 3-edge-col. $c : E(G) \to \{1,2,3\}$

$|\{e \in E(A,\bar{A}) : c(e) = i\}| \bmod 2$

does not depend on i

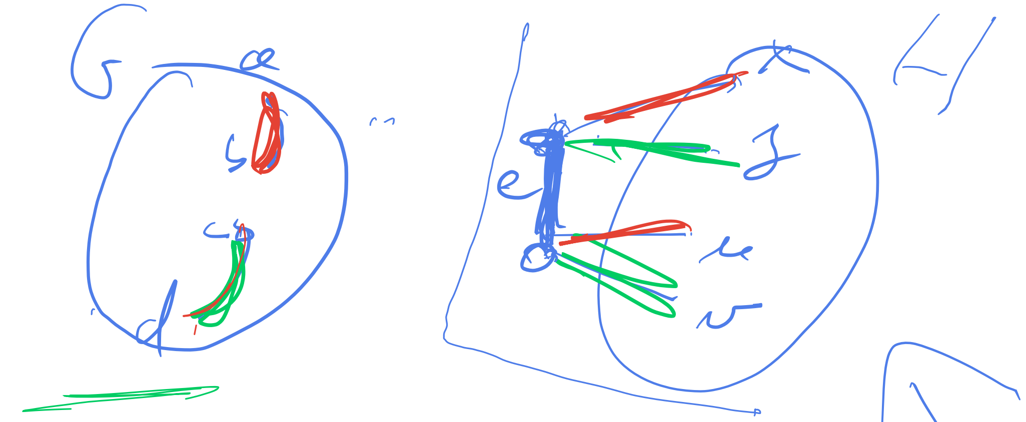
Exercise

We define two useful operations on cubic graphs. A Δ-Y transformation is a contraction of a triangle to a single vertex, a Y-Δ transformation is the inverse operation. (Observe that these operation preserve the 3-regularity.) For a cubic graph G, prove that G is a snark iff G' obtained by a series of Y-Δ and Δ-Y transformations from G is a snark.

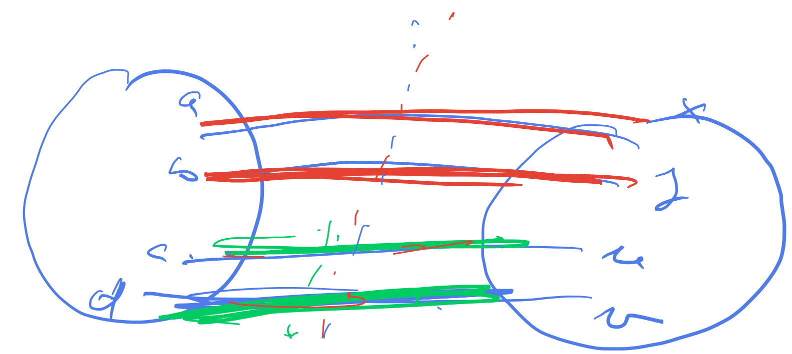
Note: The simplicity of the above two constructions, in particular the fact that only one of the smaller graphs needs to be a snark, together with possibility to reduce the "big conjectures" to cyclically 4-edge-connected graphs, explain why some authors choose to demand that the snarks are free of 2-cuts and non-trivial 3-cuts.

Snarks with a 4-cut – Isaacs' dot product

Let G, H be graphs, ab, cd edges of G, e an edge of H, let x, y be the other two neighbours of one end of e, u, v the other two neighbours of the other end. To form the *Isaacs' dot product* $G \cdot H$ of G and H we delete edges ab and cd from G, e with its end-vertices from H, and add edges ax, by, cu, dv.



Theorem 25 (Isaacs, 1975)**.** *If G and H are snarks then so is $G \cdot H$. If both G and H are cyclically 4-edge-connected and if the vertices a, b, c, d are all different, then $G \cdot H$ is also cyclically 4-edge-connected.*



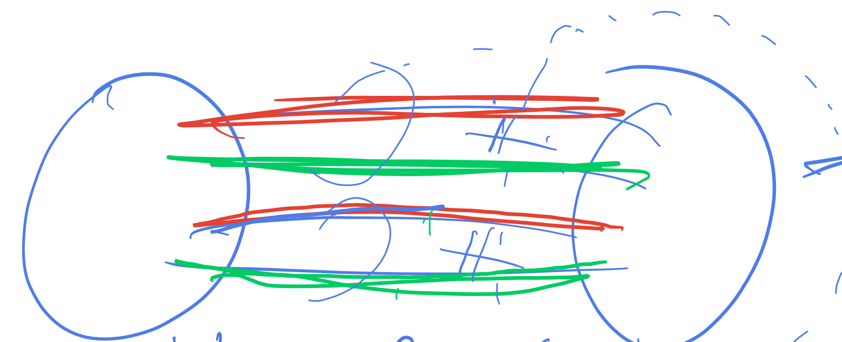
(1) $\Rightarrow f(cu) = f(dv) = ?$

$\Rightarrow$ G is not a snark

Proof. Suppose we have an edge 3-coloring f of $G \cdot H$. We distinguish two cases.

(1) $f(ax) = f(by)$ (2) $f(ax) \neq f(by)$

…



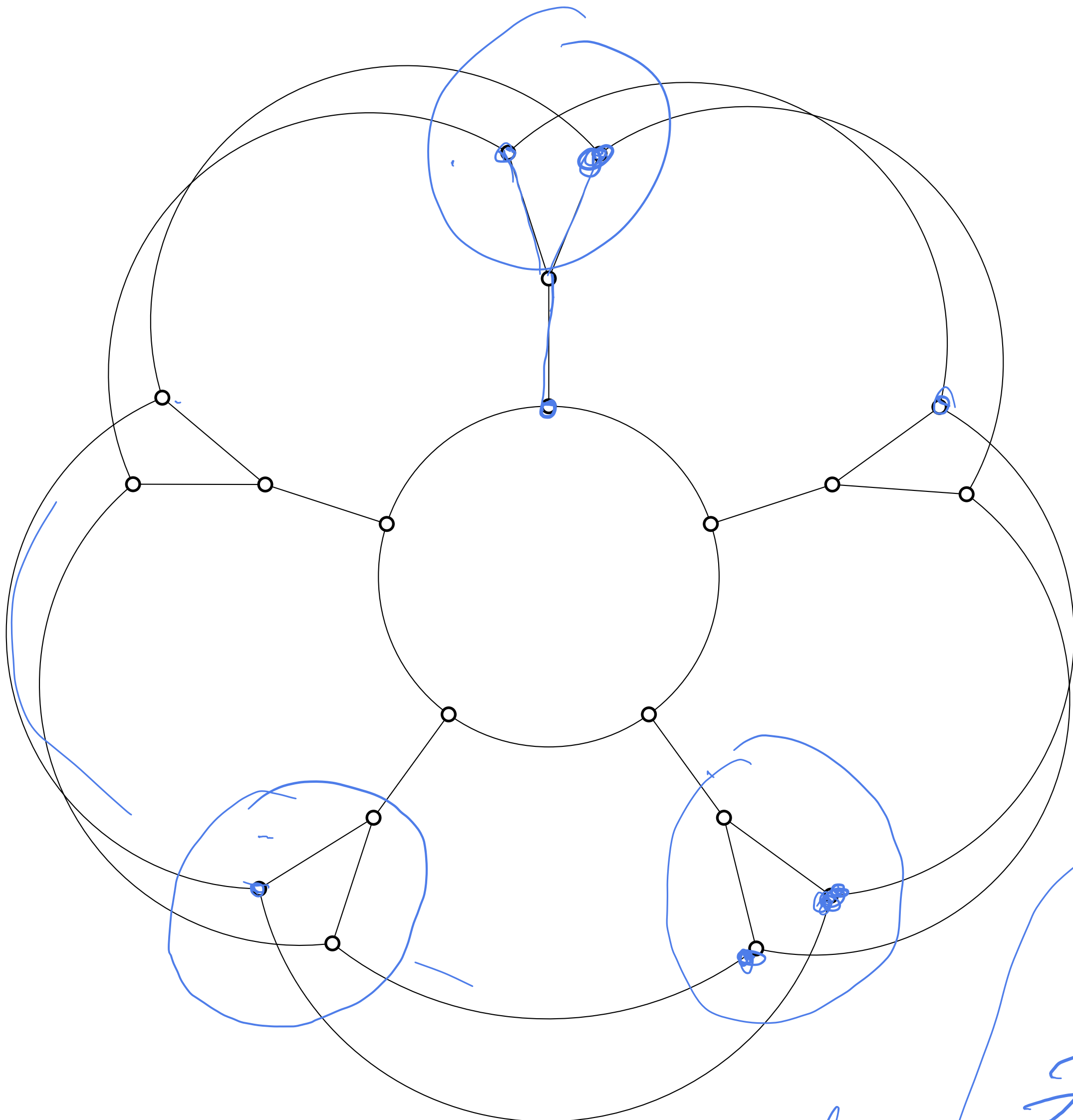
H is 3-e.c.

$\boxed{10+10-2 = 18 \text{ vertices}}$

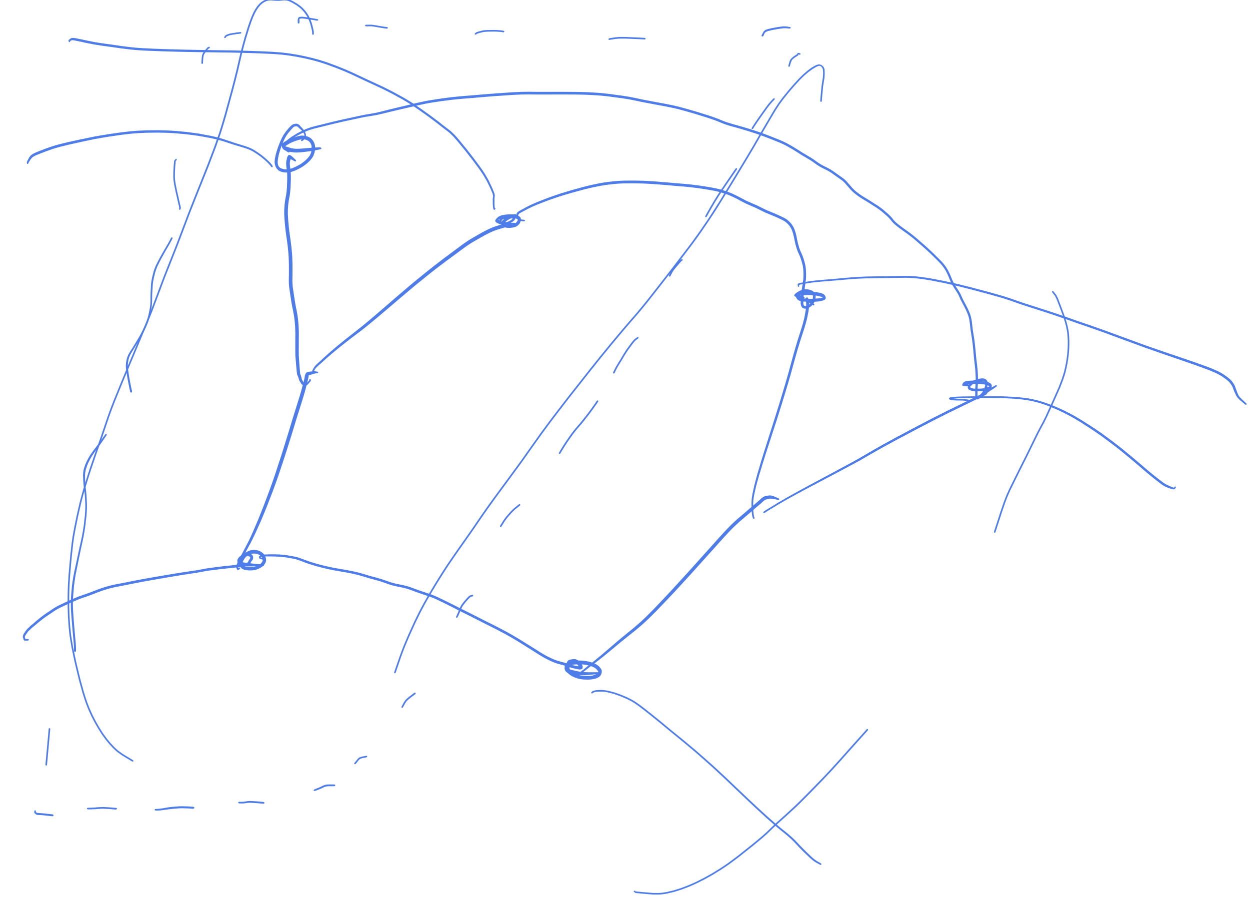
(2) $\Rightarrow \{f(cu), f(dv)\} = \{f(ax), f(by)\}$

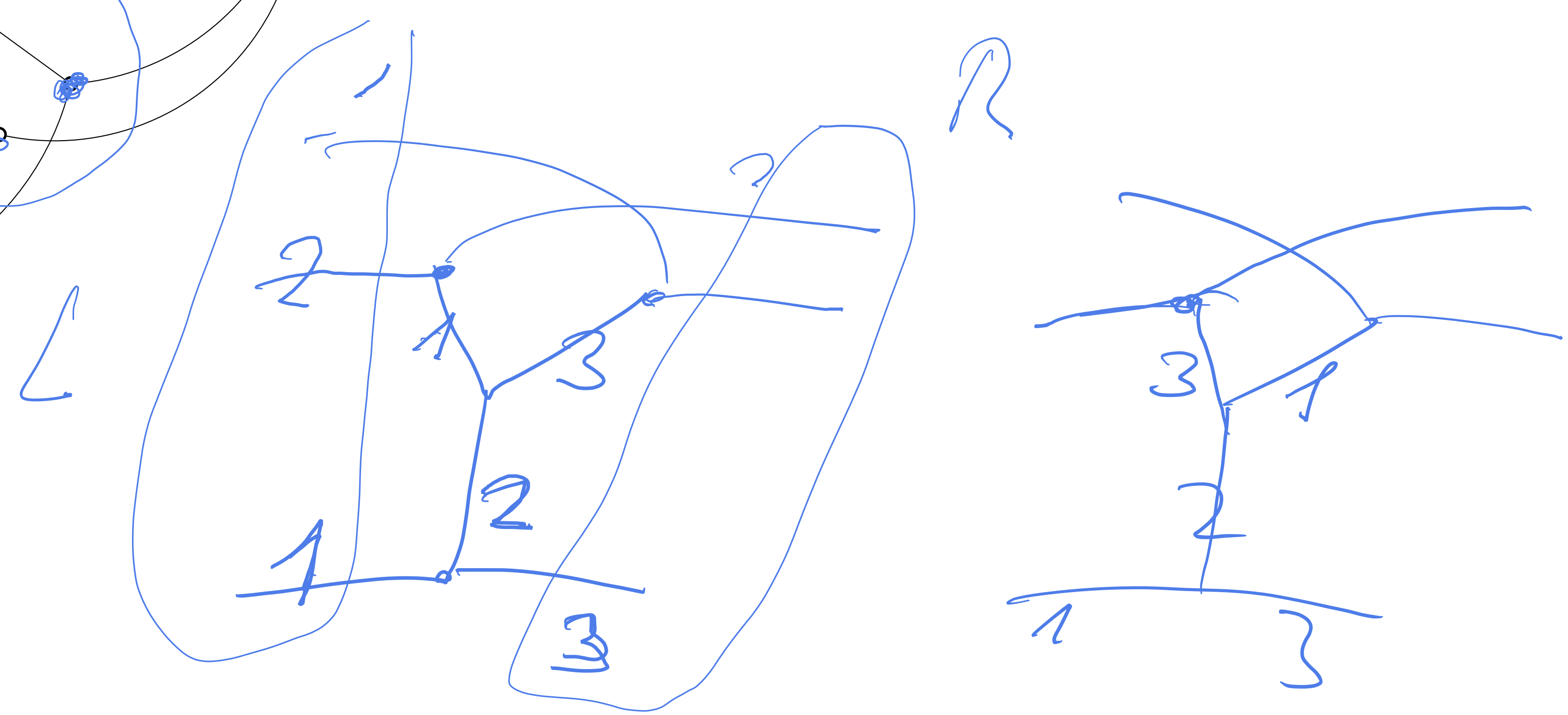
Example: Blanuša snarks

Flower snarks



J_n





Let n be odd. To describe a graph J_n, we start with three copies of C_n, we denote its vertices by i_1, i_2, i_3 for $i = 1, \ldots, n$. Replace edges $n_2 1_2$ and $n_3 1_3$ by $n_2 1_3$ and $n_3 1_2$. Finally, for each i we add a new vertex i and join it by an edge to i_1, i_2, i_3. On Figure **??** we can see J_5 (this particular graph is sometimes called the flower snark). and J_3 — is just a Y-Δ transformation of Pt (equivalently, it is $\mathrm{Pt} \equiv K_4$).

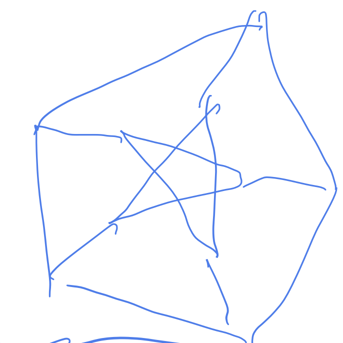
Theorem 26 (Isaacs, 1975)**.** *If n is odd then J_n is a snark. If $n \geq 7$ then J_n is cyclically 6-edge-connected.*

vertex k-connected

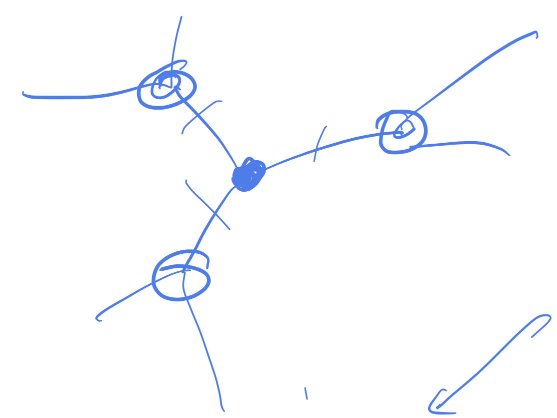
$\geq k$

edge

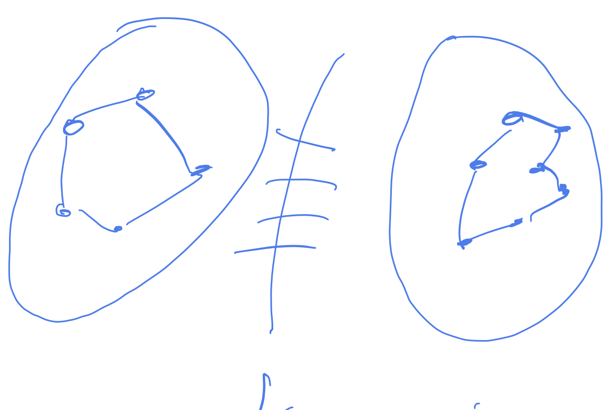
$\geq k$



c 5-e c

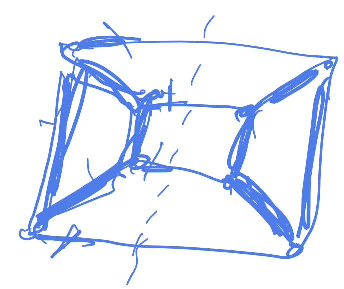


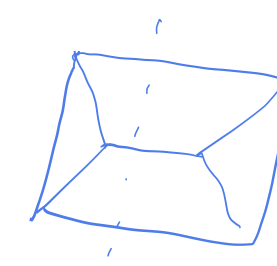
For what k is it cycl. k-edge-conn.?



$\geq k$

cycl. k-edge-connected

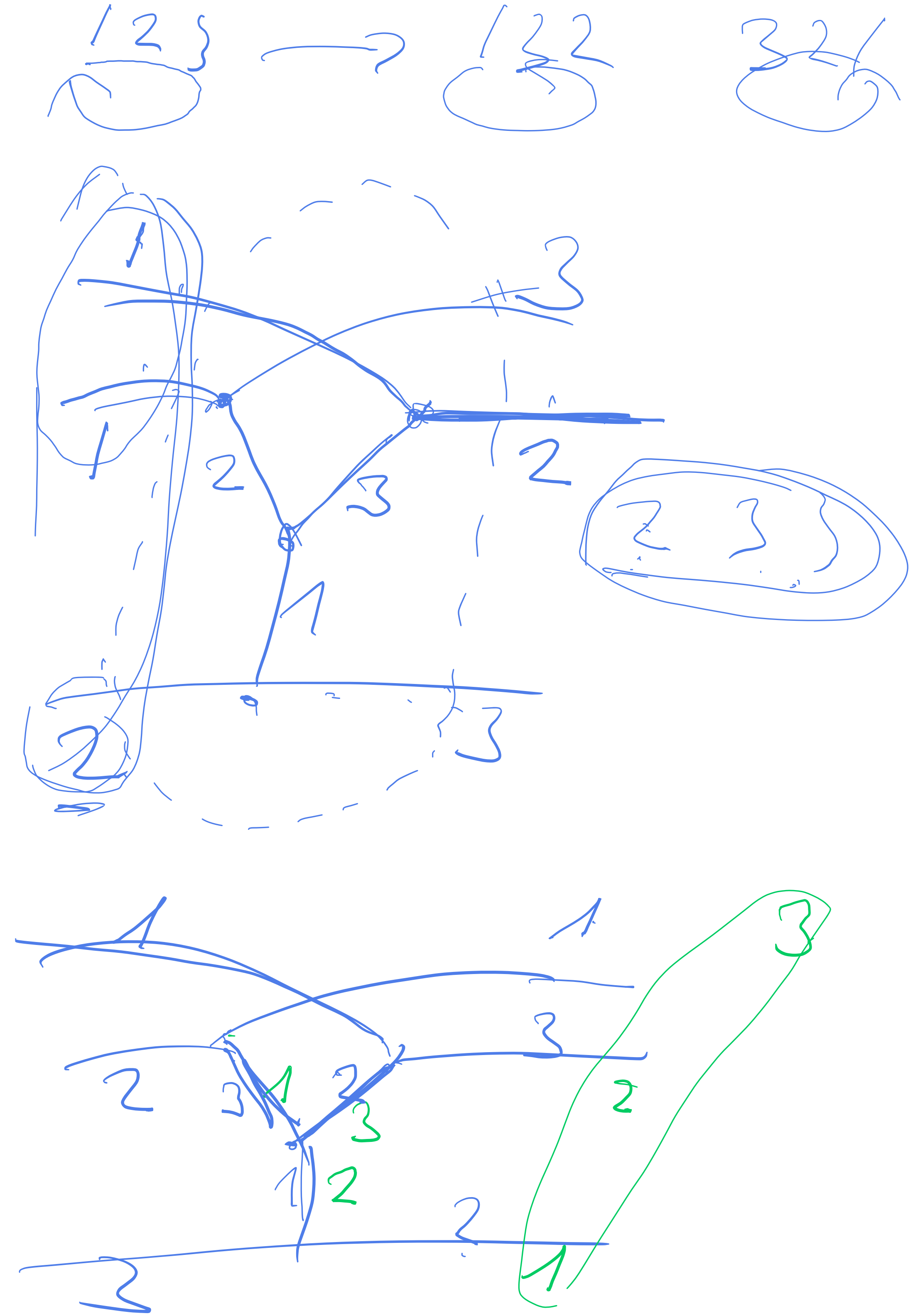




Proof. Suppose J_n can be edge-colored using three colors. Let B_i denote the subgraph induced by vertices i, i_1, i_2, i_3 and the incident edges (see Fig. **??**). We divide the edges of this subgraph into three triples, Left, Right, and Top. (Of course the Right edges of B_i are the Left edges of B_{i+1}.) Clearly not all edges of of L can be of the same color, as then it is not possible to color T. Thus there are two possibilities.

(1) Edges of L use one color twice. Say, they use colors 1, 1, and 2 in some order. It is easy to check that then edges of R use colors 2, 3, and 3, in some order. In the next block we will use 1, 1, 2 on the right, and so on. As n is odd, we get a contradiction.

(2) Edges of L use all three colors. Again, it is simple to explore the two possibilities how to extend the coloring on R: both are obtained from the coloring of L by a cyclic shift (i.e., a permuta-



tion formed by one 3-cycle). In between the blocks B_n and B_1 we introduced a transposition by the construction of the graph. Thus if there is an edge 3-coloring, then we can write an identity as a composition of 3-cycles and one transposition, which is a contradiction.

TODO: cyclic connectivity?

$\square$

Superposition construction (Kochol)

- G: a graph with all degrees 1 or 3
- a flow on G: a nowhere-zero $\mathbb{Z}_2^2$-flow where we ignore Kirchhoff's condition at degree 1 vertices.
- Observation: let E_1 be the edges incident to degree 1 vertices, let φ be a flow. Then $\varphi(E_1) = 0$.
- (k_1, k_2, k_3)-supervertex: a graph as above, with E_1 split into three nonempty subsets of sizes k_1, k_2, k_3.
- (k_1, k_2)-superedge: a graph as above, with E_1 split into two nonempty subsets of sizes k_1, k_2.
- proper superedge: a superedge, where the sum over each of the two parts is nonzero.

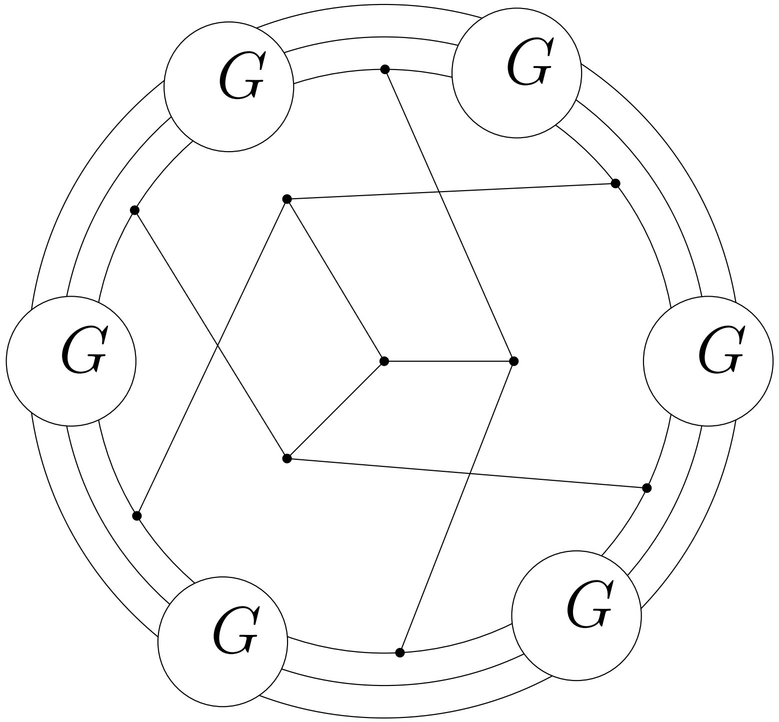
Given:

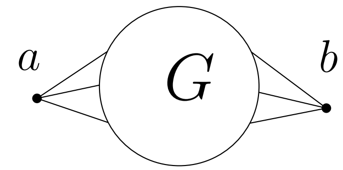
- A snark G,
- a list of *supervertices* G_v for $v \in V(G)$
- a list of *proper superedges* G_e for $e \in E(G)$

Conclusion: The superposition is a snark.

Corollary: There is a family of cyclically 6-edge-connected snarks.

Corollary: There is a family of cyclically 5-edge-connected snarks with arbitrarily high girth.



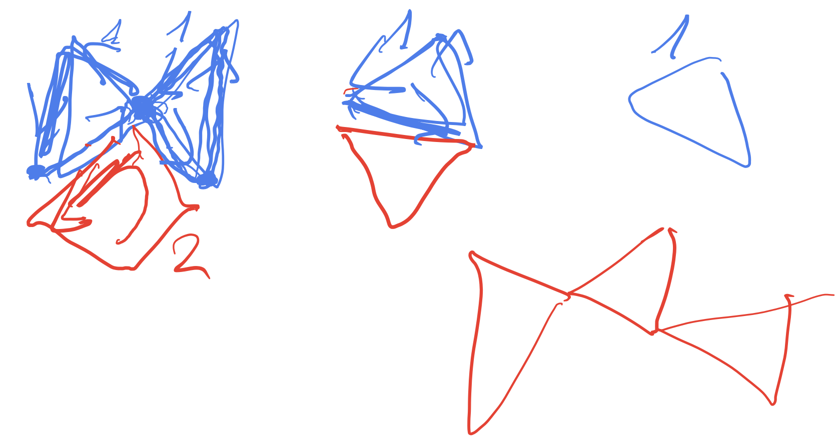
where 

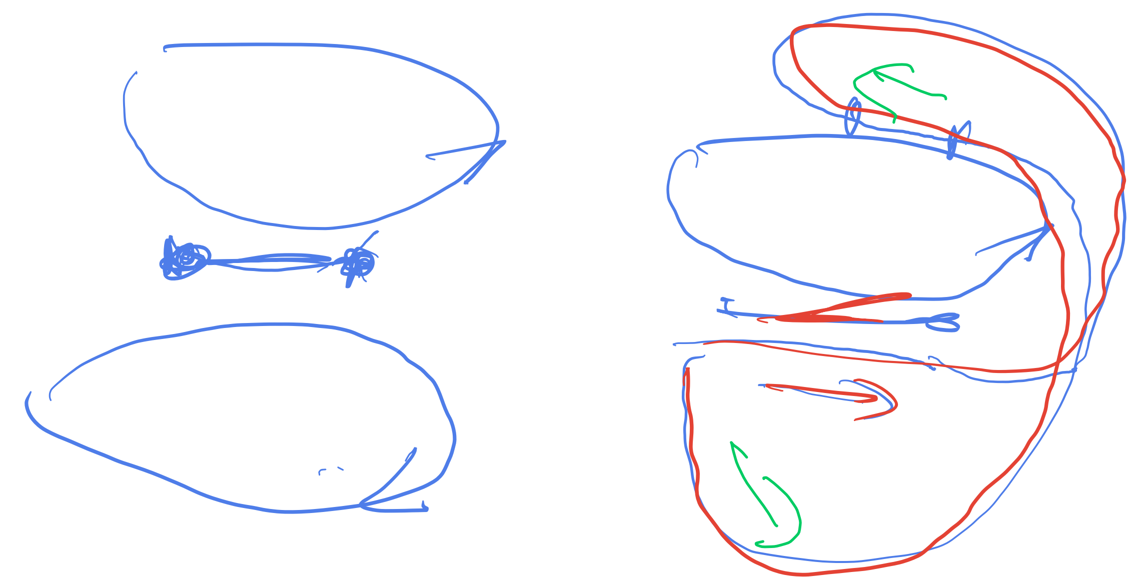
is the Flower snark depicted below.

Cycle double cover conjecture

Conjecture 1. *Let G be a bridgeless graph. Then there is a collection of cycles such that every edge is covered by exactly two of them.*

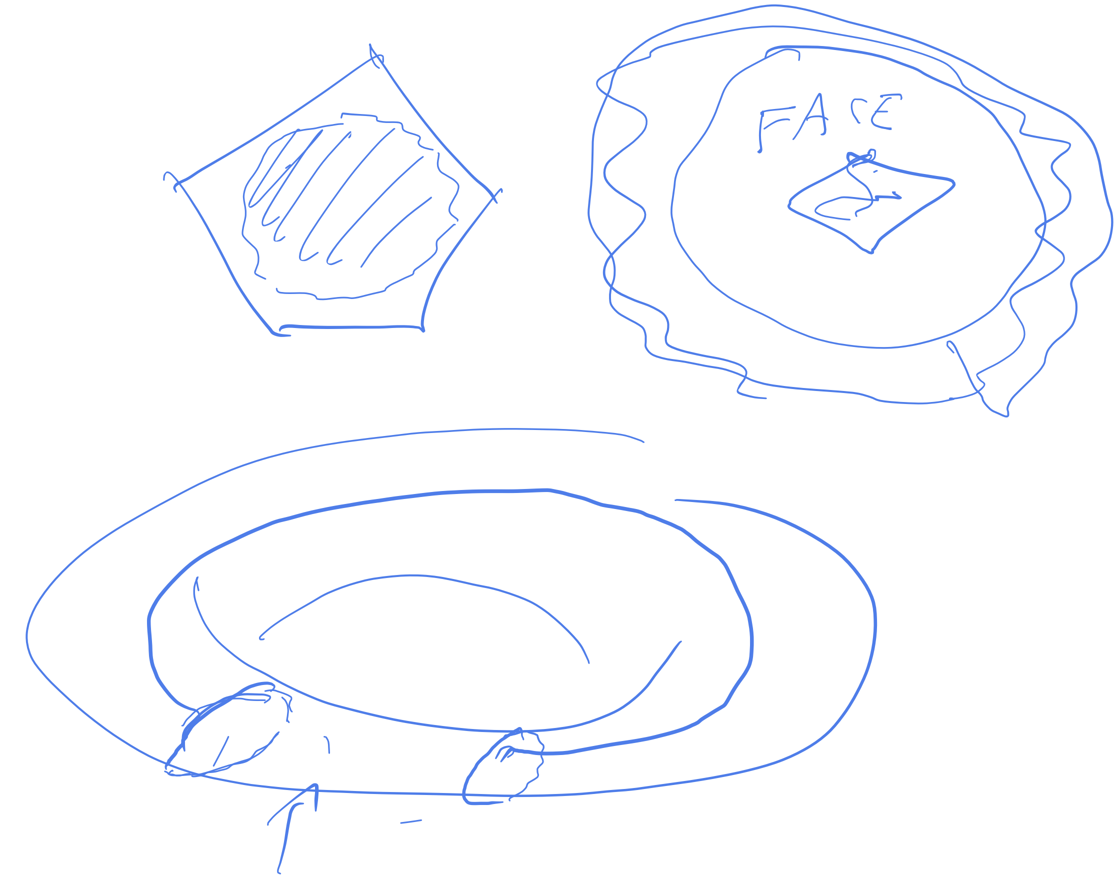
- equivalently: "a collection of circles"
- stronger conjecture (k-**CDC**): "a collection of at most k cycles"
  Conjectured for $k = 5$, unknown for any $k \geq 5$.
- equivalently: "a collection of circles that can be k colored"
- stronger conjecture (oriented CDC): "a collection of oriented cycles such that every edge is covered once in each direction"
- even orientable 5-CDC can be true





Relation with embedding

- 2-cell embedding of a graph: every face is homeomorphic to a disk
- circular 2-cell embedding: moreover, the boundary of each face is a circle
- 2-cell embedding on any surface implies CDC
- 2-cell embedding on an *orientable* surface implies OCDC

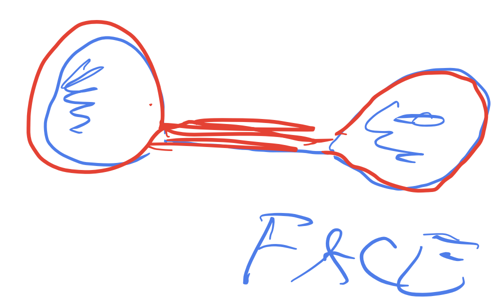


Conjecture 2. *Let G be a connected bridgeless graph. Then G has a circular 2-cell embedding on some surface.*

- If G is 3-regular, then the above is equivalent to CDC.

for 3-regular graphs

CDC



Theorem 27. *The following is equivalent for a graph G.*

$\Longleftrightarrow$ G is 3-edge colorable

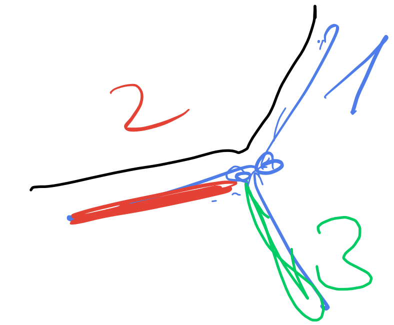
$c : E(G) \to \{1,2,3\}$

$C_{12} = c^{-1}(\{1,2\})$

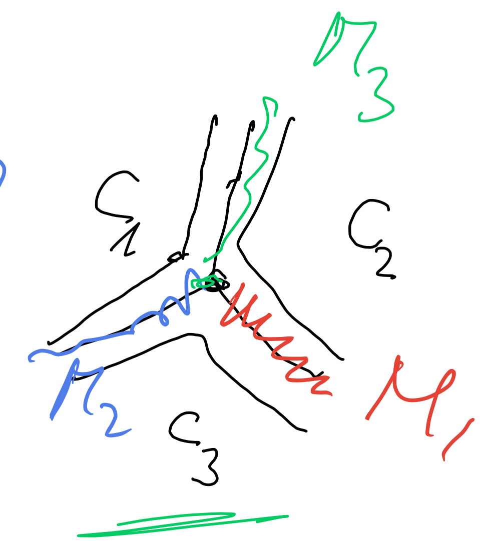
1. *G has a 4-NZF.*
2. *G has has a 3-CDC.*
3. *G has has a 4-CDC.*
4. *G has has a 4-OCDC.*

$\hookrightarrow$ is a cycle

C_{23}, C_{31} ... C_{12}, C_{23}, C_{31} is a 3-edge col



Proof. Sketch: 1 and 2 are easily equivalent. 3 $\Rightarrow$ 2: Given a CDC C_1, C_2, C_3, C_4, the collection $C_1 \,\Delta\, C_2$, $C_1 \,\Delta\, C_3$, and $C_1 \,\Delta\, C_4$ is also a CDC. 2 $\Rightarrow$ 4: Given a CDC C_1, C_2, C_3, define a flow f_i with values ± 1 along C_i and 0 elsewhere. It is easy to check that $(f_1 + f_2 + f_3)/2$, $(f_1 - f_2 - f_3)/2$, $(-f_1 + f_2 - f_3)/2$, $(-f_1 - f_2 + f_3)/2$ are a 4-OCDC. □



$M_1 = E(G) - E(C_1)$

- $C_1 \,\Delta\, C_2$ is a cycle ✓
- the is in 2 $\quad e \in C_1 \cap C_2$, $\notin C_3, C_4$ $\Rightarrow e \in C_1 \,\Delta\, C_3$, $C_1 \,\Delta\, C_4$ | $e \in C_2 \,\Delta\, C_3$, $\notin C_1, C_3$ $\quad e \in C_1 \,\Delta\, C_2$, $C_1 \,\Delta\, C_3$

C_i:

f_i: ± 1

$f_i : E(G) \to \mathbb{Z}$ 2-flows

± 1 along C_i

0 elsewhere

f_1	f_2	f_3	$g_1 = \frac{f_1+f_2+f_3}{2}$	$g_2 = \frac{f_1-f_2-f_3}{2}$	$g_3 = \frac{-f_1+f_2-f_3}{2}$	$g_4 = \frac{-f_1-f_2+f_3}{2}$
1	1	0	1	0	0	-1
0	1	1	1	-1	0	0
-1	1	0	0	-1	$+1$	0

Theorem 28. *Every graph with k-OCDC has a k-NZF. The opposite implication is known only for $k \le 4$: Every graph with a k-NZF ($k \le 4$) has a k-OCDC.*

Proof. Given a k-OCDC, find a flow f_i along the cycle C_i. The flow $\boxed{\sum_{i=1}^{k} i f_i}$ is a k-NZF. The other implication is easy for $k = 2$ and already proved for $k = 4$. The remaining case $k = 3$ relies on Exercise 29. $\square$

integer

Exercise 29. *Let f be a flow on a digraph G such that $0 < f(e) \le k$ for every edge of G. Prove that there are $\{0,1\}$-valued flows $f_1, \ldots, f_k$ such that $\boxed{f = \sum_{i=1}^{k} f_i}$.*

$f_1(e) = +1$ $i = 1$

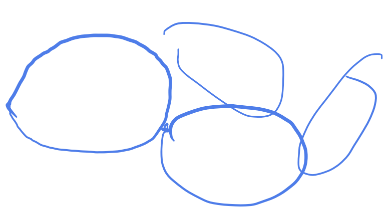
$f_2(e) = -1$ $i = 2$

$-k < \sum_{i=1}^{k} i f_i(e) < k$

$\neq 0$

$i f_i(e) - j f_j(e)$

$\boxed{i - j}$ for $i \neq j$



2-NZF

$\Downarrow$

G_T = coll. of cycles

G has 3-NZF $\Longrightarrow$ G has 3-OCDC

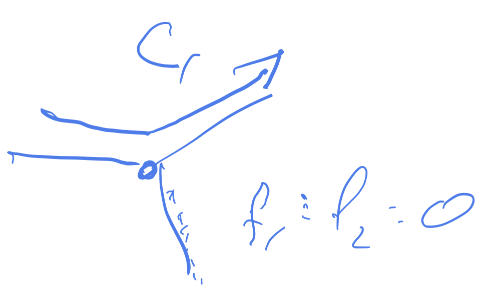
$\Updownarrow$

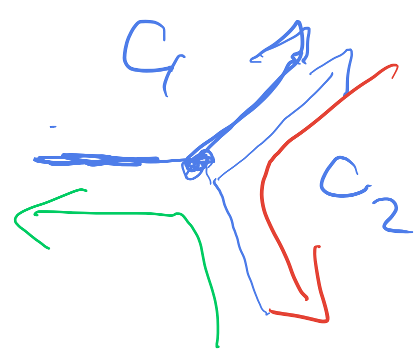
want $f : E(G) \to \{1, 2\}$

$\Downarrow$

$\underline{\exists f_1, f_2}$: $\{0,1\}$-flows s.t. ... C_1, C_2

$f = f_1 + f_2$





$f_1 - f_2, \quad f_2 - f_1 \qquad \Sigma = 0$

$0, 1 \qquad 0, -1 \qquad 0, \pm 1$

$1 \qquad -1 \qquad 0$

$0 \qquad 1 \qquad -1$

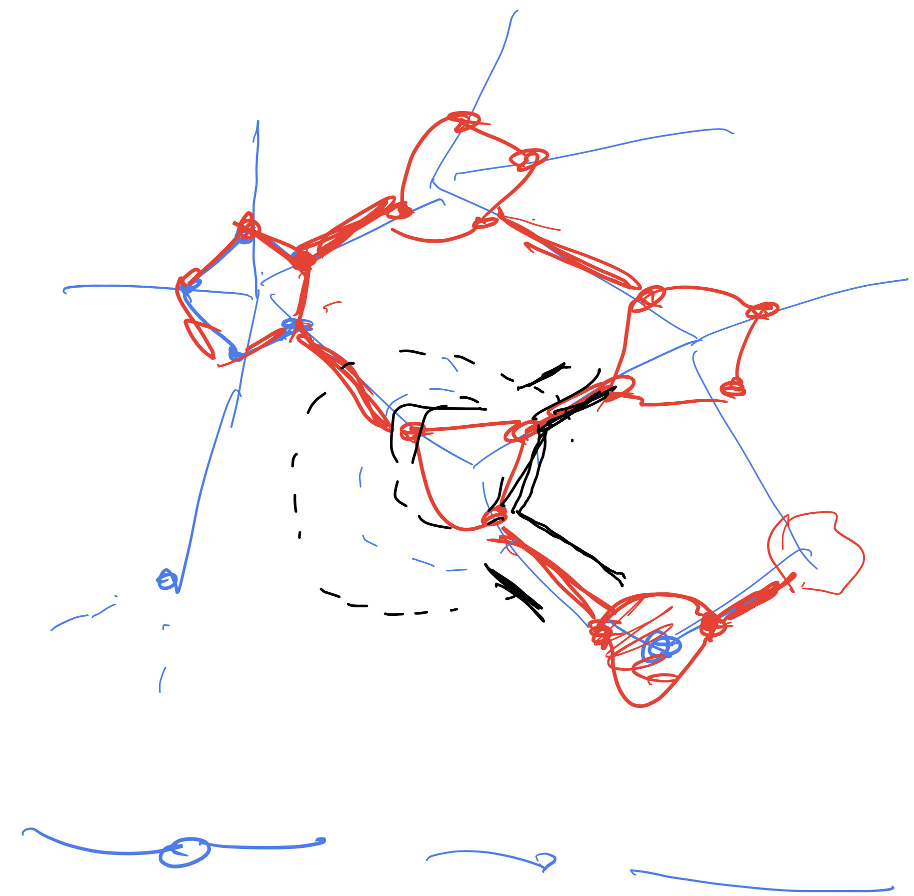
$\vdots$ good

$0 \qquad 0 \qquad 0$ bad

$f = f_1 + f_2 = 0$ ... and f is NZF

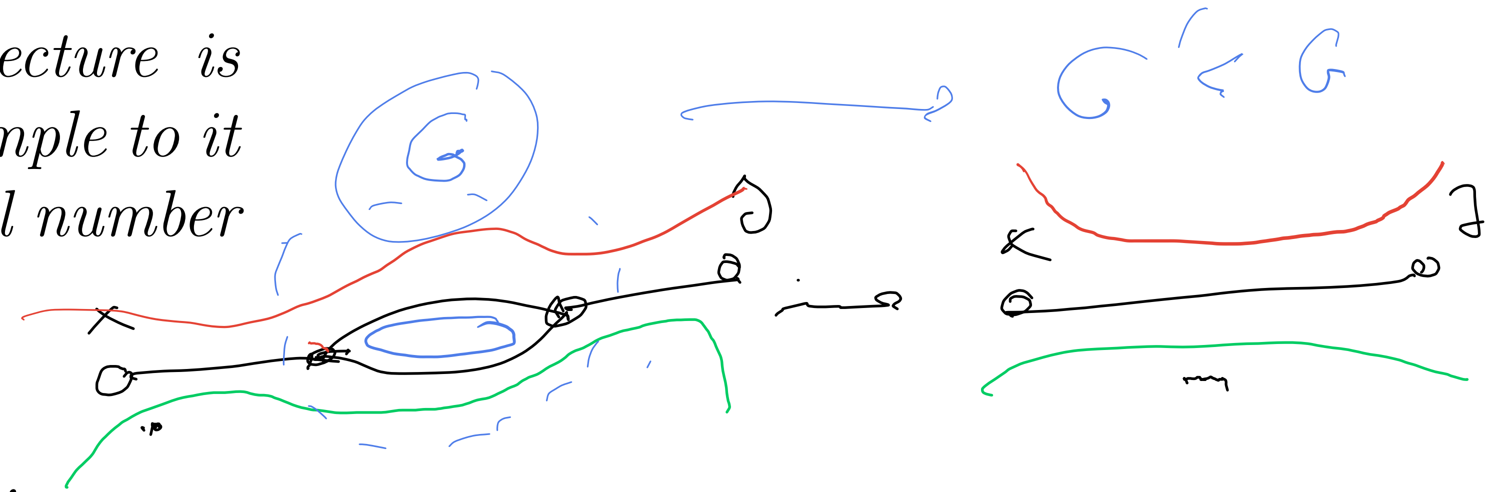
Theorem 30. *Suppose every cubic bridgeless graph has a CDC. Then every bridgeless graph has a CDC as well.*

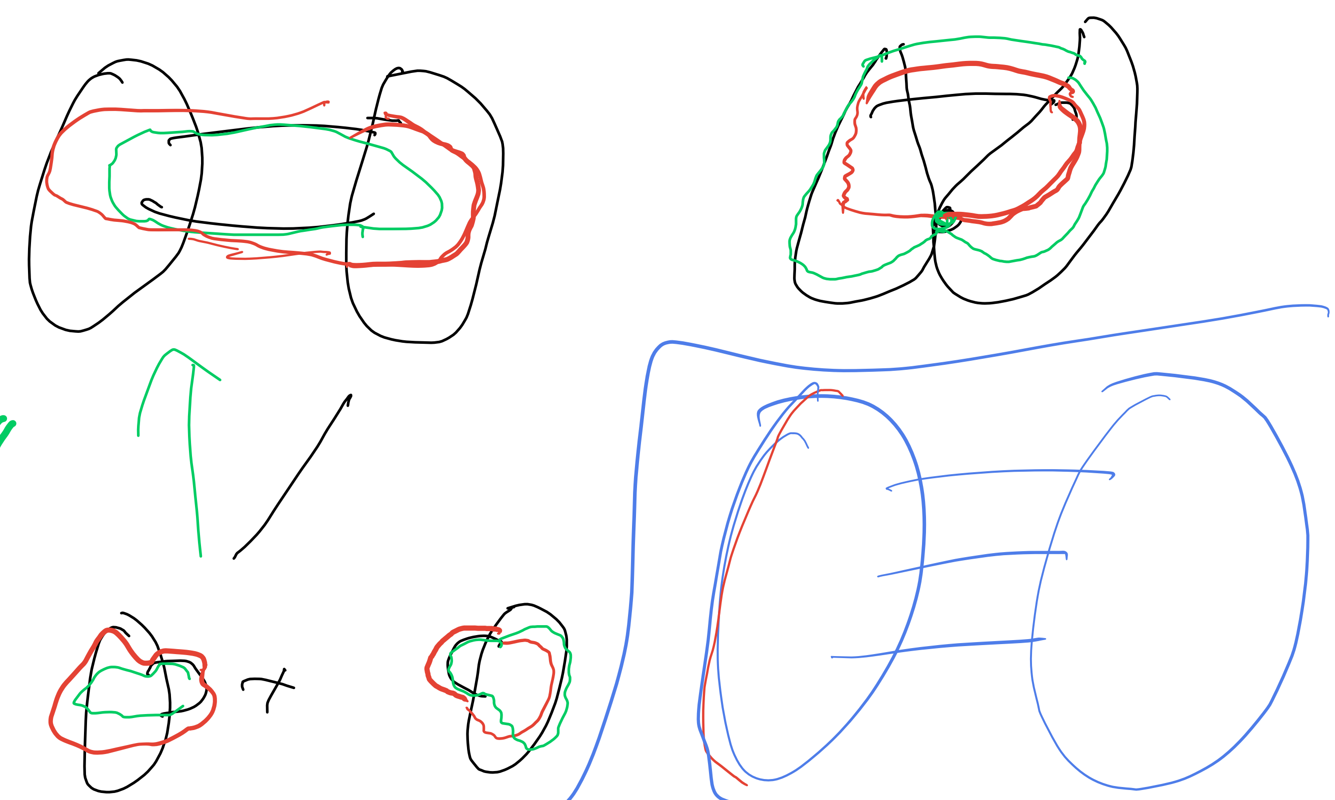
Proof. Let G be a bridgeless graph. We let $H(G)$ denote the graph obtained from G by replacing a vertex v by a circuit of length $\deg(v)$ (we allow $\deg(v) \leq 2$ as well); we let $C(v)$ be the circuit corresponding to v. If uv is an edge in G, then we add any edge between $C(u)$ and $C(v)$ in $H(G)$, choosing the order arbitrarily. It is easy to verify that $H(G)$ is bridgeless. Further, any CDC in $H(G)$ yields easily a CDC in G by contracting each of the new circuits $C(v)$. $\square$

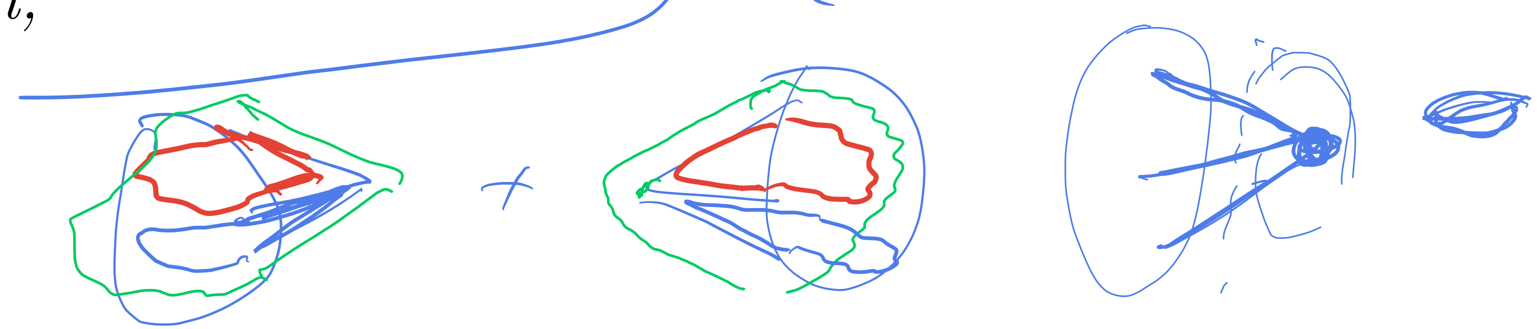


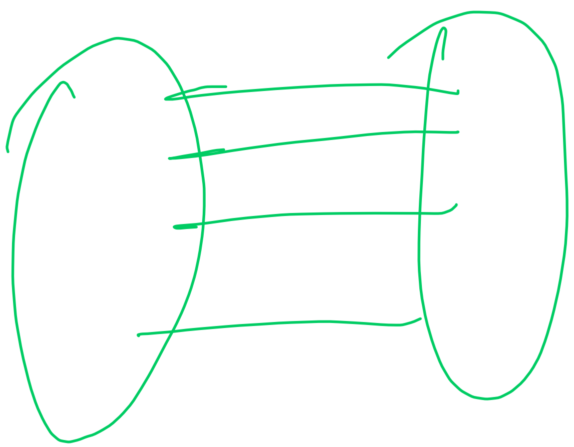
Theorem 31. *Suppose the CDC conjecture is false and let G be a minimal counterexample to it (i.e., a counterexample with the minimal number of edges). Then*

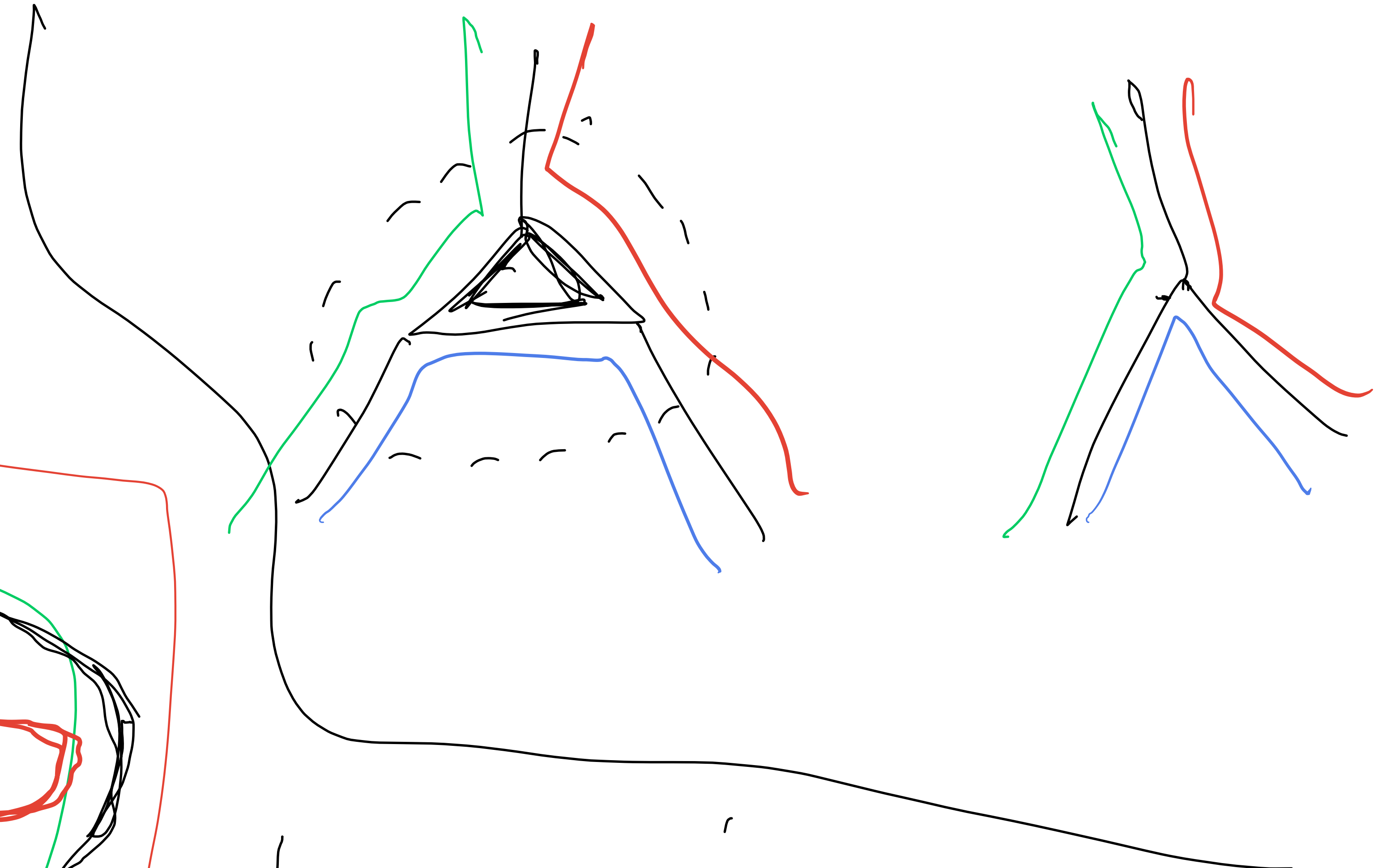
1. *G is cubic;*
2. *G does not contain two parallel edges;*
3. *G does not contain a 2-edge-cut;*
4. *G does not contain a 3-edge-cut;*
5. *G does not contain a* C_3*;*
6. *G does not contain a* C_4*;* $C_5, C_6 \dots, C_{11}$
7. *G does not contain a* C_{11}*;*
8. *G is not planar;*
9. *G does not contain a Hamilton cycle;*
10. *G does not contain a Hamilton path;*
11. *G has oddness at least 6;*

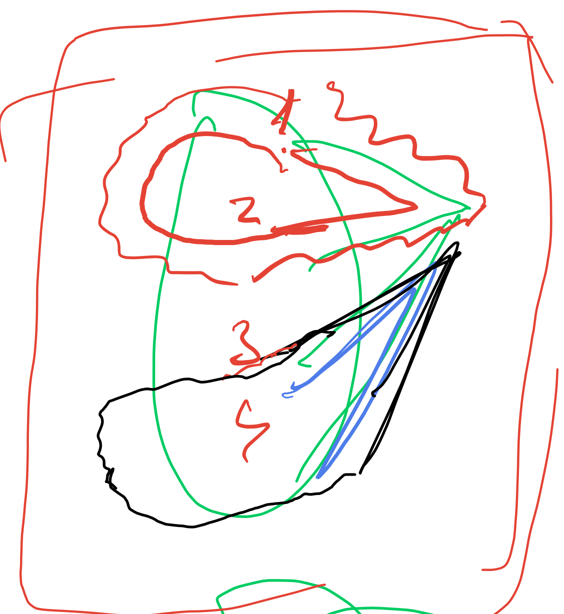


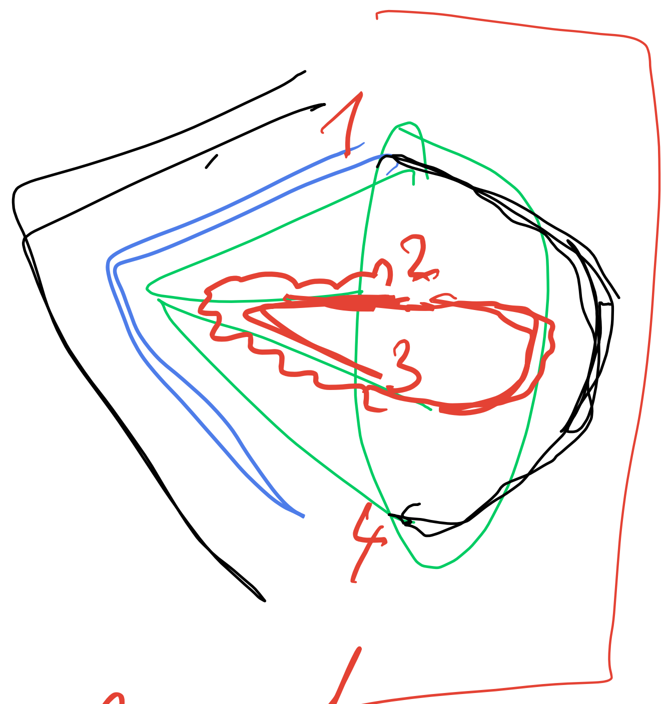




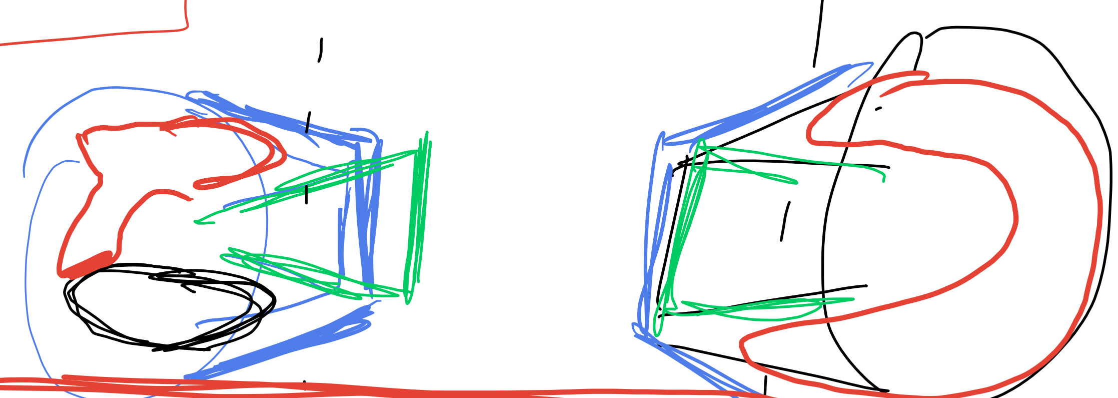








4-ef-cut doesn't work this way

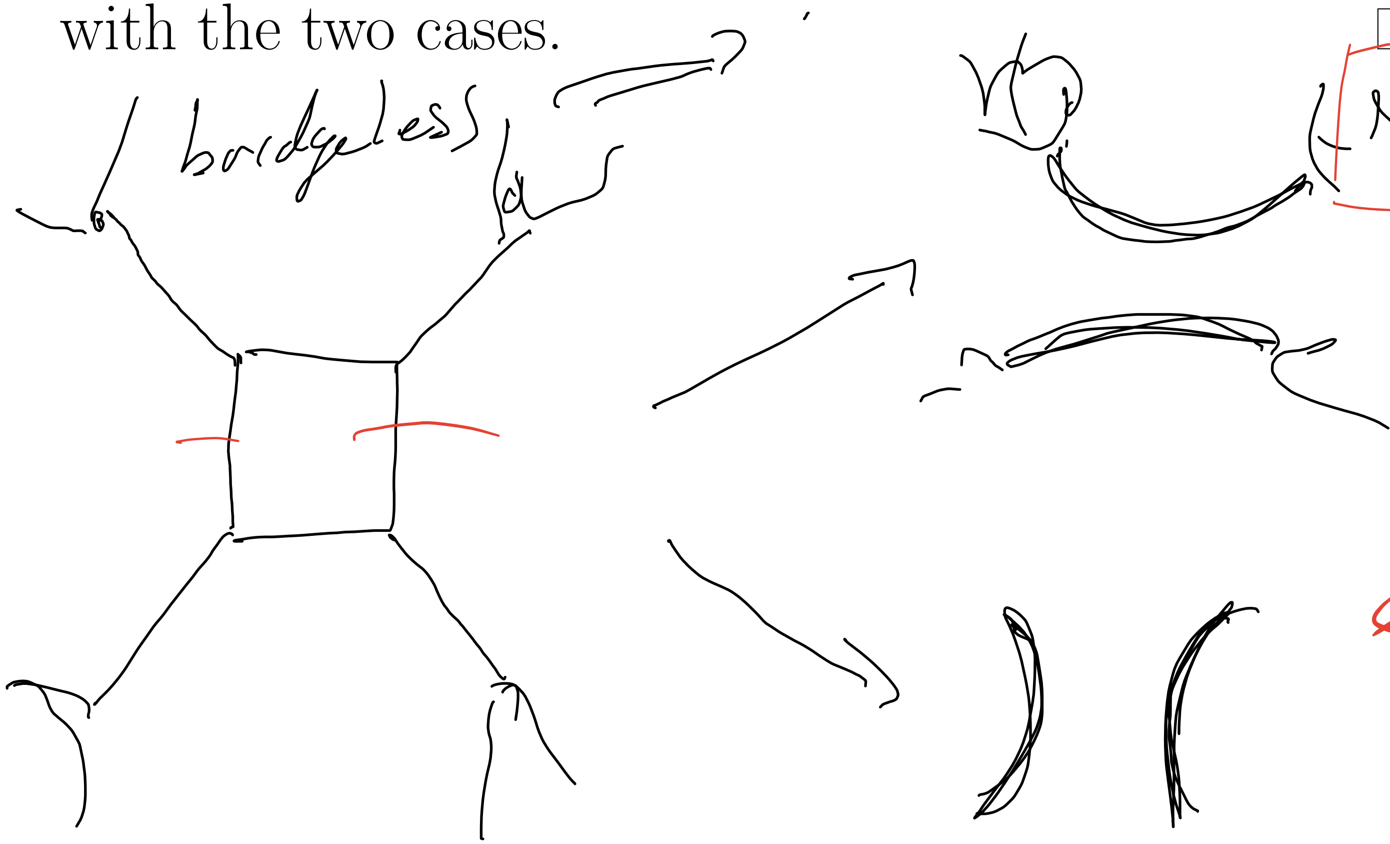


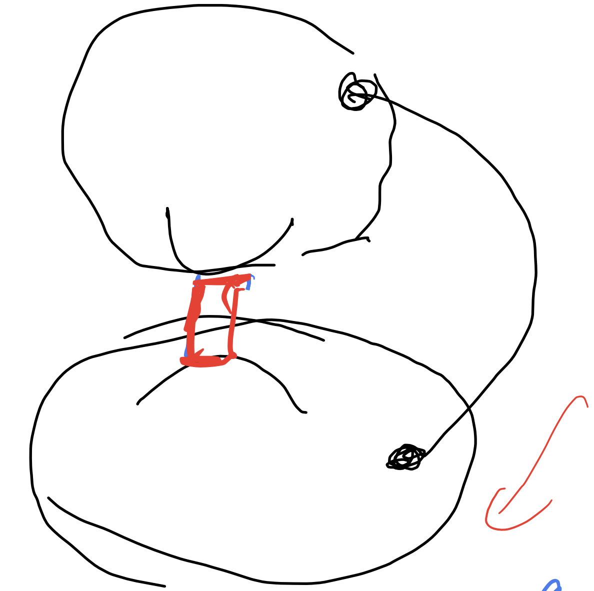
Q ? -. Is IT known ?

12. G is not 3-edge-colorable.

Proof. We will prove 1 and 6 later. 2, 3, 4, 5 are easy. 7 is the best-known in this direction so far (a computer search by Huck, 2000). 8 is easy: for a planar graph we may take the face-boundaries. We postpone 9 and 10 and ??11?? to Section **??**.

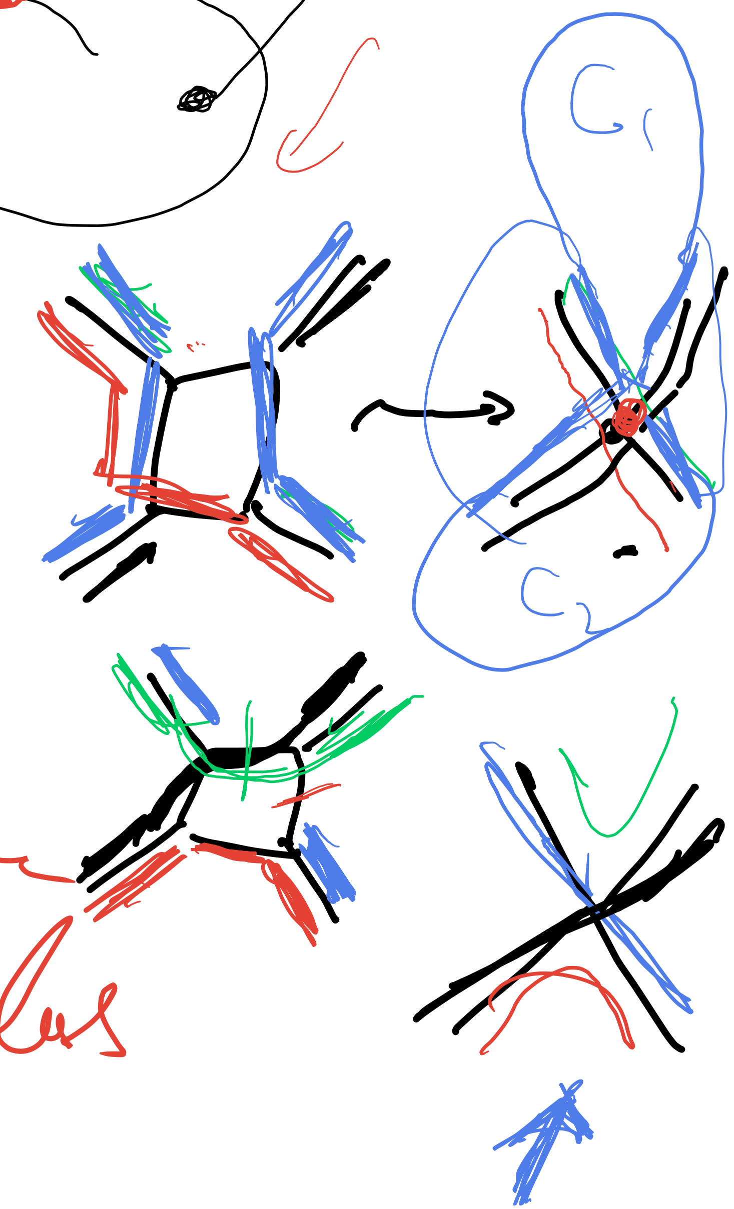
For 1: we apply the splitting lemma (and suppress vertices of degree 2) until we are left with a cubic graph. For 6: contract the 4-cycle and split, deal with the two cases. □

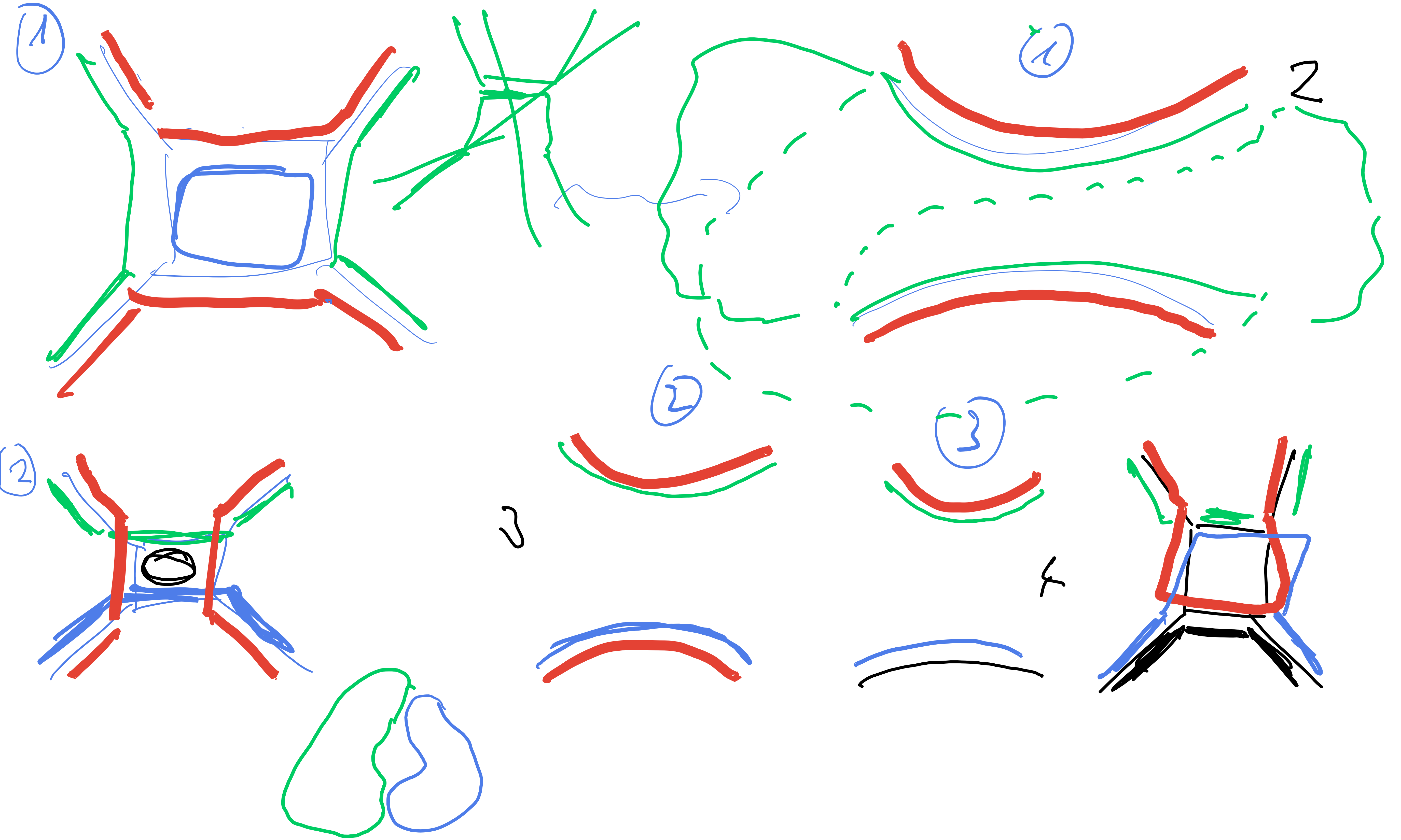


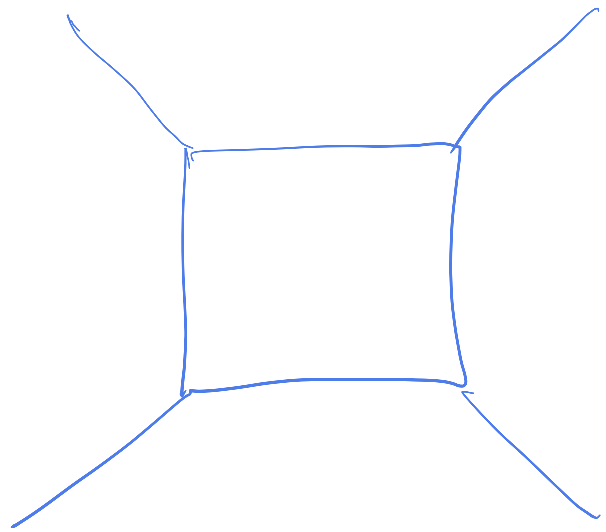


is bridgeless?

one of them is bridgeless

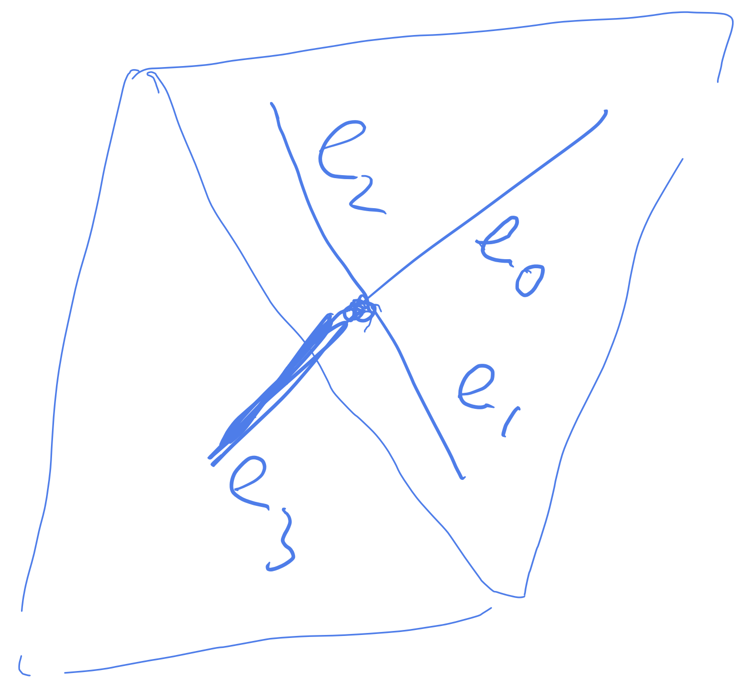



1
1
2
2
2
3
3
4



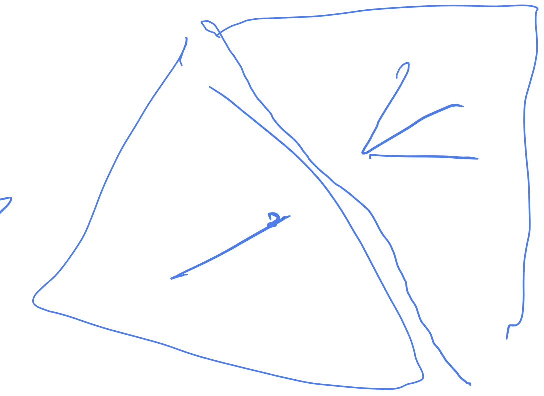
G

bridgeless



G'

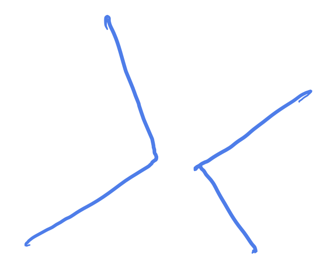
bridgeless



e_3 is a bridge in G'

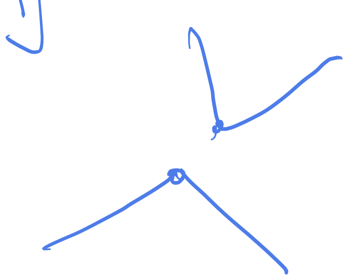
disconnected?

No.



is bridgeless

OR

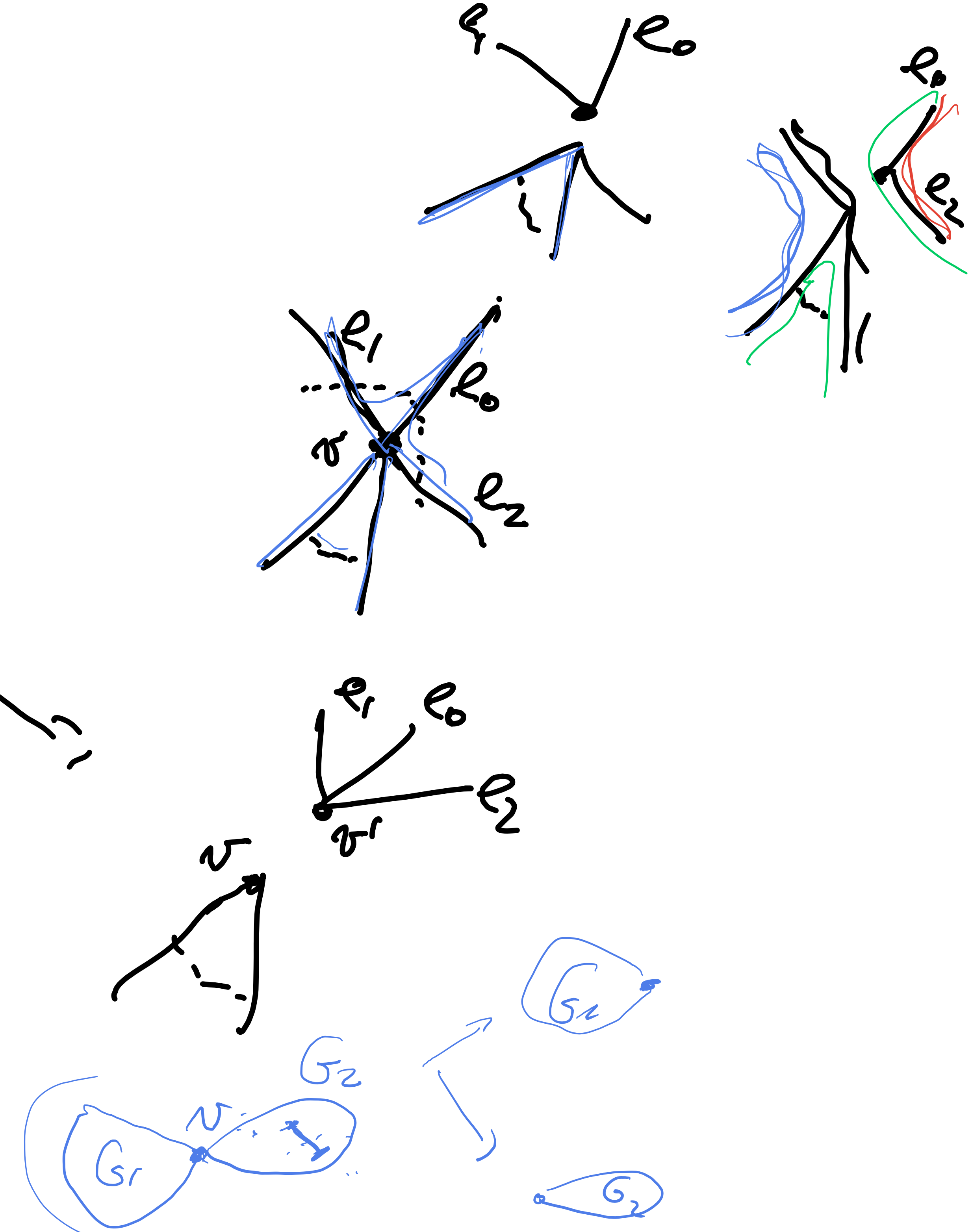


is bridgeless

Splitting lemma

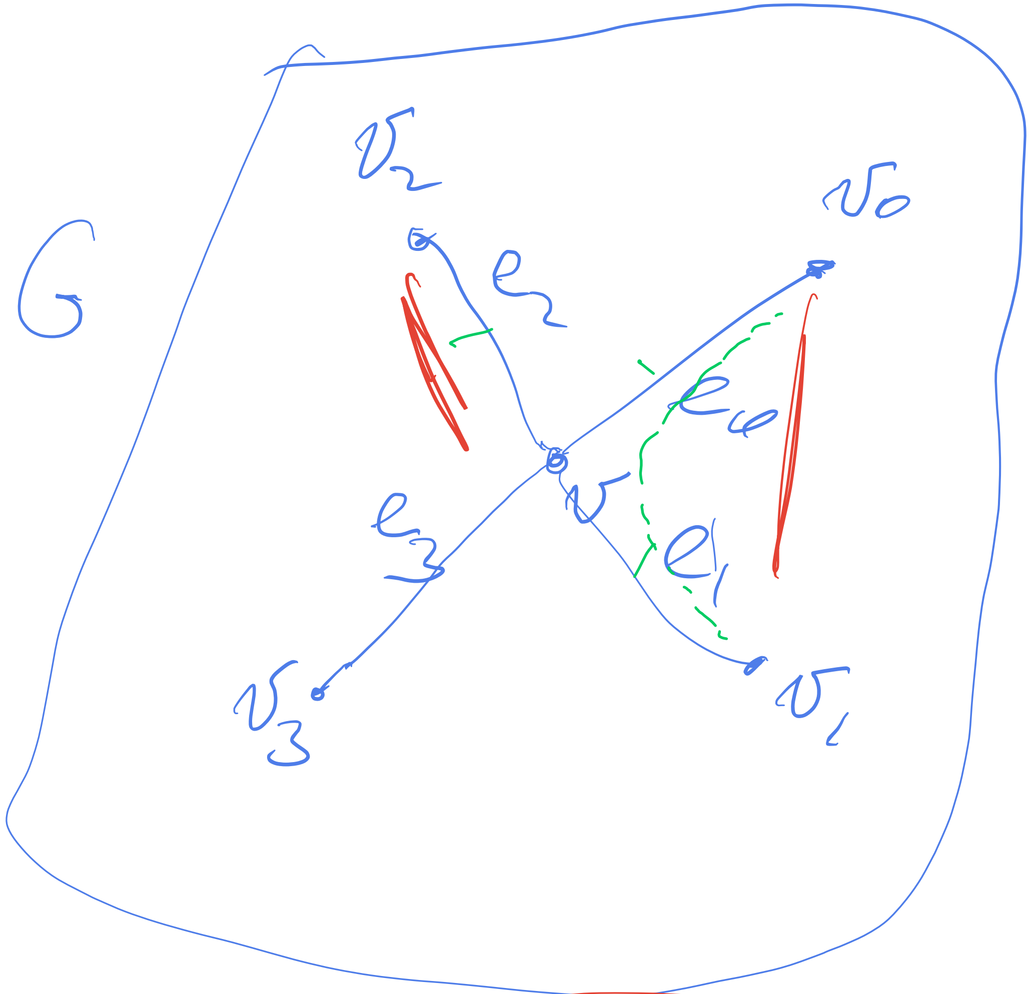
Lemma 32 (Splitting lemma, Fleischner/Mader). *Let G be a connected bridgeless graph, v a vertex with $\deg v \geq 4$, and e_0, e_1, e_2 three of its incident edges. Suppose that $G_{[v:e_0,e_1,e_2]}$ is connected. (This in particular holds whenever G is 2-connected.) Then at least one of $G_{[v:e_0,e_1]}$, $G_{[v:e_0,e_2]}$, is bridgeless connected.*

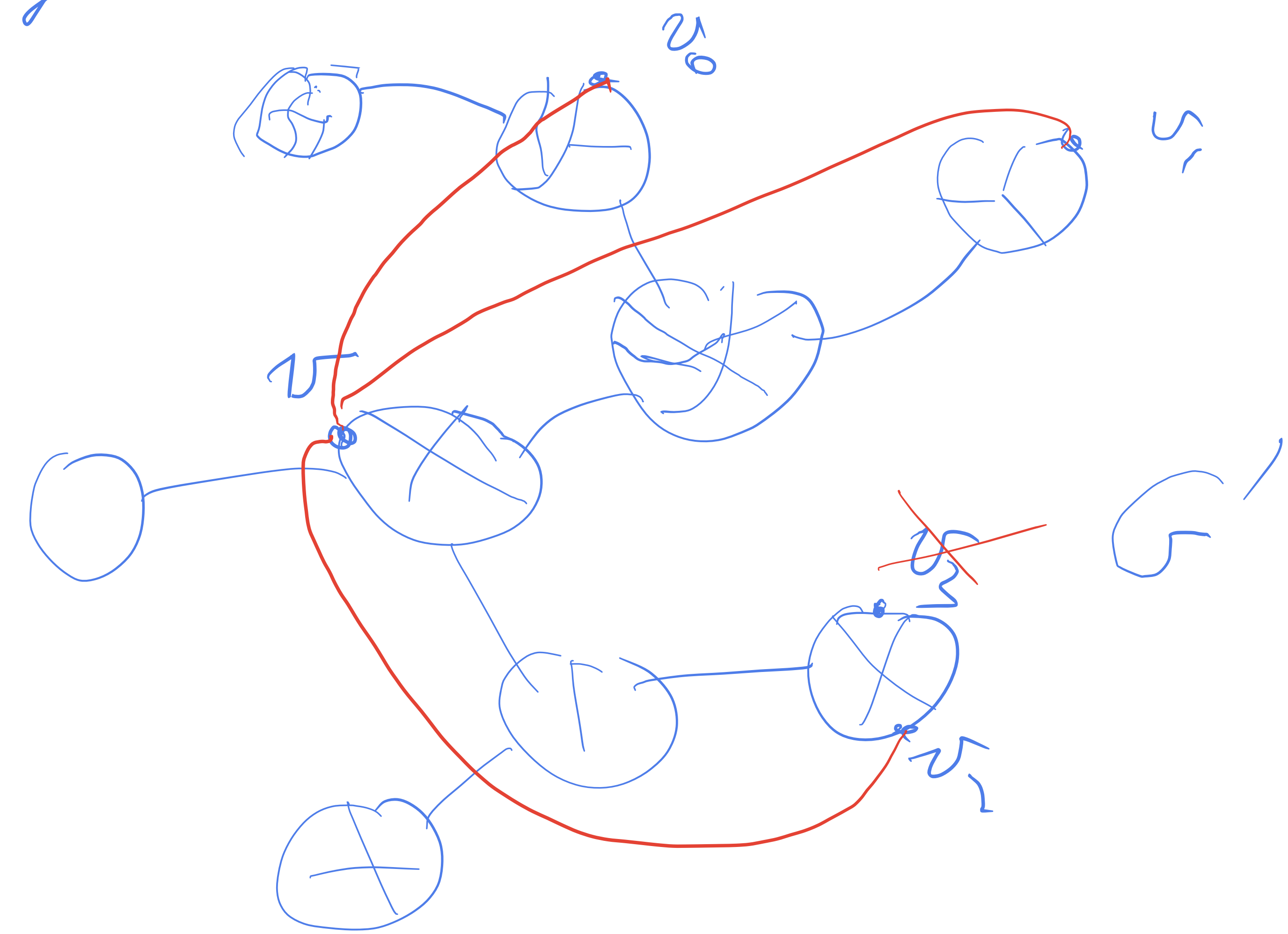
Proof. Let the edge e_i connect v with v_i. Let $G' = G - \{e_0, e_1, e_2\}$ and consider decomposition of G' into edge 2-connected blocks. Next contract each block to a vertex, what we get is a forest, say, F. Let u and u_i be the vertices of F corresponding to v and v_i $(i = 0, 1, 2)$. As G is connected and bridgeless, the same is true for $F + \{uu_i : i = 0, 1, 2\}$. (In particular, the only leaves of F are among u, u_0, u_1, u_2.) Also splitting, say, e_0, e_1 away from v corresponds to adding $F + \{uu_2, u_0u_1\}$ – for such graphs we

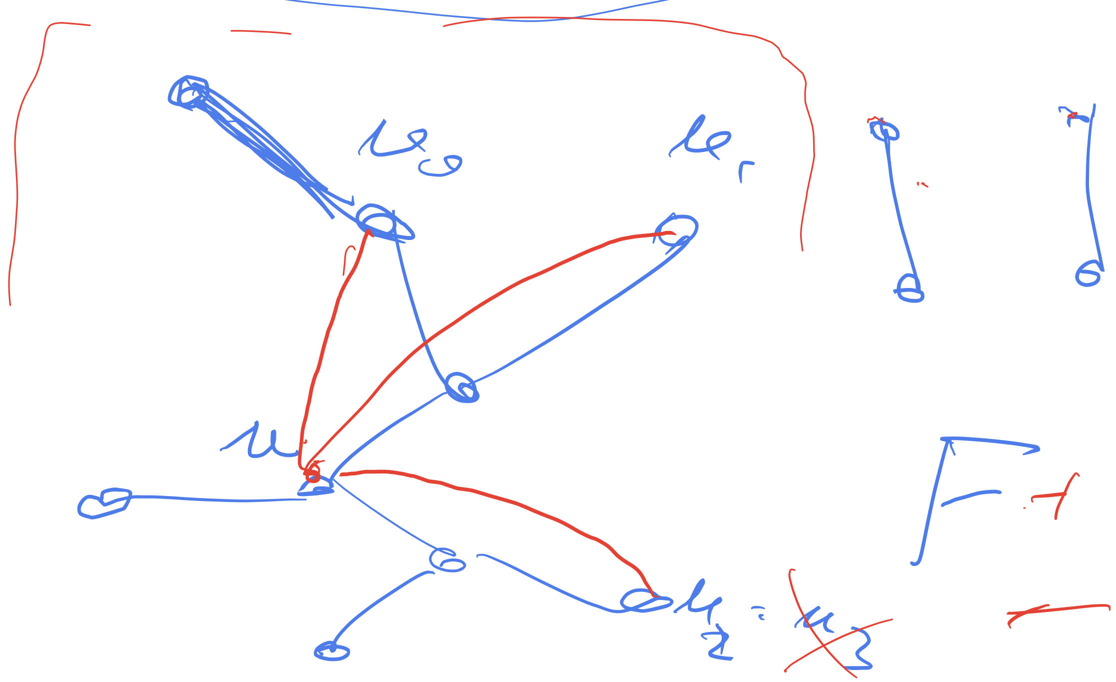


$G' = G - \{e_0, e_1, e_2\}$

edge 2-conn. blocks

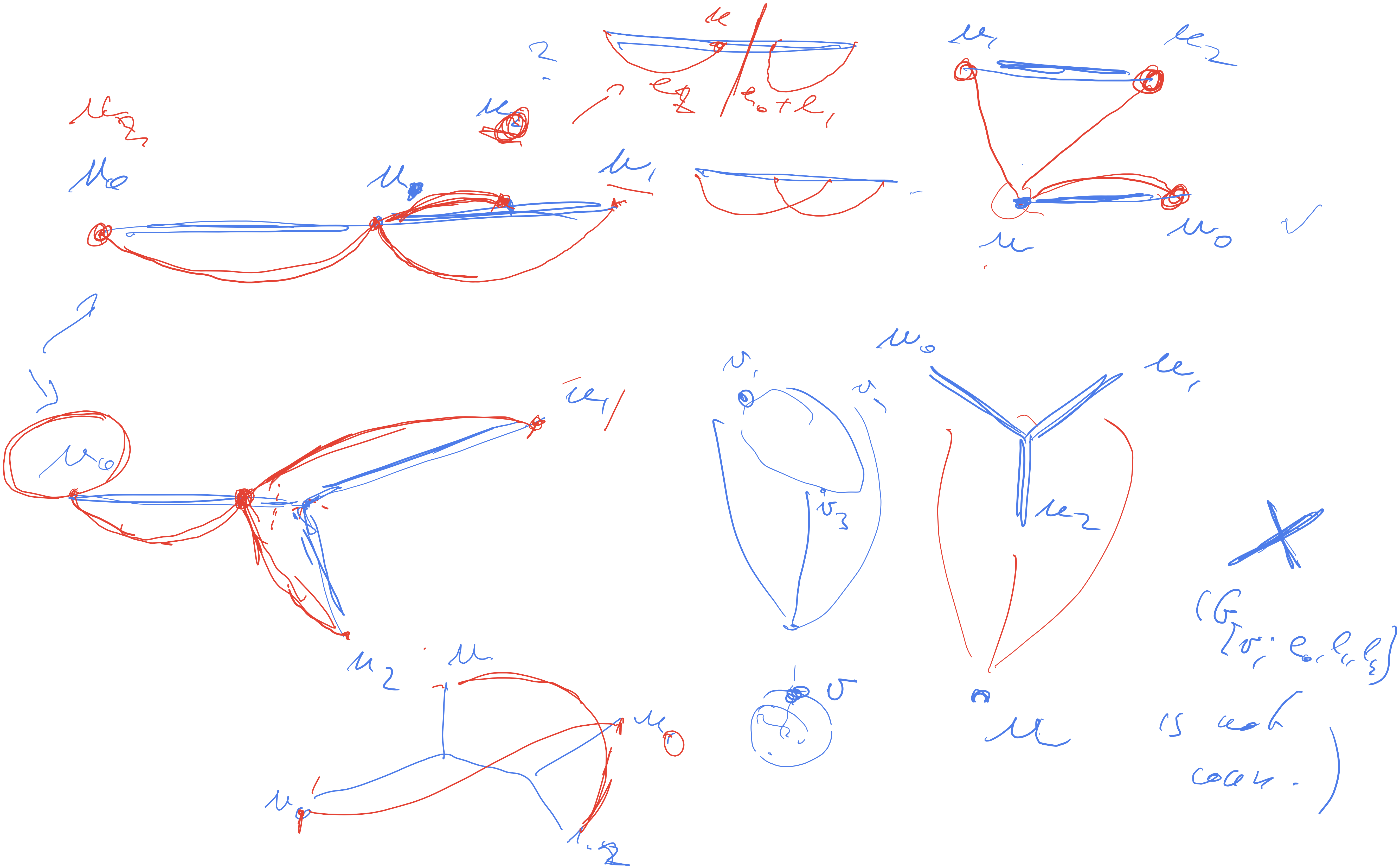






$F + e_0, e_1, e_2$ is ← conn. & bridgeless

$G = G' + |||$ is conn. & bridgeless

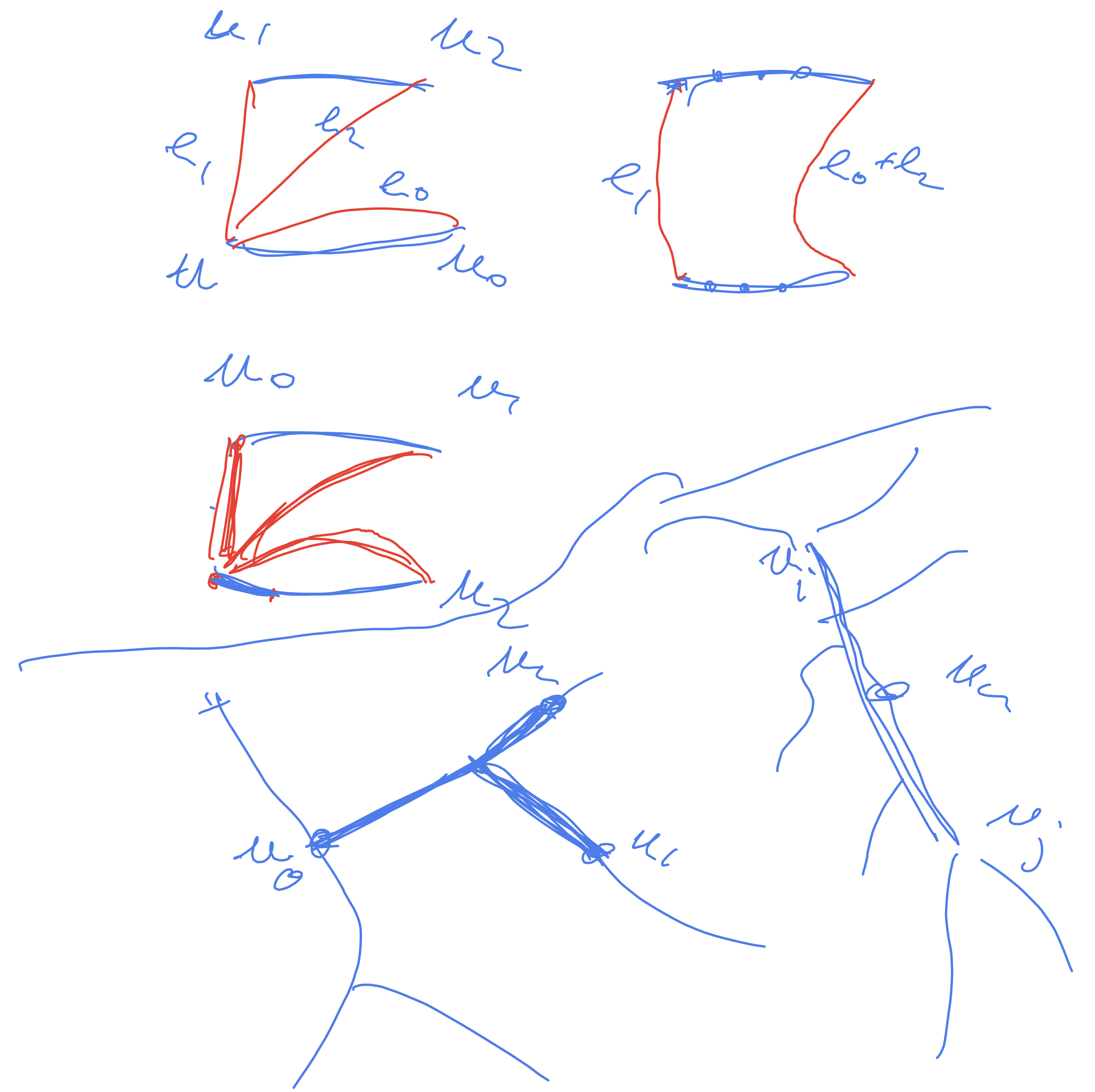


u_0
u_1
u_2
u
e_2
$e_0 + e_1$
v_1
v_3
v
(G[v; e_0, e_1, e_2]
is not
conn.)

need to check edge 2-connectivity. We have just two possibilities:

***F* is disconnected.** As $G_{[v:e_0,e_1,e_2]}$ is connected and G bridgeless, the component containing u contains also (exactly) one u_i. Moreover, this component is a path connecting u with u_i. The other important vertices (say u_j, u_k, where $\{i,j,k\} = \{0,1,2\}$) are in the other component, this component is a $u_j - u_k$ path. In this case, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity. Easily, on of these includes the desired cases (as $0 \in \{i,j,k\}$). See the first case in Figure **??**.

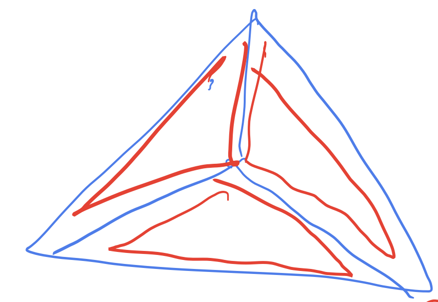
***F* is connected.** Let T be the minimal subtree containing u_0, u_1, u_2. Let $w \in T$ be such that F is T plus a $w - u$ path. There is (at least one) i such that w is in a $u_i - u_j$ and in a $u_i - u_k$ path (again, $\{i,j,k\} = \{0,1,2\}$). Again, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity, and at least



one of these is what we search for. See the second case in Figure **??**. $\square$

Shortest cycle cover problem

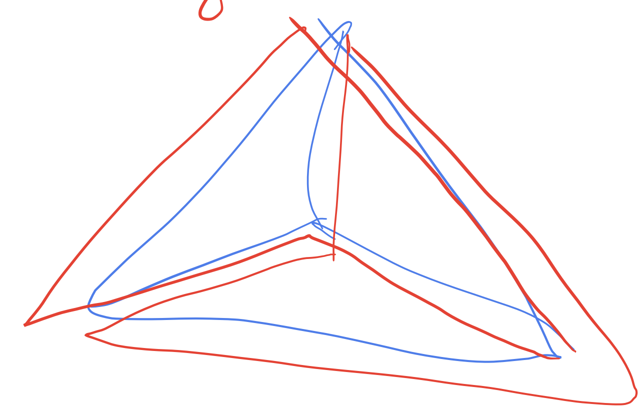
We briefly remark a related problem: the *shortest cycle cover problem*. Given a bridgeless graph G we care about a collection of cycles that covers every edge of G *at least once*. We denote by $scc(G)$ the minimal total length of such collection. Jaeger's 8-flow gives easily a 4-cover by 7 cycles; it follows that $scc(G) \le 4m$. This can be certainly improved; the best known general result is $scc(G) \le \frac{5}{3}m$ (Jamshy and Tarsi). (Better results are known for some classes of graphs, in particular for cubic graphs.) It is conjectured that $scc(G) \le \frac{7}{5}m$ and this would, if true, imply the CDC conjecture. $1.4 \times m$



$3+3+3=9$

$scc(K_4) \ge 6$

8



$4+4=8$

$\varphi : E(G) \to \boxed{\mathbb{Z}_2^3 \setminus \{(0,0,0)\}}$

$(\_,\_,\_)$

$(\varphi_1, \varphi_2, \varphi_3)$

supp φ_1 is a cycle C_1

φ_2 C_2

φ_3 C_3

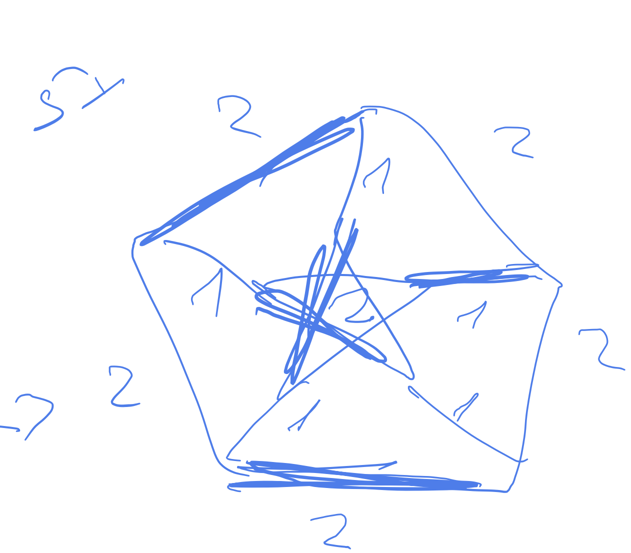
$C_1 \cup C_2 \cup C_3$ is a cc

$\le 3 \cdot n = 2 \cdot m$

$n = |V(G)|$

$m = |E(G)| = \frac{3}{2} n$

Berge-Fulkerson conjecture



Conjecture 3 (Berge, Fulkerson)**.** *If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.*

Notes:

- true in 3-edge-colorable graphs
- true for the Petersen graph
- corollary: five matchings that cover all edges
- open: constant number of matchings that cover all edges

G is 3-e-c. $\Rightarrow E(G) = M_1 \cup M_2 \cup M_3$
$\qquad\qquad\qquad\qquad\quad M_4 \quad M_5 \quad M_6$

Petersen thm: G is bridgeless $\Rightarrow \forall e \in E(G)$ is ~~covered by~~ contained in a P.M.

Matching polytope and applications

We will look at various sets of edges geometrically. That is, we consider $\mathbb{R}^{E(G)}$ as a euclidean vector space (which it is) and study various polytopes in it. For a set $M \subseteq E(G)$, we define c_M – the *characteristic vector of* M – by $c_M(e) = 1$ if $e \in M$, and $c_M(e) = 0$ otherwise.

Definition 33. *The* matching polytope *of a (multi)graph* G *is defined by*

$$MP(G) = \operatorname{conv}\{c_M : M \textit{ is a matching in } G\}\,.$$

It is not hard to see that all points c_M (for a matching M) are in fact vertices of $MP(G)$. Note that we consider non-perfect matchings too, so the zero vector is a vertex of every matching polytope.

For many application it is desirable to obtain description of the matching polytope as an intersection of halfspaces. An application for a problem related to Berge–Fulkerson conjecture will follow shortly, for an (original) application in combinatorial optimization consider the task to find a maximal matching in a graph with weighted edges. This is the same as solving a linear program over the matching polytope, and we can do this using ellipsoid method. (We only need to provide an efficient representation of the matching polytope, for details see XXX.)

For each $f \in MP(G)$ and $v \in V(G)$ we have $\sum_{e \in \delta(v)} f(e) \leq 1$ (we sum over all edges incident with one vertex), as this inequality holds for all vectors c_M. This, however does not describe $MP(G)$ completely (Exercise!).

Next, we observe that for each vertex set X of odd size, each matching uses at most
$(|X|-1)/2$ edges induced by X. Consequently, for

each such X we have inequality

$$\sum_{e \in E(G[X])} f(e) \leq \frac{|X|-1}{2},$$

satisfied for each $f = c_M$ and so for each $f \in MP(G)$. This is already enough to describe the matching polytope.

$f_e = f(e)$

$\sum_{v \in e} f(e)$

→ $f(S) := \sum_{e \in S} f(e)$

Theorem 34 (Edmonds)**.** *For every graph G we have $MP(G) =$*

$$\{f \in \mathbb{R}^{E(G)} : f(\delta(v)) \leq 1 \; \forall v \in V(G)$$
$$f(E(G[X])) \leq \frac{|X|-1}{2}$$
$$\forall X \subseteq \cancel{E}\,V(G) \text{ of odd size}\}.$$

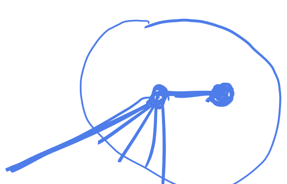
$\delta(v)$

PMP(G)

$\supseteq MP(G)$

Theorem 35. *Let $PMP(G)$ be the polytope of perfect matchings that is $PMP = \mathrm{conv}\{c_M :$*

odd X



$1 \cdot |X| = \sum_{v \in X} f(\delta(v)) = 2\underbrace{f(E(G[X]))}_{\leq \frac{|X|-1}{2}} + f(\delta(X))$

M *is a perfect matching in* $G\}$. *Then* $PMP(G) =$

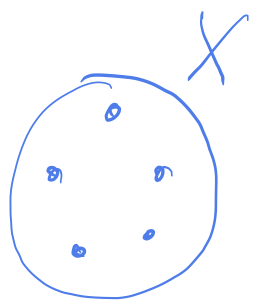
$$\{f \in \mathbb{R}^{E(G)} : \quad f(\delta(v)) = 1 \ \forall v \in V(G)$$
$$\boxed{f(\delta(X)) \geq 1}$$
$$\forall X \subseteq E(G) \ \textit{of odd size}\}\,.$$

$\Rightarrow$ f is fract. PM

$f(\delta(v)) = 1 \quad \forall v$

$\deg(v) = r$

$\frac{1}{r} \cdot r = 1$



$|X|$ odd $\Rightarrow$ $|\delta(X)| \geq r$

$\frac{1}{r} \cdot |\delta(X)| \geq 1$

$f(\delta(X))$

$\notin PMP$

r-graphs

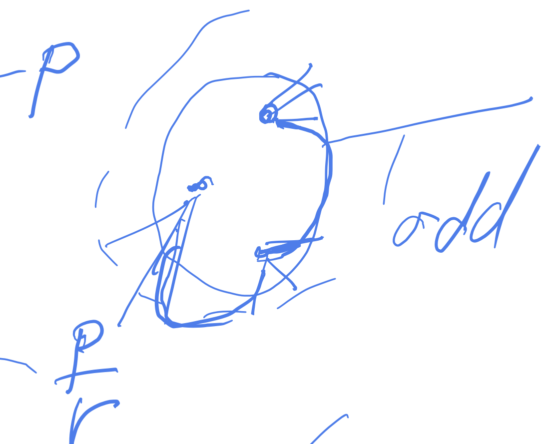
We say a graph G is an r*-graph*, if G is r-regular, and for every odd set of vertices X the size of the edge-cut $\delta(X)$ is at least r. For example, a 3-graph is the same as a bridgeless cubic graph (Exercise!).

Application 1 Every r-graph has a uniform cover by perfect matchings. That is, there is a list of perfect matchings such that each edge is in the same number of them. (Easily, this number must be $1/r$.)

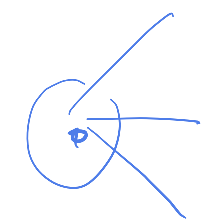
Proof. Let G be the graph and let $f(e) = 1/r$ for each edge of G. We will show that f is in the perfect matching polytope $PMP(G)$. Obviously the sum around each vertex equals 1. Now for each odd set X the size of $\delta(X)$ is at least r, which gives the other condition ☐

$\deg = r$

$|\delta(X)| > 1$ (G bridgeless)

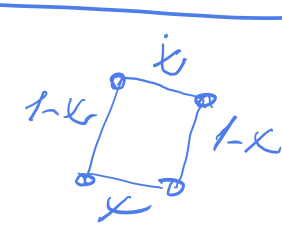


$|\delta(X)| \neq 2$



$|\delta(X)| \geq 3$

$f \in PMP = \mathrm{conv}\{\ldots\} = \sum \frac{p_i}{q} c_{M_i}$



$x\, c_{M_1} + (1-x)\, c_{M_2}$

G is 3-reg.

$|X|$ odd $\Rightarrow$ $|\delta(X)|$ odd

$x = \frac{\sqrt{2}}{2}$

Corollaries of Application 1

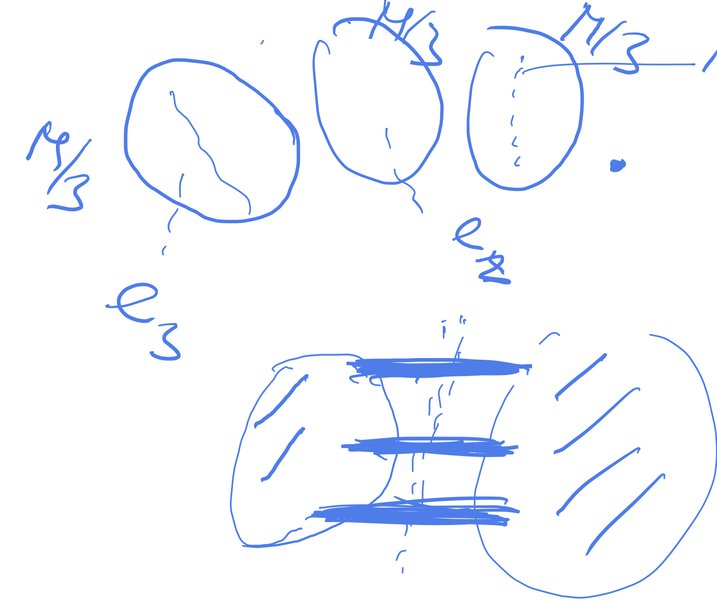
1) Every bridgeless cubic graph has a uniform cover by perfect matchings.

2) Every bridgeless cubic graph has a perfect matching. (This of course has easier proofs.) It also has a perfect matching using any given edge. (This, too, can be proved by an application of Tutte's theorem, but it's always good to have another proof technique.)

3) Every bridgeless cubic graph has a perfect matching that contains no odd cut of size 3. Indeed, every matching that is a part of the uniform cover works. Consequently, every such graph has a 2-factor that does not contain a triangle.

$(r = 3)$ M = # of PM's in the uniform cover

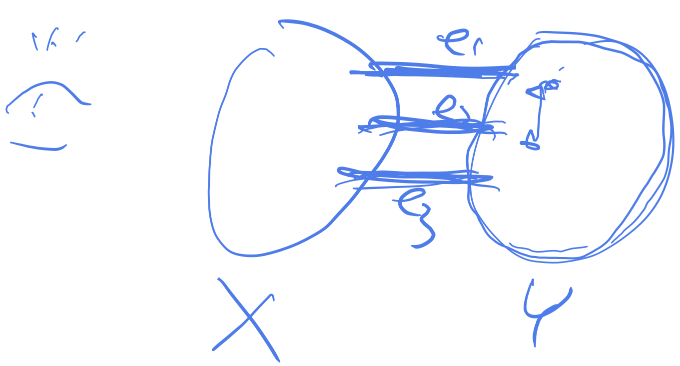
the e is in $\frac{M}{3}$ of them



$F = G \setminus P$

2-regular spanning subgr.

2-factor



$|X|, |Y|$ odd

$\exists$ PM containing ≥ 1 of e_1, e_2, e_3

(1 or 3)

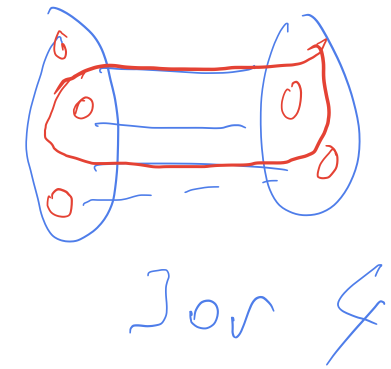
$\forall$ edge is in $\frac{1}{3}$ of the PM

$f = \sum \alpha_i c_{M_i}$ & $\sum \alpha_i = 1$ $\alpha_i \geq 0$

$f \in \mathbb{Q}^{E(G)}$ & α_i can be found by solving system of lin. eq's $\Rightarrow$ $\alpha_i \in \mathbb{Q}$, WLOG $\alpha_i = \frac{p_i}{q}$

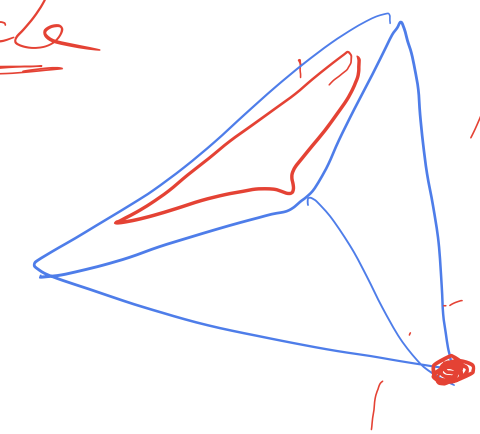
5-edge-connected factor

A more complicated result of Kaiser and Škrekovski says that every graph contains a 2-factor that intersects every 3-cut and every 4-cut. As a corollary we get the following result that is often useful for dealing with properties of flows and cycles in graphs.

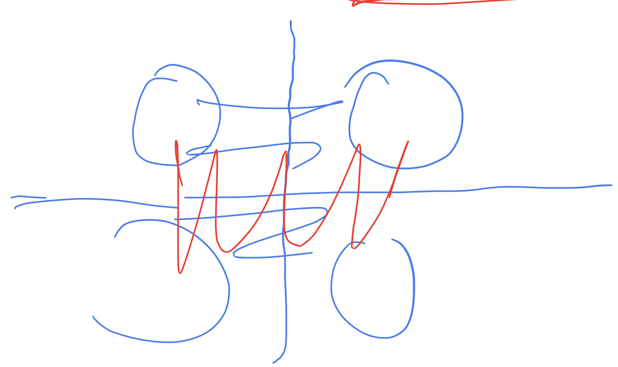


Theorem 36 (Kaiser, Škrekovski)**.** *Let G be a 3-edge-connected graph. Then G contains a cycle C such that the graph G/C (where each component of C is contracted to a vertex) is 5-edge-connected.*

2-factor → ? cycle



(The proof is essentially a cut-uncrossing argument.)



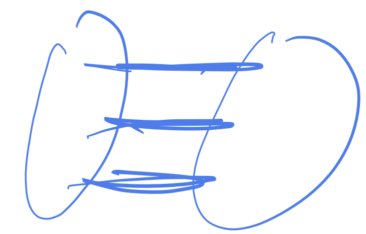
Application 2

Theorem 37 (Kaiser, Král̆, Norine)**.** *Every bridgeless cubic graph G has perfect matchings M_1, M_2 such that $|M_1 \cup M_2| \geq \frac{3}{5}|E(G)|$.*

true for 3-edge-col.

$0.6 < 0.666\ldots = \frac{2}{3}$

Proof. First use Application 1, namely the third corollary: Let M be a perfect matching that contains no odd cut of size 3. Define $f(e) = 1/5$ for $e \in M$ and $f(e) = 2/5$ elsewhere.



not in M

M -- PM $\rightsquigarrow$ $f \in \mathbb{R}^{E(G)}$

$f(e) = \frac{1}{5}$ $e \in M$

$= \frac{2}{5}$ $e \notin M$

We check that $\boxed{f \text{ is in } PMP_G.}$ The sum around each vertex is 1. If X is an odd-size vertex set, then $|\delta(X)|$ is odd, therefore either 3, or at least 5. In the latter case, $\sum_{e\in\delta(X)} f(e) \geq 5 \cdot \frac{1}{5} = 1$, which we need. In the former case, we know by the choice of M that exactly one of the edges in $\delta(X)$ is in M, therefore $\sum_{e\in\delta(X)} f(e) = \frac{1}{5} + \frac{2}{5} + \frac{2}{5} = 1$.

1) $f(\delta(v)) = 1$

2) 

$|\delta(X)|$ odd > 1

$= 3$, ≥ 5

$f(\delta(X)) \geq 5 \cdot \frac{1}{5} = 1$

As f is in the perfect matching polytope, f is a convex combination of c_{M_i} for some perfect matchings M_i. Put $S = E(G) \setminus M$. By definition of f, we

$f(\delta(X)) = \frac{1}{5} + \frac{2}{5} + \frac{2}{5}$

have $f(S) = \frac{2}{5}|S|$, hence $c_{M_i}(S) \geq \frac{2}{5}|S|$ for some M_i involved in the convex combination for f. Now $|M \cup M_i| = |E(G)| \cdot (\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{5}) = \frac{3}{5}|E(G)|$. $\square$

The above may be generalized as follows. For a graph G define $m_i(G)$ to be the maximum fraction of edges that can be covered by a union of i perfect matchings – that is

$$m_i(G) := \max\{\frac{|M_1 \cup \cdots \cup M_i|}{|E(G)|} : M_i \text{ are perfect matchings}\}$$

So we found that $m_2(G) \geq 3/5$ for every 3-graph G, and this bound is attained for the Petersen graph. [KKN] did further find that $m_3(G) \geq 27/35$ for a 3-graph G. If Berge-Fulkerson conjecture is true, we have $m_5(G) = 1$.

Exercises: 1. Prove that a 3-graph is the same as a bridgeless cubic graph. **2.** Find upper and lower bounds for $m_3(G)$ when G is a cubic bridgeless graph. (Note that $m_3(G) \geq 27/35$ is the best known so-far.)

3. Find some bounds on $m_i(G)$ for a general i, and use this to estimate number of perfect matchings needed to cover all edges of a graph G.

Now we give the postponed proof of Theorem 35.

Proof. Let P_G be the polytope defined by the inequalities (**??**). Easily $PMP_G \subseteq P_G$, as all vertices of the perfect matching polytope (i.e., all c_M for a perfect matching M) satisfy the inequalities (**??**). For the other inclusion, we proceed by contradiction: we take the graph G with smallest $|V(G)|+|E(G)|$, and one vertex f of P_G such that $f \notin PMP_G$.

We have $0 < f(e) < 1$ for each edge e of G. If $f(e) = 0$ for some edge e, we let $G' = G - e$ and f' to be the restriction of f to $E(G')$. It is easy to check that $f' \in P_{G'}$, and as G' is smaller than G, we have $P_{G'} = PMP_{G'}$ and f' is a convex combination of characteristic vectors of perfect matchings of G'. When we take these matchings as perfect matchings of G (by extending the characteristic vector by a 0 in the coordinate indexed by e), we get $f \in PMP_G$, a contradiction.

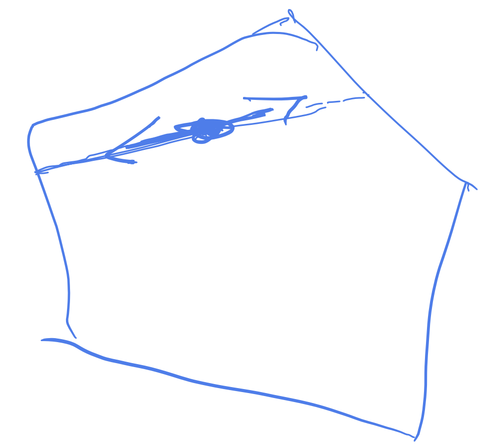
On the other hand, if $f(e) = 1$ for some edge $e = uv$, then we put $G' = G - u - v$. Again, we let $f' = f|_{E(G')}$ and we check that $f' \in P_{G'} = PMP_{G'}$. By extending all the perfect matchings that occur in the convex combination for f' by the edge e we get perfect matchings whose convex combination is f, again a contradiction.
G has no vertices of degree ≤ 1. G certainly does not have isolated vertices (by inequality (**??**)), and if v is a vertex incident only with an edge e, then $f(e) = 1$, which we already disproved. Consequently, $|E(G)| \geq |V(G)|$.
Case 1. $|E(G)| = |V(G)|$ G is 2-regular, thus a disjoint union of circuits. None of these is odd (otherwise we let X be the set of vertices of an odd circuit and get a contradiction with inequality (**??**)). For even circuits it is easy to ... (Exercise!).
Case 2. $|E(G)| > |V(G)|$ As f is a vertex of a

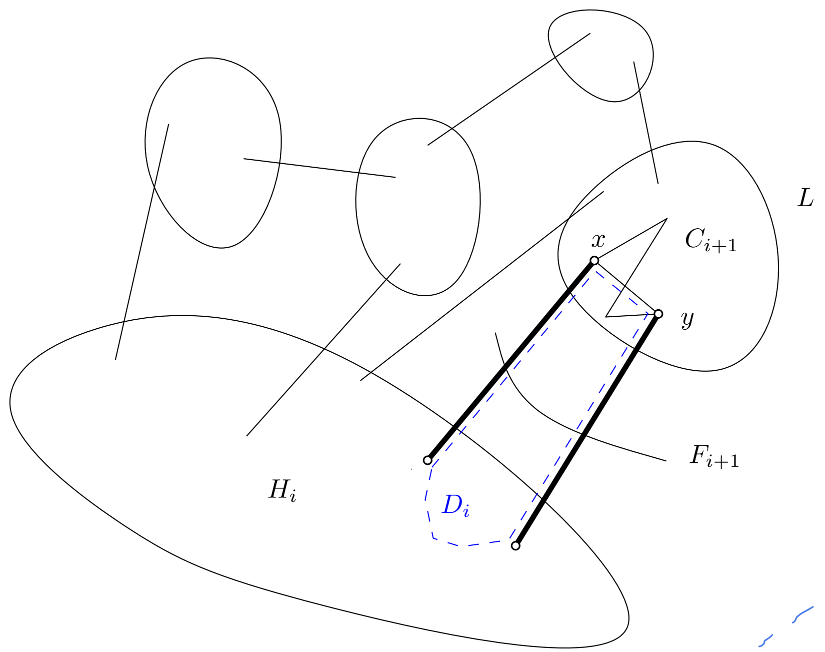
polytope in $\mathbb{R}^{E(G)}$, at least $|E(G)|$ of the inequalities are satisfied with an equality. (Exercise!) Thus, one of them must be (*) $\sum_{e\in\delta(X)} f(e) = 1$ for some $X \subseteq V(G)$, such that $1 < |X| < |V(G)|$ and $|X|$ is odd. As $|X|$ is odd, every perfect matching of G contains an edge of $\delta(X)$. This together with (*) implies that each of the sought-for matchings involved in the representation of f contain exactly one edge of $\delta(X)$. This suggest that we may want to treat X as a single vertex: if there is a representation for f, then this change of the graph will transform them in matchings.

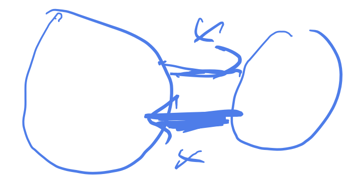
To put this formally, we let $G_1 = G/X$ – all vertices of X are identified to a single vertex, we keep possible multiedges) – and $G_2 = G/\bar{X}$ (where $\bar{X} = V(G) \setminus X$). Again, let f_i be the restriction of f to the edge-set of G_i ($i = 1, 2$). It is easy to check that $f_i \in P_{G_i}$, which implies (Exercise!) that there are perfect matchings $(M_{i,k})_{k=1}^{N}$ of G_i such that

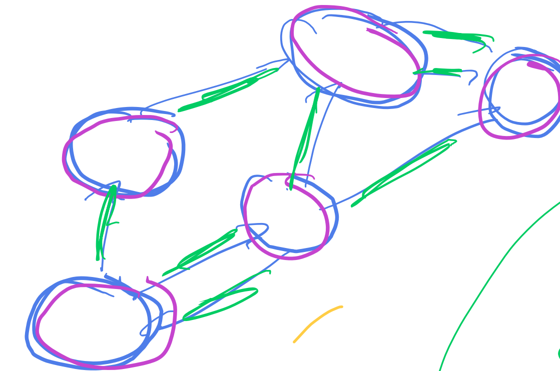
$$f_i = \frac{1}{N} \sum_{k=1}^{N} c_{M_{i,k}} . \tag{2}$$

Recall that each $M_{i,k}$ contains exactly one of the edges of $\delta(X)$ (we abuse the notation slightly, we identify the edges of $\delta(X)$ in G, and the corresponding edges of G_1, G_2). Moreover, if e is one of these edges, then the number of perfect matchings $M_{i,k}$ of G_i for which $e \in M_{i,k}$ is $Nf_i(e)$ (just look at the e-th coordinate of (2)). However, $Nf_1(e) = Nf_2(e) = Nf(e)$ (recall f_i was defined as a restriction of f to $E(G_i)$). Consequently, we may pair up the matchings of G_1 and of G_2 to agree on the edges of $\delta(X)$, indeed we may assume that $M_{1,k}$ and $M_{2,k}$ contain the same edge from the cut Z. We put $M_k = M_{1,k} \cup M_{2,k}$. It is easy to check that f is the average of c_{M_k}, which finishes the proof. $\square$









$f = 1$

$g : E(G) \to \mathbb{Z}_3$

$g \neq 0$ on green

$f = 0$ on green

R. Diestel: Graph Th.

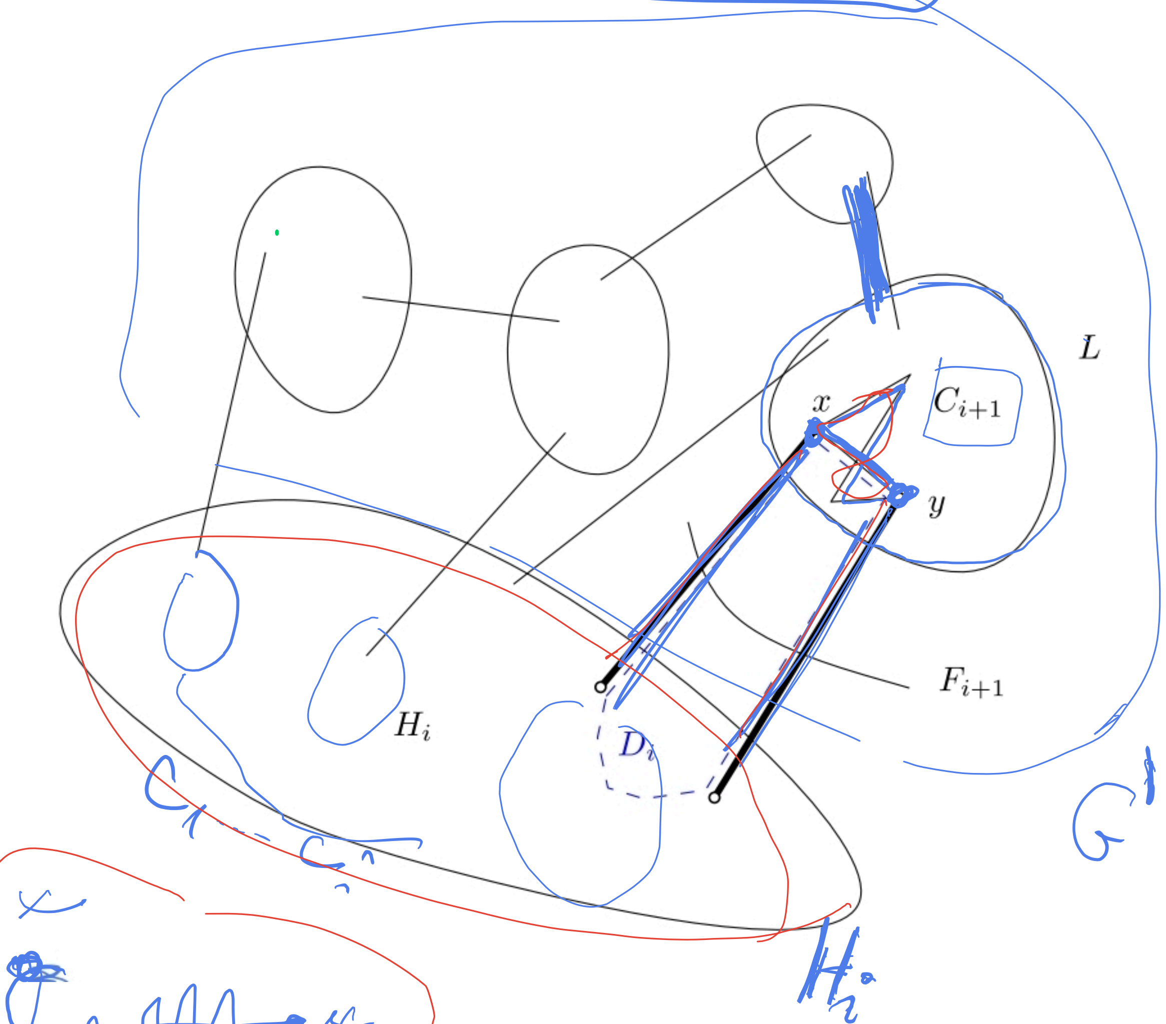
Theorem 38 (Seymour)**.** *Every bridgeless graph G has a 6-NZF.* (we saw: 8-NZF, open: 5-NZF)

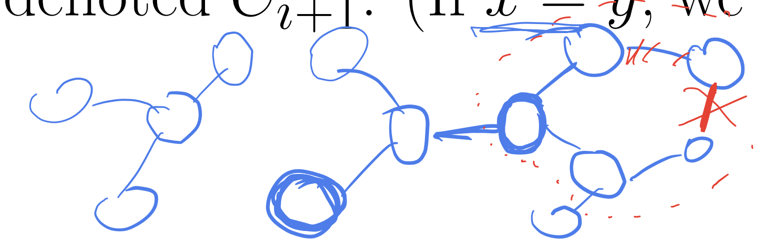
Proof. Equivalently, we will show it has NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow. First, we can assume that G is 3-edge-connected (with the same proof as in the case of 8-NZF). We will find a spanning cycle C and carefully chosen edges between various components of C. The plan is to use a $\mathbb{Z}_2$-flow f with support $E(C)$ and a $\mathbb{Z}_3$-flow g that is NZ outside of $E(C)$.

$H_0 = C_0$ — C_i is a cycle or a single vertex

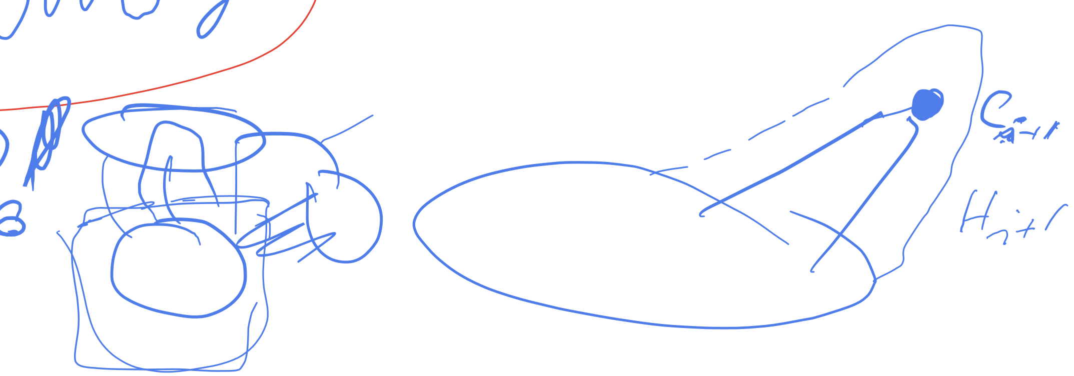
We will recursively define subgraphs $(H_i)_{i\geq 0}$ of G, cycles $(C_i)_{i\geq 1}$ and sets of edges $(F_i)_{\geq 1}$. To start, let C_0 H_0 be any vertex of G. If H_i is defined, we consider a decomposition of $G' = G - V(H_i)$ into 2-edge-connected components–blocks. (If $V(G) = V(H_i)$, we stop and put $n := i$.) The structure of this decomposition is such that after contracting each of the blocks, we obtain a forest. We take any leaf of this forest at let L be a block of G' corresponding to it.

We observe that $|\delta_{G'}(L)| \leq 1$ (by the choice of L), while $|\delta_G(L)| \geq 3$ (as G is 3-edge-connected). This implies there are at least two edges connecting L with H_i, we let F_{i+1} be the set of some two of them, and x, y be the ends of those edges in L. As L is 2-edge-connected, there are two edge-disjoint $x-y$ paths, and their union is a connected cycle, let it be denoted C_{i+1}. (If $x = y$, we may choose C_{i+1} to be $\{x\}$



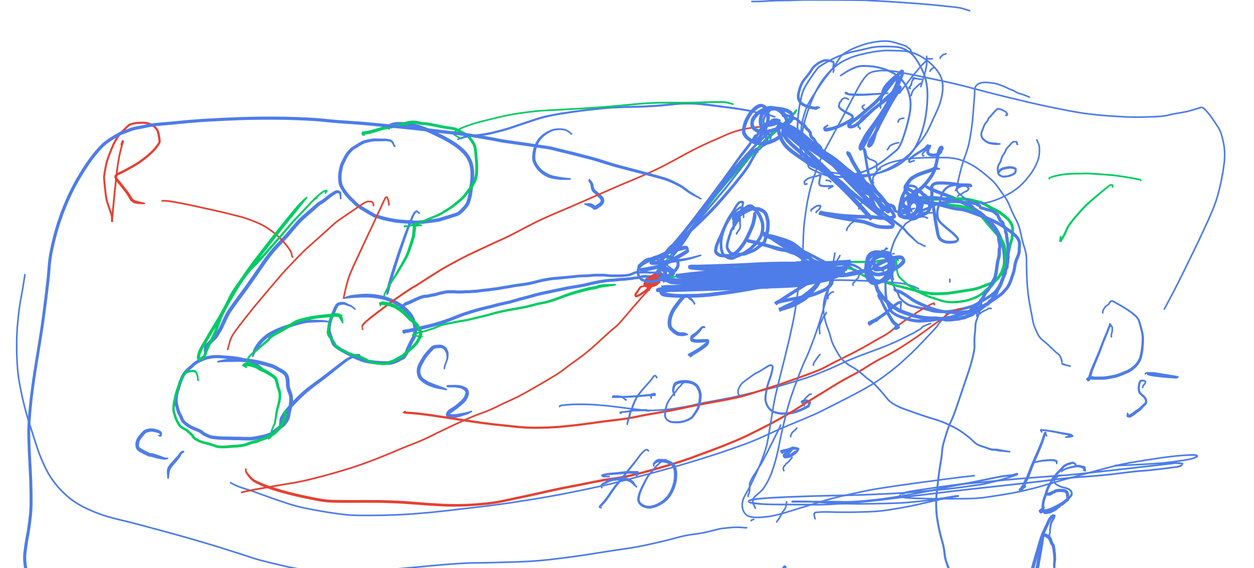


$G - V(H_i) = G'$



empty.) We put $H_{i+1} = H_i + C_{i+1} + F_{i+1}$ (We do not add spanned edges.)

We let $C = \cup_{i=1}^n C_i$, $F = \cup_{i=1}^n F_i$, $H = H_n$. All edges of G are of three types: $E(C)$, F, and the rest, denoted by R. As claimed above, C is a ~~spanning~~ cycle, so it is easy to take a $\mathbb{Z}_2$-flow with support $E(C)$. We now define a $\mathbb{Z}_3$-flow that is non-zero on $R \cup F$



(but it may have isolated vertices, also vert. of degree > 2

We observe (by induction on i) that all graphs H_i are connected, so we take a spanning tree $T \subset H$ and let g_n be a $\mathbb{Z}_3$-flow that equals 1 on $E(G) \setminus E(T)$. Next, we define $g_{n-1}, \ldots, g_0$ so that each g_k is a $\mathbb{Z}_3$-flow that is nonzero on R and on F_j with $j > k$. If g_{i+1} is already defined, we consider a cycle D_i containing both edges of F_{i+1}, some $x - y$ path in C_{i+1} and any path in H_i that connects the other ends of the edges of F_{i+1}. Let φ_i be a $\mathbb{Z}_3$-flow that is nonzero on D_i. Consider flows $\boxed{g_{i+1} + \alpha\varphi_i}$ $\neq 0$

$\neq 0$ on R, on some edges of $C \cup F$ it will be 0

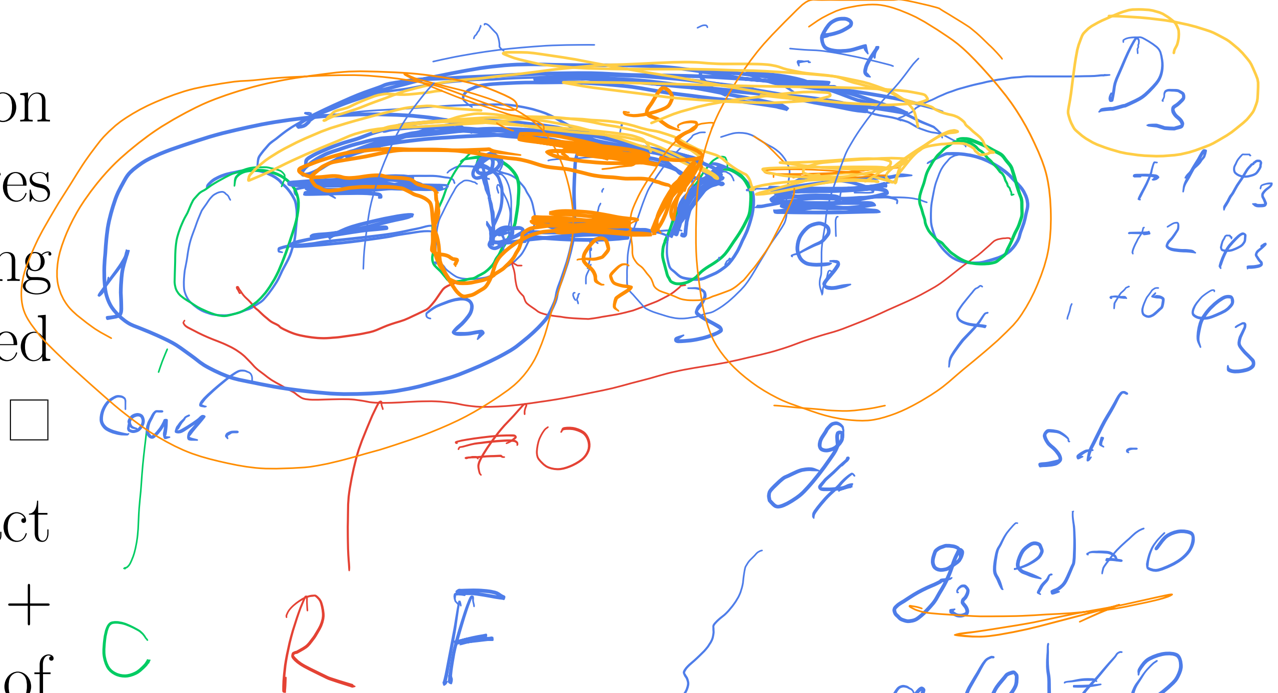
$E(D_i) \cap R = \emptyset$

$= F_i$ + some edges in H_i + part of C_{i+1}

for $\alpha = 0, 1, 2$. At least one of them is nonzero on both edges of F_{i+1}, while we didn't change edges of R neither of $F_{>i+1}$. Consequently, the mapping $g = g_0$ is nonzero on $R \cup F$ and (f, g) is the desired $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF on G. $\square$

Notes: 1) Recall the standard proof of the fact that graph of maximum degree at most k is $(k+1)$-colorable. The second phase of the above proof is an analogue of this for $k = 2$. Indeed, if the graph G/C (each component of C is contracted to a vertex) is planar, then we are using the fact that the dual $(G/C)^*$ is 2-degenerate. As we saw, the argument works even for non-planar graphs. The nontrivial part is, of course, to find the cycle C such that G/C has this degenerate property.

2) It's tempting to try and use similar ideas to get a 5-flow conjecture. For this, one may say the above proof in an alternative way: we find a 2-flow f and



$+1\varphi_3$
$+2\varphi_3$
$+0\varphi_3$

g_4 s.t.

$g_3(e_1) \neq 0$
$g_3(e_2) \neq 0$

$g_3 = g_4 + \alpha\varphi_3$

D_2, φ_2 $g_2 = g_3 + \alpha_2\varphi_2$

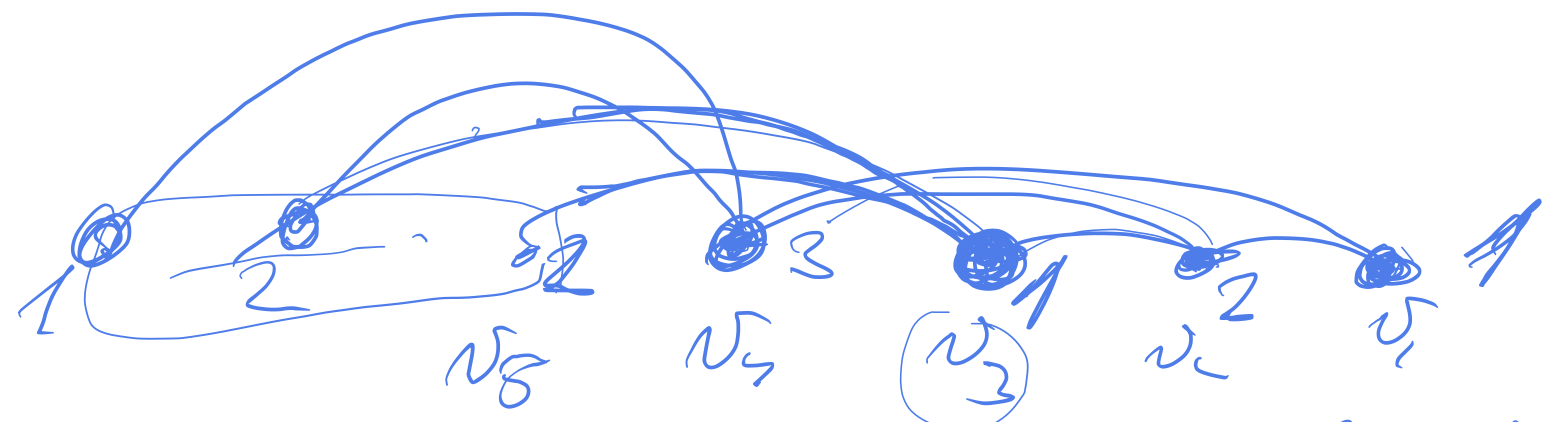
$g_2(e_3), g_2(e_4) \neq 0$

$g_2(e_1) = g_3(e_1) \neq 0$
$g_2(e_2) = g_3(e_2) \neq 0$

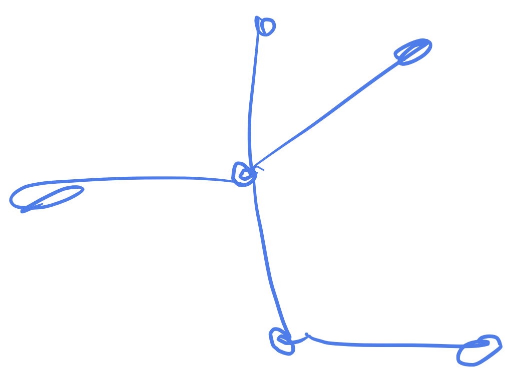
If G is $\overset{k}{2}$-degenerate $\Longrightarrow$ $\chi(G) \le 3^{\,k+1}$

$\forall H \subseteq G \;\; \exists v \in H : \deg_H v \le 2$

degeneracy $= \max\limits_{H} \min\limits_{v} \deg_H v$



"degree to the left" ≤ 2



tree is 1-degenerate

$\Downarrow$

2-colorable

$\chi(G) \le \Delta(G) + 1$

a 3-flow g that are not both equal to zero at the same edge. Then $2g + f$ is a NZ 6-flow. Now one may try to find a 2.5-flow g instead that is a real-valued flow such that $1 \le g(e) \le 1.5$ for each edge e for which $f(e) = 0$. This would indeed produce a 5-flow. Exercise: discuss why does the above approach fail.

Exercises: 4. Describe PMP_G when G is a disjoint union of even circuits.

5. Let P be a polytope $\{x \in \mathbb{R}^d : Ax \le b\}$. Let V denote the vertices of P. Let x be a point of P.

(a) x is a convex combination of at most $d + 1$ elements of V.

(b) If A and b have rational entries then x is a convex combination of some elements of V *with rational coefficients.*

(c) There is a list $v_1, \dots, v_n$ of vertices from V (possibly with repetition), such that $x = (v_1 + \cdots +$

(f', g') is NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow

$2g+f$ is a NZ 6-flow

f — 2-flow $-1, 0, 1$

g — 3-flow $-2, -1, 0, 1, 2$

$2g$ $-4, -2, 0, 2, 4$

$-5, -3,$ $\cancel{0}$ 5

g : $\mathbb{R}$-flow

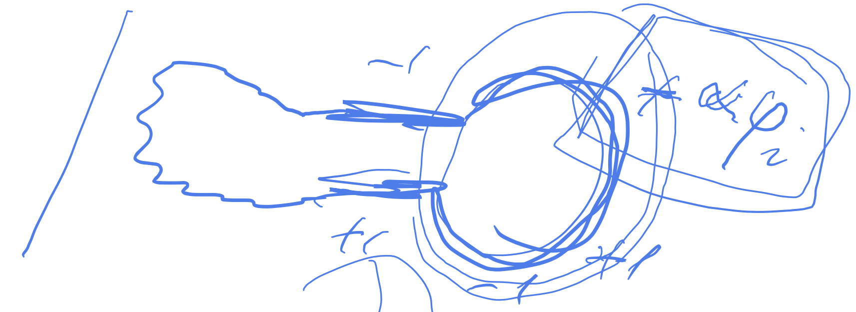
$\forall e : -1.5 \le g(e) \le 1.5$

& $f(e) = 0 \to$ $|g(e)| \ge 1$?

$-4 \le 2g + f \le 4$

$f: -1, 0, 1$ ——— $f(e) = \pm 1$ for $e \in C$
$\ 0$ for $e \notin C$

g



$h = f + 2g$ is an $\mathbb{R}$-flow s.t. the $1 \le |h(e)| \le 4$

$\Updownarrow$

$\exists$ a $\mathbb{Z}$-flow s.t.

$-4 \le f + 2g \le 4$

$|f + 2g| \ge 1$

always $g(e) > 0$

$f = 0 \quad 0.5 \le g \le 2$

$g = 1$ or 2

$f = -1 \quad 1 \le g \le 2.5$

$f = +1 \quad 0 \le g \le 1.5$

① find g_u s.t. $g_u(e) = 1$ for $e \notin E(T)$

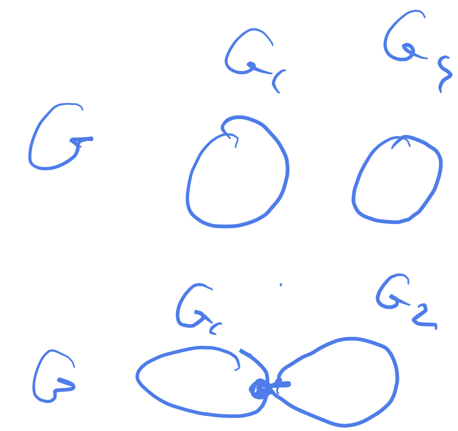
...... cannot do for $\mathbb{Z}$-flows

$0 \le g(e) \le 2$

$\ne 0$ for $e \in T$

$v_n)/n$.

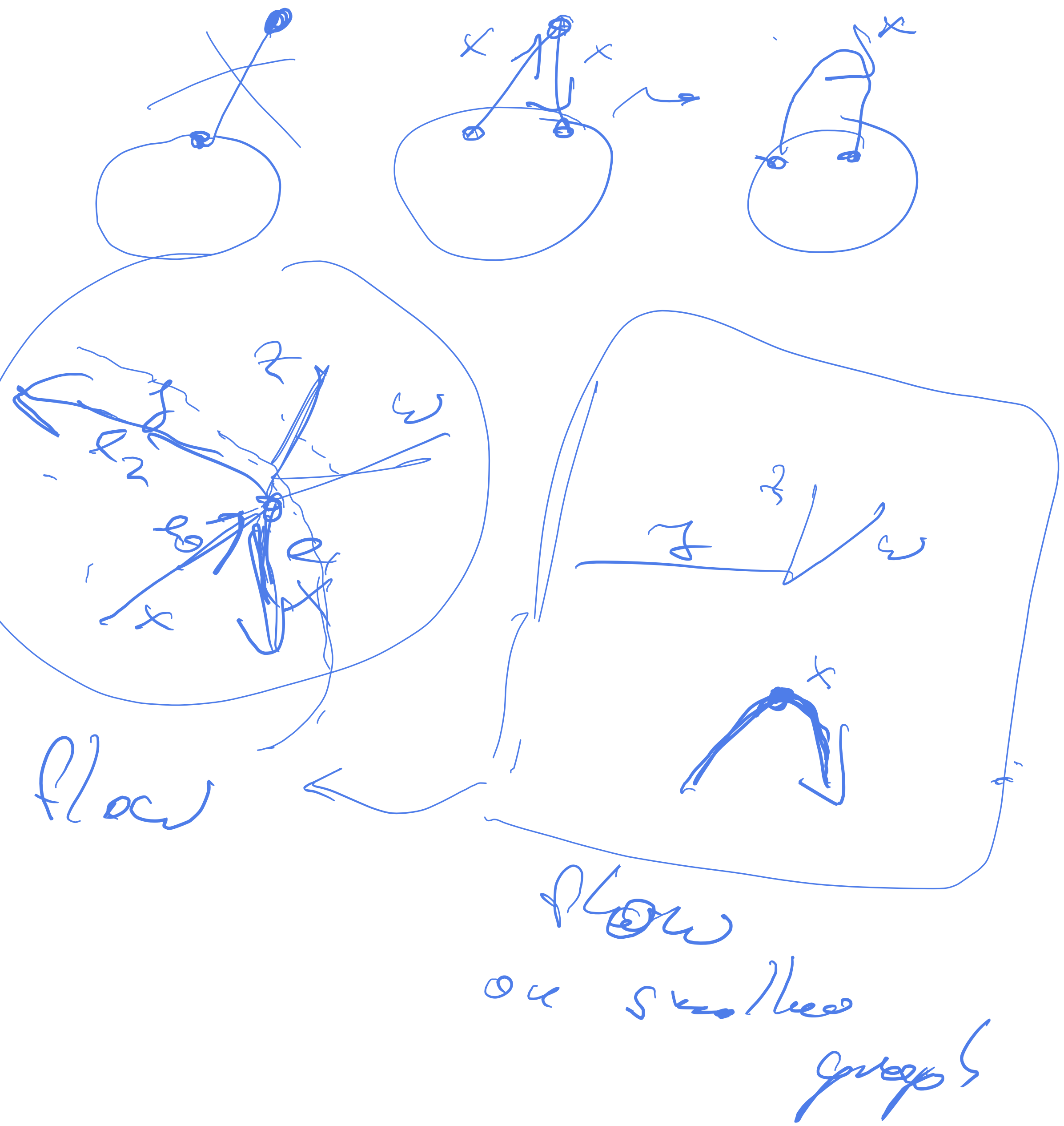
6. * Try to modify the proof of 6-NZF theorem to work for 5-NZF (as indicated in the notes below the proof). Describe what makes this approach fail. (If you succeed in proving the existence of 5-NZF, let humanity know! – see the list of open problems ...)



In the last section we saw Seymour's proof of the existence of NZ 6-flow. Tutte's 5-flow conjecture is still elusive, let us however look at some simple observations. In the following, G is a minimal counterexample to the conjecture. Explicitly: G is a bridgeless graph that admits no NZ 5-flow and among such graphs G has the smallest $|V(G)| + |E(G)|$.

(1) G is 2-connected Suppose not; then $G = G_1 \cup G_2$ where graphs G_1 and G_2 share ~~just~~ at most one vertex, and both are bridgeless. By minimality of G, both G_1 and G_2 admit a NZ 5-flow, thus G has it, too.

(2) G is cubic Suppose not, let v be a vertex such that $\deg v \neq 3$. If $\deg v = 1$, then G has a bridge, contradiction. If $\deg v = 2$, then we can suppress this vertex (contract one of its incident edges). The graph we obtain is smaller, so has a NZ 5-flow, which is easily extended back to G. Finally, let $\deg v \geq 4$, let (as in the Fleischner's lemma), e_0, e_1, e_2 be three of the incident edges. As G is 2-connected, the lemma implies that one of graphs $G_i = G_{[v:e_0e_i]}$ $(i = 1, 2)$ is bridgeless. After suppressing the newly created vertex of degree 2, we get a graph G_i' that has the same number of vertices as G but one edge less – thus it admits a NZ 5-flow f_i. It is easy to extend it back to G_i and then to G, which yields contradiction.

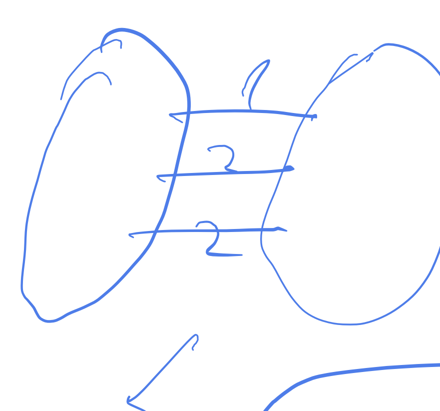


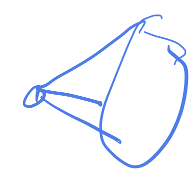
(3) G is edge 3-connected Suppose not, let $A \subseteq V(G)$ be such that $|\delta(A)| = 2$, say $\delta(A) =$

$\{e, e'\}$. Now $G' = G/e$ is smaller then G, thus it admits a NZ 5-flow f. We extend it to G by letting $f(e) = \pm f(e')$ (the sign is chosen according to the orientation of e, e'). As we saw already in several occasions, this extension yields a flow.

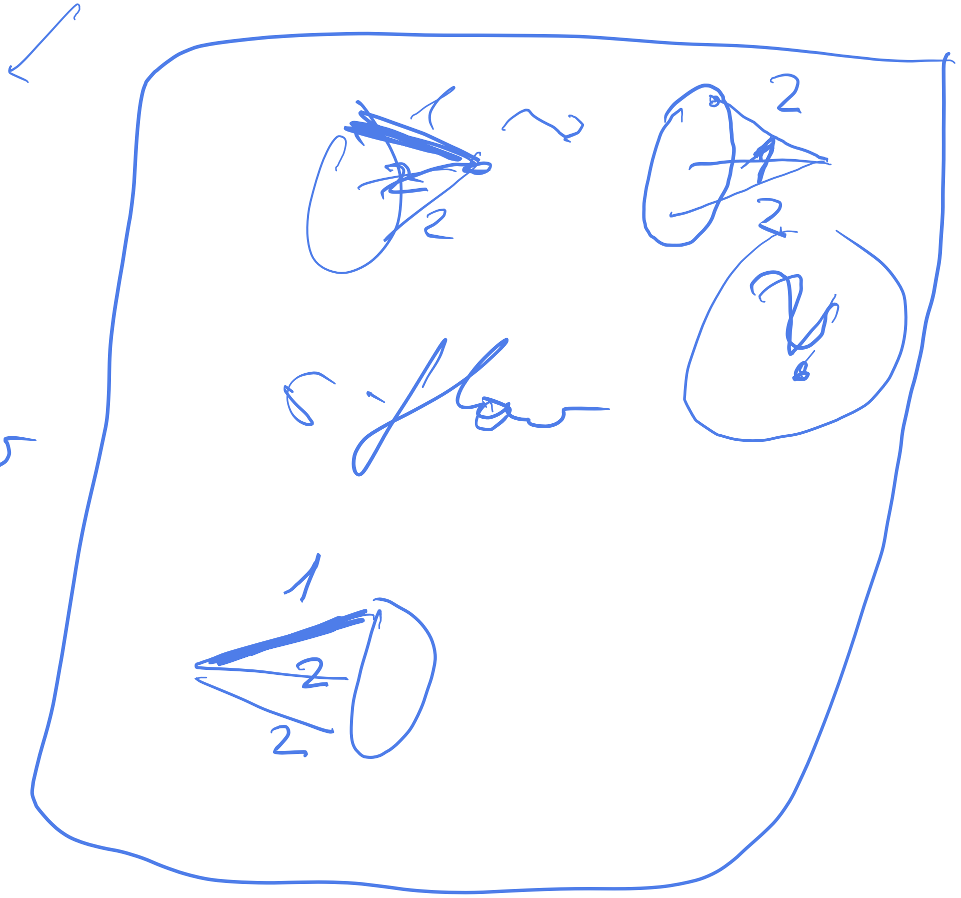
(4) G is cyclically edge 4-connected Note that a graph G is called *cyclically edge k-connected*, if $|\delta(A)| \geq k$ whenever A is a set of vertices such that both A and $\bar{A}$ contain a circuit. (Exercise: determine the cyclic edge connectivity of the Petersen graph.)

Suppose G fails the above definition with $k = 3$ that is there is A such that $|\delta(A)| = 3$. Put $G_1 = G/A$, $G_2 = G/\bar{A}$ – both G_1 and G_2 are smaller than G, thus admit a NZ 5-flow. Now it is possible

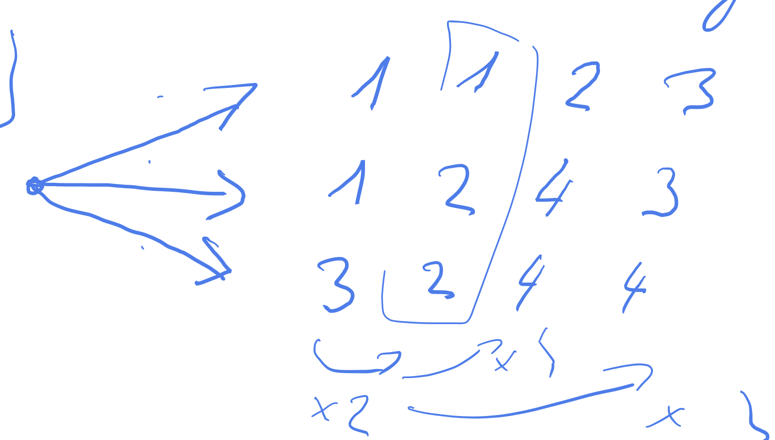




$\bar{5}$-flow



$a, b, c \in \mathbb{Z}_5 \setminus \{0\}$



to show [Sekine and Zhang] that

$$F_G(x) = \frac{F_{G_i}(x) \cdot F_{G_2}(x)}{F_{K_2^3}(x)}.$$

> 0

> 0

(Here $K_2^3 = C_3^*$ is the graph with two vertices and three parallel edges.)

Using this with $x = 5$ (CHECK) gives us that G has a NZ 5-flow, a contradiction.

(5) G **is cyclically edge 6-connected** [Kochol 2004]

(6) G **is has no circuit of length less than 9** [Kochol 2006]

---

define $\partial\varphi$ – boundary operator, "net outflow"

define group connectivity

mention two formulations (forbidden value on edges, mapping with a given boundary) consequences: implies normal connectivity and NZ flow lemma: G a digraph, H its subgraph. Suppose H is Γ-connected

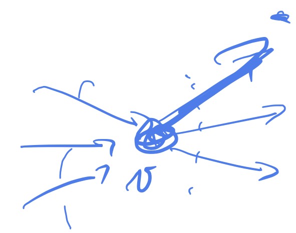
$\varphi : E(G) \to \Gamma$

digraph

ab. group

$\partial\varphi : V(G) \to \Gamma$

$\partial\varphi(v) = \varphi^+(v) - \varphi^-(v)$?



$= \sum_{e=v?} \varphi(e) - \sum_{e=?v} \varphi(e)$

$\partial\varphi(v) = 0 \ \forall v \iff \varphi$ is a flow

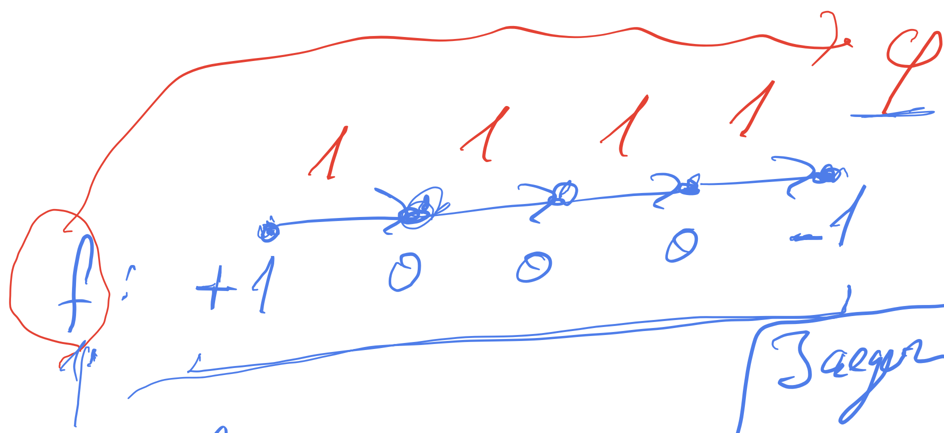
$\sum_v \partial\varphi(v) = 0$

Df G is Γ-connected iff $\forall f : V(G) \to \Gamma$ s.t. $\sum_{v \in G} f(v) = 0$ $\exists \varphi$ (NZ) $\partial\varphi = f$

$E(G) \to \Gamma$, not a flow

$\forall e : \varphi(e) \neq 0$

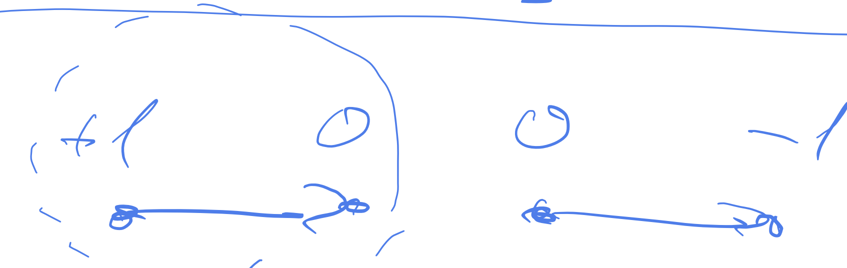
$(3$-e-c. $\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3$-connected$)$



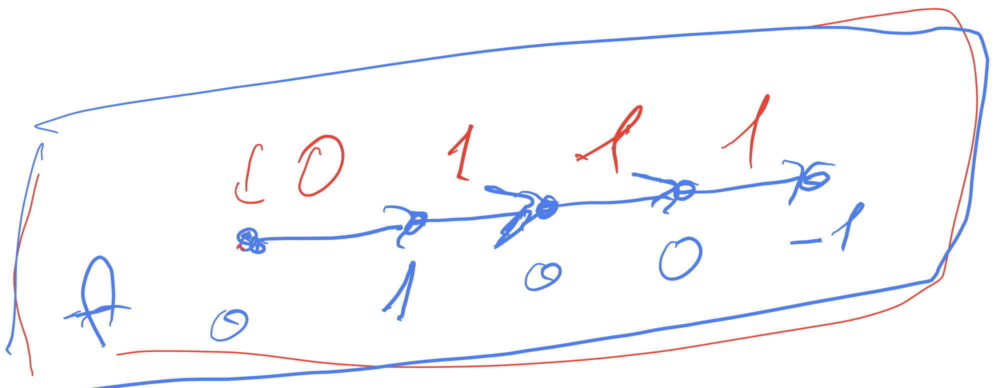
$\Gamma = \mathbb{Z}$

boundary

Jaeger, Linial, Payan, Tarsi



not Γ-connected



$\exists!\ \varphi : \partial\varphi = f$ & it is not ...

ex. unique

Thm G 4-edge-conn. $\Rightarrow$ G is $\mathbb{Z}_2^2$-conn.

Pf T_1, T_2 ... disj. sp. trees

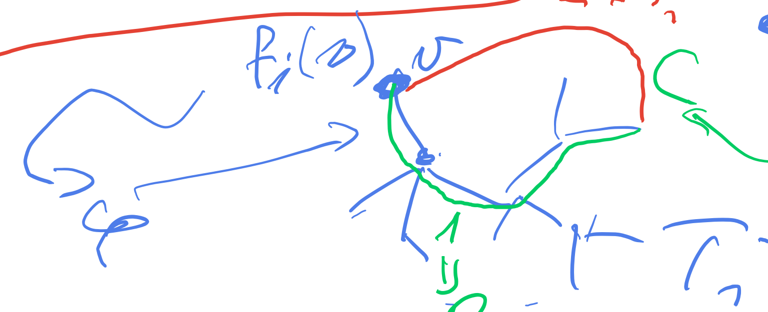
φ_1 φ_2 want: $\forall e\ \ \varphi_1(e) \neq 0$ or $\varphi_2(e) \neq 0$

$f = (f_1, f_2) : V(G) \to \mathbb{Z}_2^2$

$\forall i = 1,2$: • create $\varphi_i : E(T_i) \to \mathbb{Z}_2$ s.t. $\partial\varphi_i = f_i$

$e \notin T_i$ • modify $\varphi_i : E(G) \to \mathbb{Z}_2$ s.t.

$\forall e \notin E(T_i)\ \ \varphi_i(e) = 1$



c: +1 on C, 0 elsewhere

$\partial(\varphi_i + c) = \partial\varphi_i = f_i$

C_4 is $\mathbb{Z}_5$-connected (Γ-conn. $\forall \Gamma$ of size > 4)

given f with $\sum f = 0$

1) choose any $\varphi : E(C_4) \to \mathbb{Z}_5$ s.t. $\partial \varphi = f$

(such φ exists as C_4 is connected)

2) c: $+1$ around C_4

consider $\boxed{\varphi + x.c}$ for $\boxed{x = 0, 1, 2, 3, 4}$

$\partial(\quad) = f \quad \forall x$

$\forall e$ in C_4 $\exists!\, x$ s.t. $\boxed{\varphi(e) + x = 0}$

$\Rightarrow \geq 1$ x is good for each edge

min. counterexample to 5-flow conj.: has no C_4

Alt. def. of Γ-conn.

(Γ-connected $\Rightarrow \exists\, \Gamma$-NZF)

① $\forall f$ s.t. $\sum f(v) = 0$ $\exists$ a Γ φ: $\partial\varphi = f$ … $\forall v$ $\partial\varphi(v) = f(v)$

$f:V\to\Gamma$

flow: $\forall v$ $\partial\varphi(v) = 0$

Put $f(v) = 0$ $\forall v$

$\Updownarrow$

② $\forall h: E(G) \to \Gamma$ $\exists$ flow φ s.t. $\forall e$ $\varphi(e) \neq h(e)$

Put $h(e) = 0$ $\forall e$

$\Updownarrow$

③ $\forall f$ s.t. $\sum f(v) = 0$ $\forall h: E(G) \to \Gamma$ $\exists \varphi$: 1) $\partial\varphi = f$

& 2) $\forall e$ $\varphi(e) \neq h(e)$

and G/H (a graph obtained from G by identifying all vertices of H to a single vertex) has a nowhere-zero Γ-flow. Then G has a nowhere-zero flow as well.

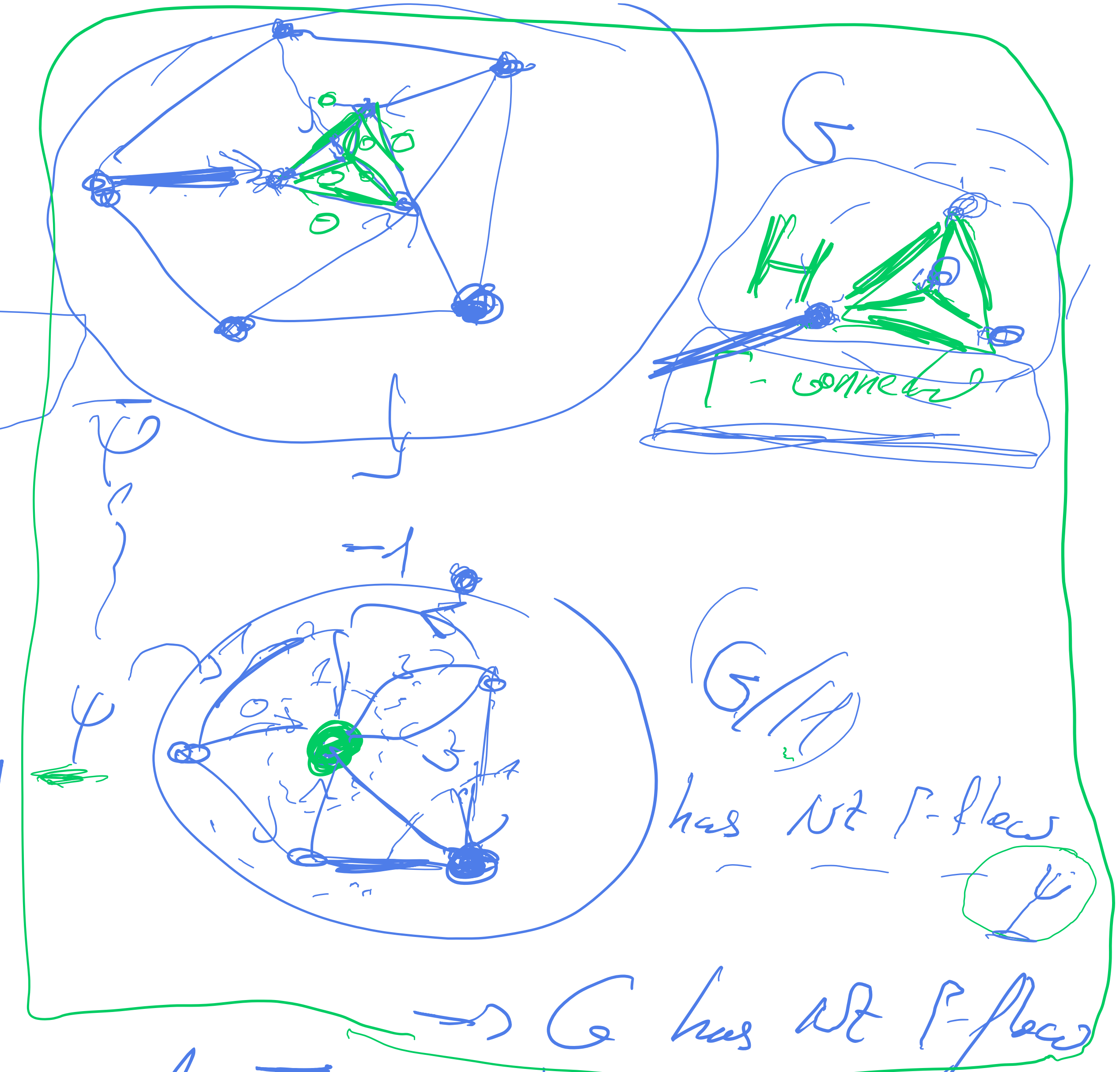
bad properties: there is a graph that is $\mathbb{Z}_5$-connected but not $\mathbb{Z}_6$-connected. JLPT (LMT) (Exercise!)

There is a graph that is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected. (LM and RS and ~~??~~ BRH)

The boundary operator is also crucial for a proof we are about to present.

Theorem 39 (Seymour, 1979?). *Every bridgeless graph has a nowhere-zero $\mathbb{Z}_6$-flow.*

Seymour actually provided two proofs. Besides the original papers, they can be read in [Diestel] and [Seymour-appendix]. We are going to present a new proof based on [DRS]. As $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, we are going to look for two flows, a $\mathbb{Z}_2$-flow φ_2 and a $\mathbb{Z}_3$-flow φ_3. If we manage to find such flows that are



consider $\bar{\psi}$ on $E(G)$ (replacing by 0 to $E(H)$)

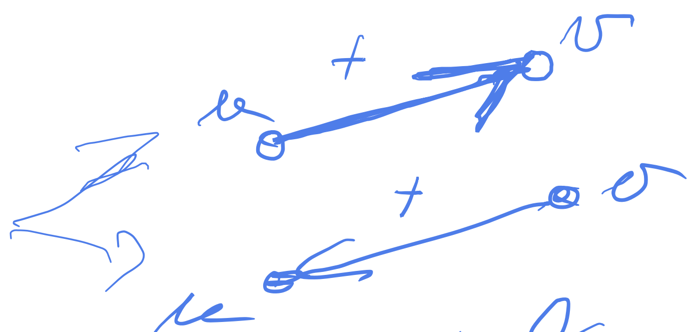
$\bar{f} = \partial \bar{\psi}$

$\bar{f}(v) = 0$ for $v \in V(G) \setminus V(H)$

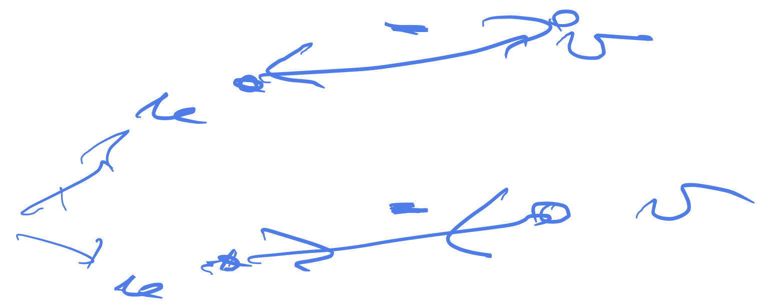
$f = \bar{f}|_{V(H)} \rightsquigarrow \exists\, \varphi : E(H) \to \Gamma \setminus \{0\} : \partial\varphi = f$

$\bar{\psi} - \varphi$ is a NZ Γ-flow on G

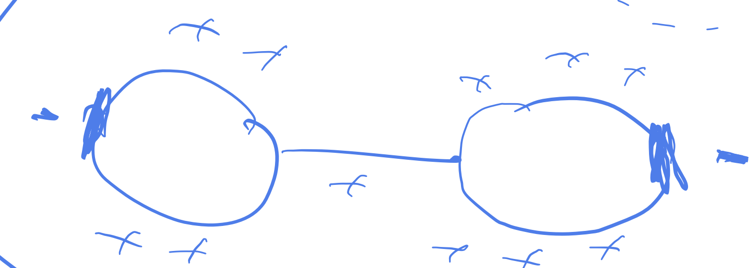
Conj. (Bouchet) Every bidirected graph has a NZ $\mathbb{Z}_6$-flow.



normal
pos. edge



neg. edge



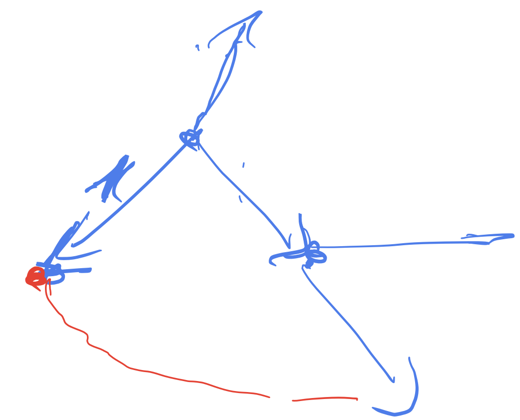
signed graph!

not simultaneously zero, we are done. In fact, this is the same for all proofs of $\mathbb{Z}_6$, there are differences in how we construct these flows.

We are going to create them recursively. Roughly speaking, we will grow a tree on which $\varphi_3 \neq 0$. Then we use the rest of the graph to make φ_3 a flow — but at the risk of making it zero at some edges. We use φ_2 to make sure that φ_2, φ_3 are not zero simultaneously and then use the tree to ensure that φ_2 is a flow.

TODO: make this more clear.

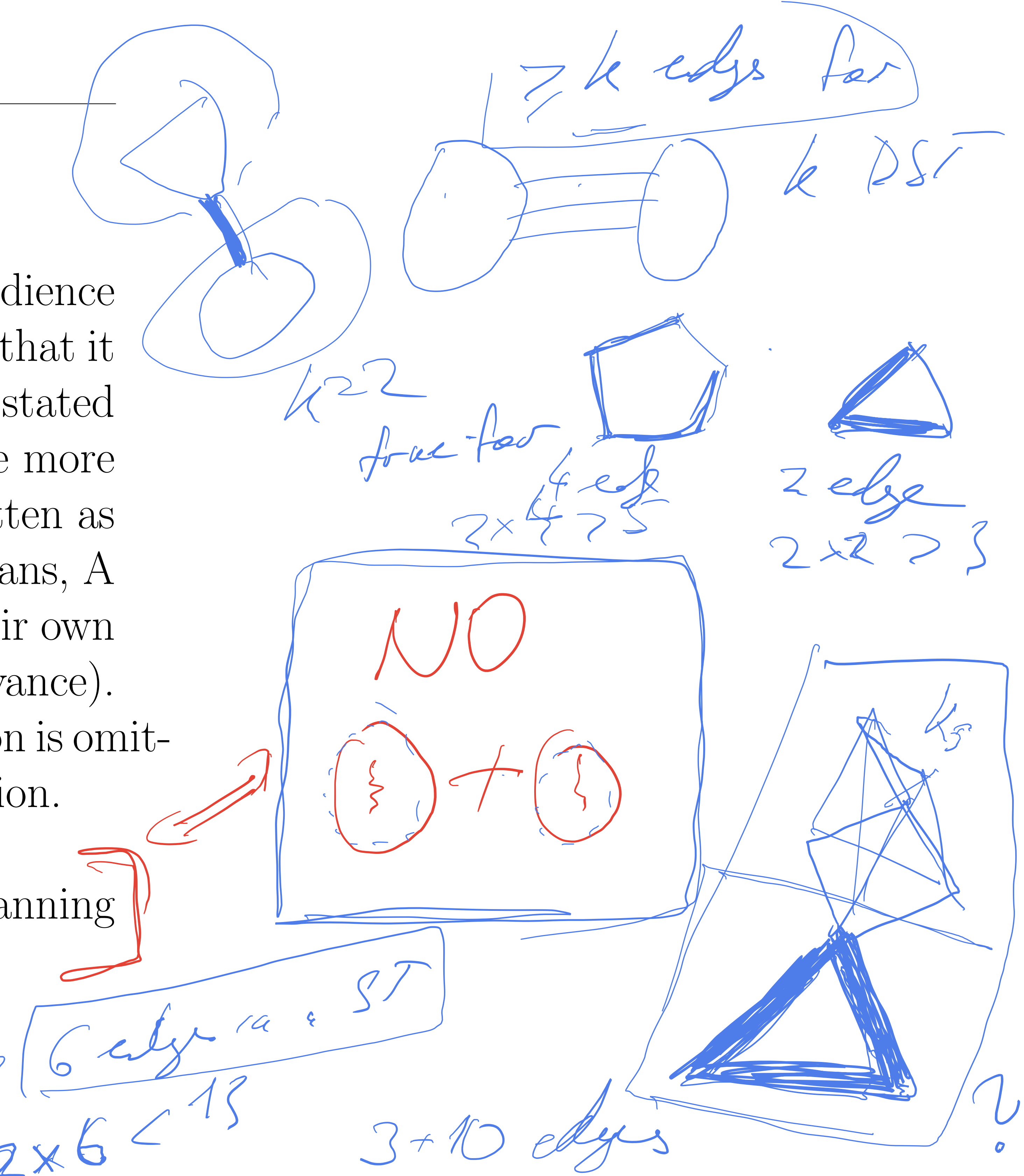
In order to formalize this, we will actually prove a somewhat technical lemma, that will be very easy and natural to prove by induction: the main two differences: we prove that we can in fact prescribe the values around a "root vertex". And we allow some vertices where $\partial\varphi_2$, $\partial\varphi_3$ are not zero to capture more easy the recursive building of the tree.



~~¡input manuscript¿~~

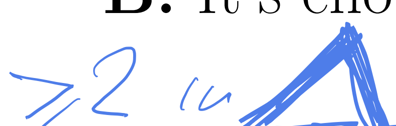
1 Intermezzo – musing about disjoint spanning trees

For this part of the class, large part of the audience was about to be missing – so the topic is such that it can be safely skipped, the main result will be stated again, when it will be needed. To capture the more leisurely pace of this class, this section is written as a dialogue. The idea is that two mathematicians, A and B, are trying to discover the result on their own (although A seems to know something in advance). Frequently, some argument/picture/calculation is omitted, to let the reader take part in the discussion.



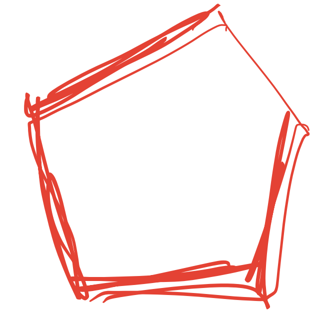
A: What does a graph need to have a spanning tree?

B: It's enough to be connected.



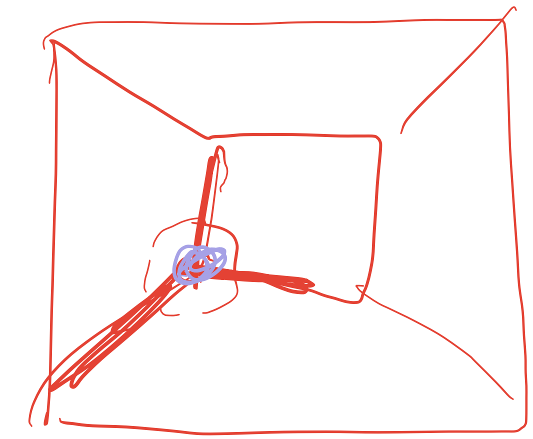
A: What about two (edge-) disjoint spanning trees?

B: What about 2-connected?



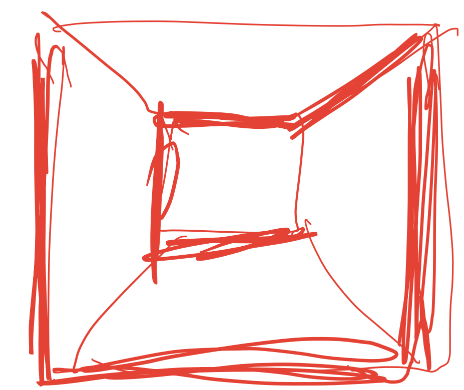
A: Let's try some graph – a circuit perhaps?

B: Ummm.



A: Let's try higher-connected graphs. Say, a cube.

B: Well, none of the trees can contain a vertex of degree 3, so both must be hamiltonian paths. And . . . cube has no such disjoint hamiltonian paths.



A: True. Can you say it simpler? (Counting proofs are nice!)

B: I see, the cube does not have enough edges!

8 vert., 12 edges

sp.t. ... 7 edges

$2 \times 7 > 12$

A: Can you generalize it?

B: I suppose a graph with k disjoint spanning trees and n vertices must have at least $k(n-1)$ edges.

A: Is this enough?

B: No, it still needs to be connected. Wait, it is even k-edge-connected.

≥ 2

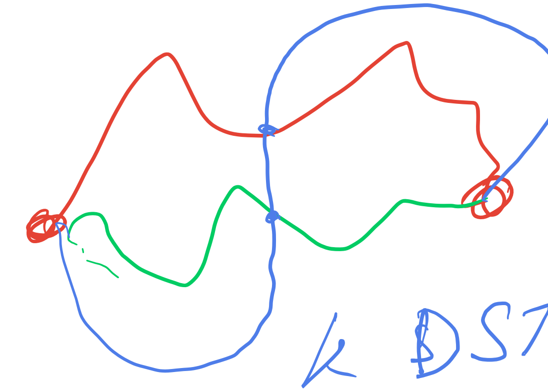
A: And this is sufficient?

B: I don't know …

A: It's not. Now when you know, find a counterexample :-)

B: Well … If I started with a graph that does not have k disjoint spanning trees (like the cube for $k = 2$), I could start adding edges somehow …

A: Sounds right. But you better not add them among the old vertices, or it may start to have 2

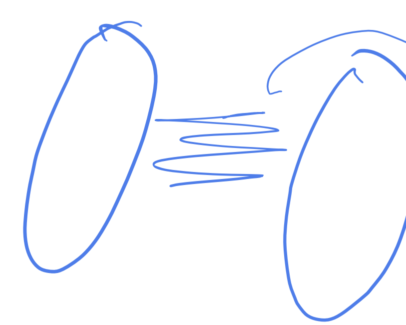


k DST

$\Downarrow$

$\forall u \neq v \implies k$ edge disj. path $u \to v$

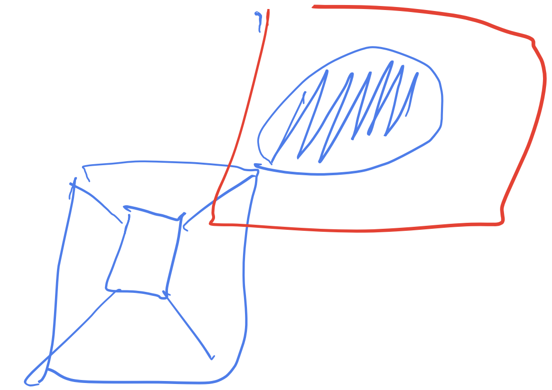
Given graph G
find nice set of edges
s.t. $\#$ST has a lot of these



≥ 1

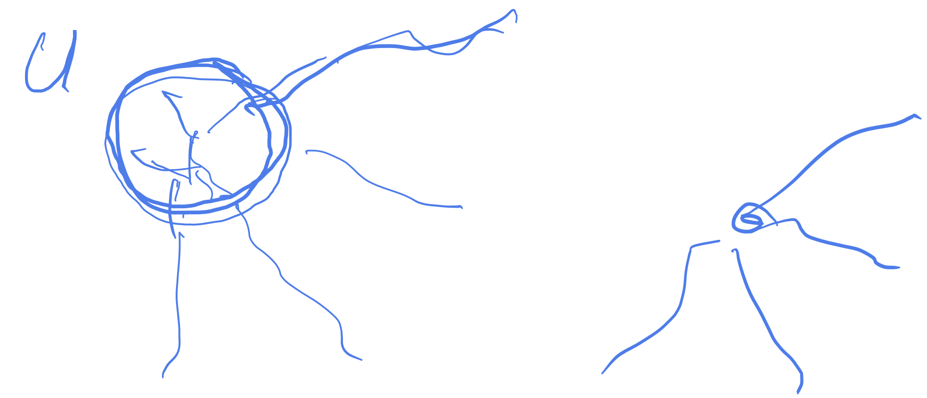
DST (disjoint spanning trees).

B: I see. What about attaching a large clique at a vertex?



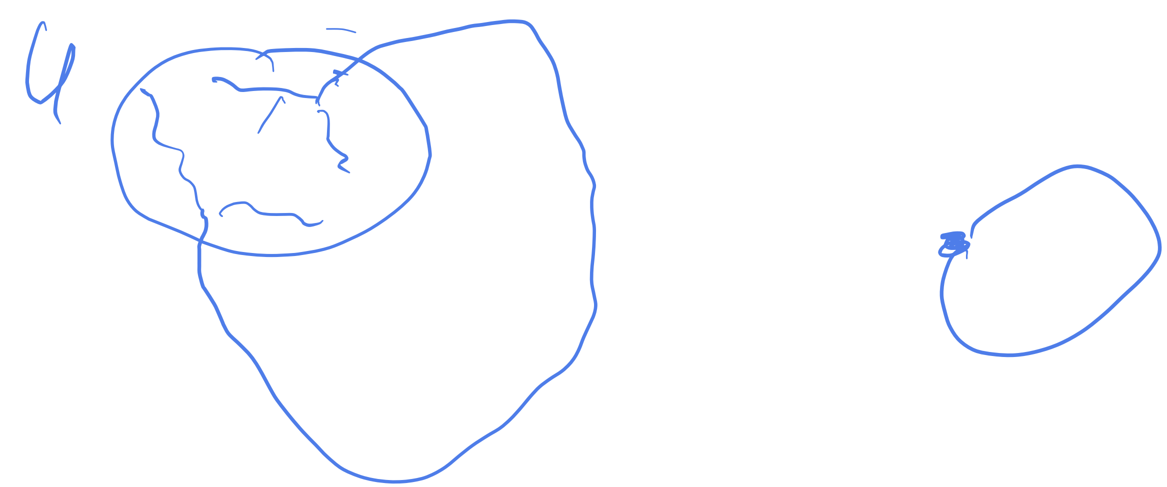
A: Yes, that's it. Can you do it without a cut-vertex?

B: If I let the cube and the clique have two vertices in common, it will have 2 DST . . .



A: Think about this: if U is a subset of $V(G)$, and you identify all vertices of U to a single vertex (preserving multiplicities). How do spanning trees of G look in G/U.

B: They are spanning trees again!



A: Careful!

B: . . . except they may contain a cycle . . .

A: And are they spanning?

B: Ummm, yes, they are. I see ... If $G[U]$ has lots of edges, we may think that G has 2 DST. But G/U can still be a cube, so the 2 DST of G would become two disjoint spanning subgraphs of G/U, the cube ...

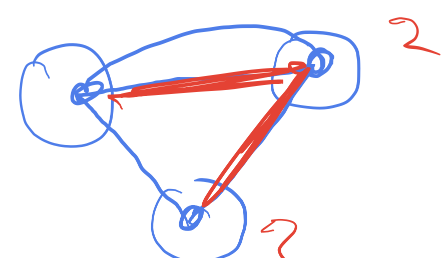
A: Can this be generalized?

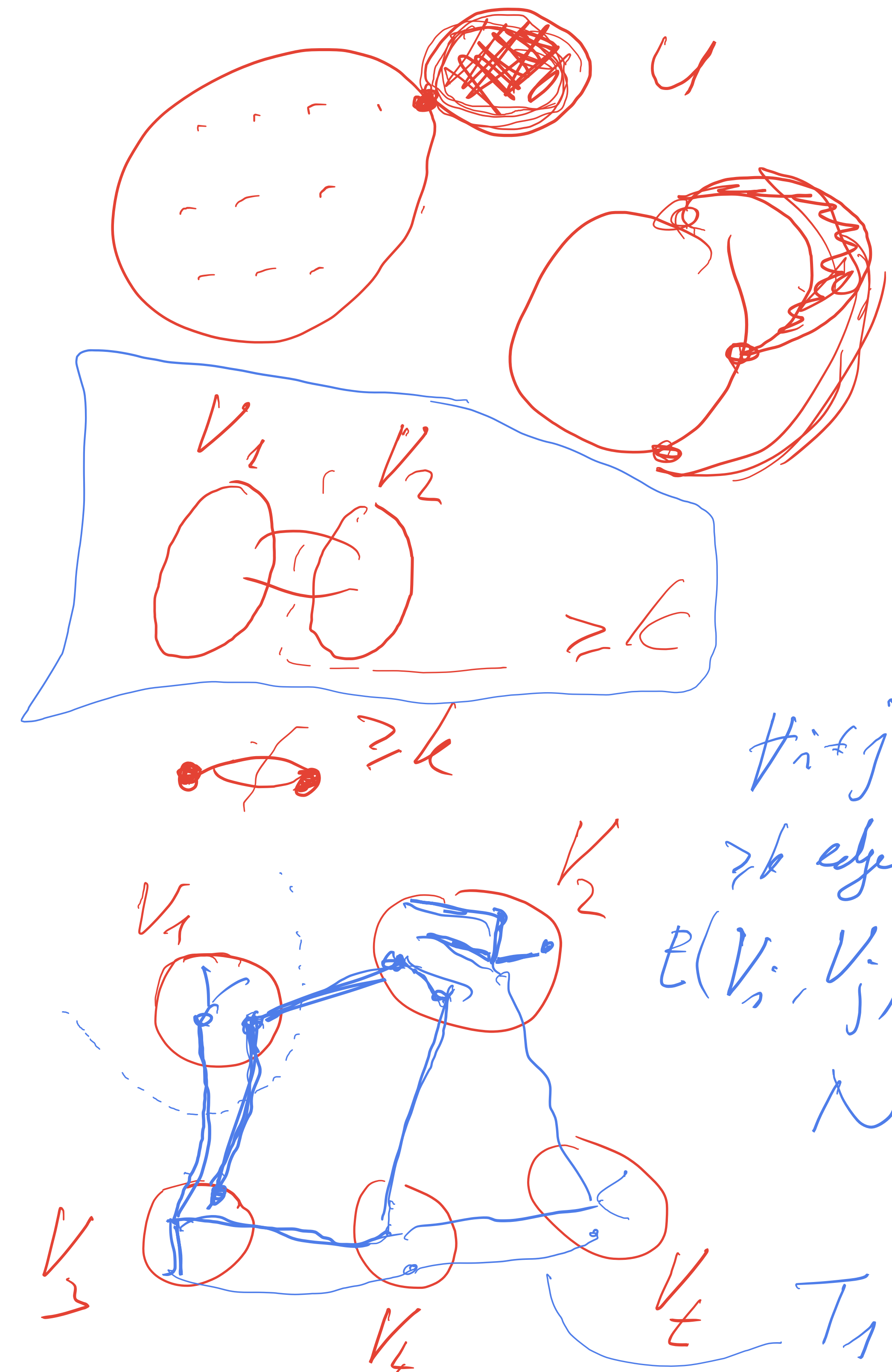
B: Perhaps we can contract more than one set?

A: Yes! The right notion is a partition. If a partition $\mathcal{P}$ consists of disjoint sets $V_1, \ldots, V_t$ that cover $V(G)$, then $G/\mathcal{P}$ means G where vertices of each set V_i were identified to a single vertex, preserving all the edges between distinct sets.

B: I see, and again, a spanning tree of G becomes a spanning tree, of $G/\mathcal{P}$. I mean spanning connected graph.

T_1/P is connected spanning in G/P





$\forall i \neq j$ $\geq k$ edges in $E(V_i, V_j)$? No.

A: Precisely. What does this say about G with k DST?

B: For every partition $\mathcal{P}$ the graph $G/\mathcal{P}$ has also k DST. In particular $G/\mathcal{P}$ must be k-edge-connected and it has enough edges.

A: Let's stick with counting edges. What exactly did we get?

B: OK. $\boxed{|E(G/\mathcal{P})| \geq k(|V(G/\mathcal{P})| - 1).}$

A: Let's try some special cases. What if $\mathcal{P}$ is the partition into 1-vertex parts?

B: I see, we get our original bound on the total number of edges.

A: Try the other extreme!

B: If $\mathcal{P}$ has just one class – it's not very illumi-

$T_1, \ldots, T_k$

G is $(2k)$-edge-connected

$\mathcal{P} = \{\{v\} : v \in V\}$

$|E(G)| \geq (|V(G)| - 1) \cdot k$

$\forall \mathcal{P} \; |E(G/\mathcal{P})| \geq k(|V(G/\mathcal{P})| - 1)$

? $\Downarrow$ $\Uparrow$ $\Uparrow$

G has k DST

nating ... I see, if $\mathcal{P}$ has two classes, we get that ... G has no cut of size $< k$!

A: So the condition with $G/\mathcal{P}$ seems rather strong ... And believe it or not, it actually characterizes graphs with k DST!

B: Awesome! We actually proved half of the equivalence already :-). Shall we try the other one?

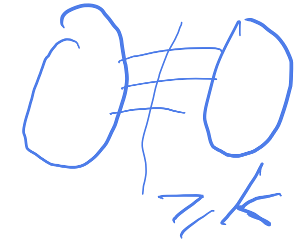
A: Sure! :-) What do you suggest?

B: We may try induction on k. Or we may try to take some small cut, find DST in both parts and connect them somehow. Or ...

(few hours later :-))

B: Or we may try something else ...

A: Let me suggest an approach. Instead of looking for k DST, let us go for k edge-disjoint spanning forests, lets call them $F_1, \ldots, F_k$.

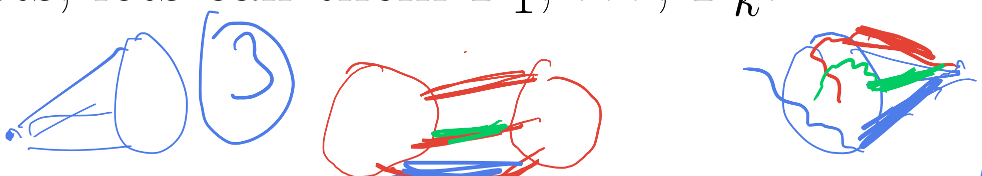


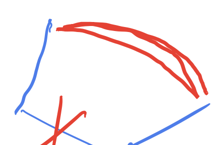
max. # of edges

1 k edge-disjoint sp. forests $F_1, \ldots, F_k$

2 k sp. trees $T_1, \ldots, T_k$ (with limited intersection & minimize intersections

(matroids)





What if no k-edge-cut?

B: Sure, just take each F_i to be edge-less. So what?

A: And now try to put in the forests as many edges as possible. Say, do $\boxed{\text{maximize } \sum |E(F_i)|}$.

B: Sounds interesting. If all edges of the graph are used, then either we have k DST, or the graph has too few edges. So there must be some edges missing.

$|E(G)| \ngeq k(|V(G)| - 1)$ ← no k DST

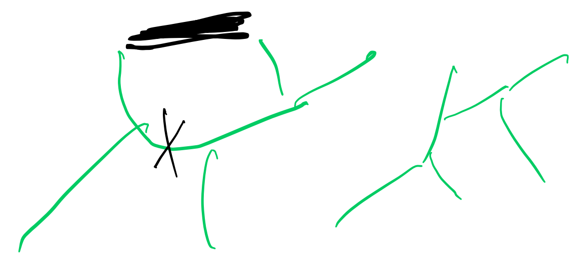
— first the cond (*) does not hold

$|E(G)| \geq k(\ ) \ngeq \sum |E(F_i)|$

otherwise I'm done

A: What can you say about such an edge?

B: It creates a ~~cycle~~ circuit in every F_i.



A: Can you used the cycles to move around?

B: What do you mean?

A: It might be useful if there are many maximal

could add e to F_i $\forall i$

k-tuples of forests.

B: I see ...

B: Yes, now I really see. We can pick some F_i, add the new edge and remove some other edge of the created cycle. This creates another maximal tuple.

A: Can you go any further?

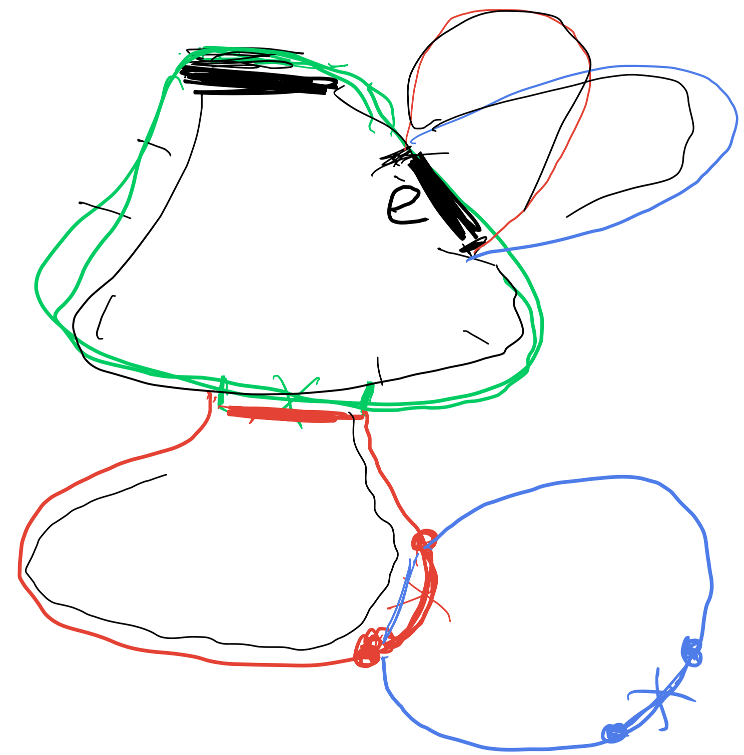
B: Well, after we remove the other edge of the cycle, we cannot add it back to any other forest, F_j, as we would create a cycle. And I suppose we can go on and on ...

A: Will it ever stop?

B: It must, the graph is finite. But I still can't see what is it good for.

A: Try to draw a picture!

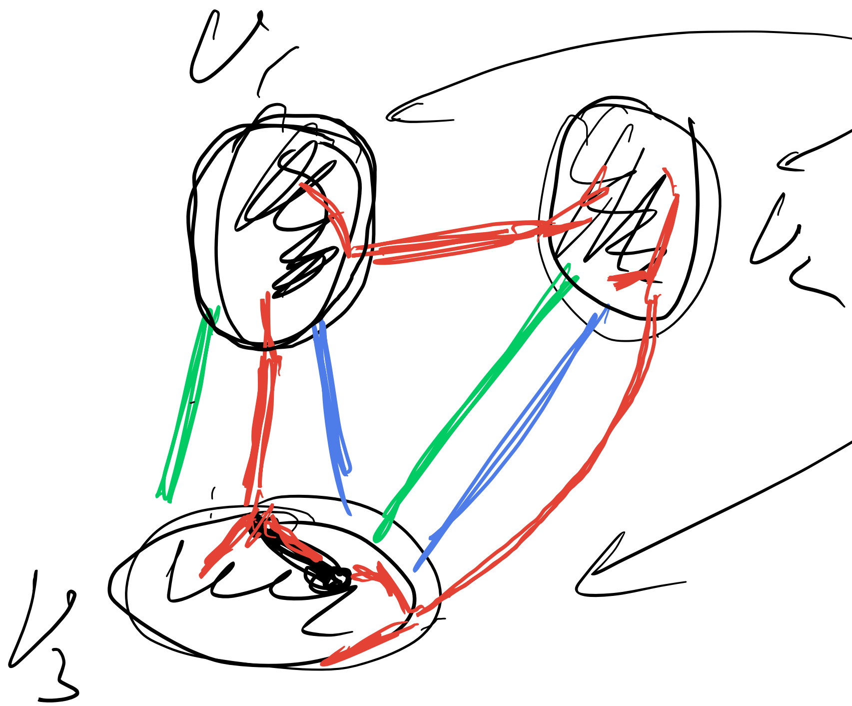
B: Umm, can it be that we get some part of G



black edges e: $\exists F_1, \dots, F_k$ of max. $\sum |E(F_i)|$ s.t. $e \notin \bigcup E(F_i)$

black $e = u,v$ $\exists$ k paths $u \to v$ (not using e)

comp. of black is $(k+1)$-edge-connected



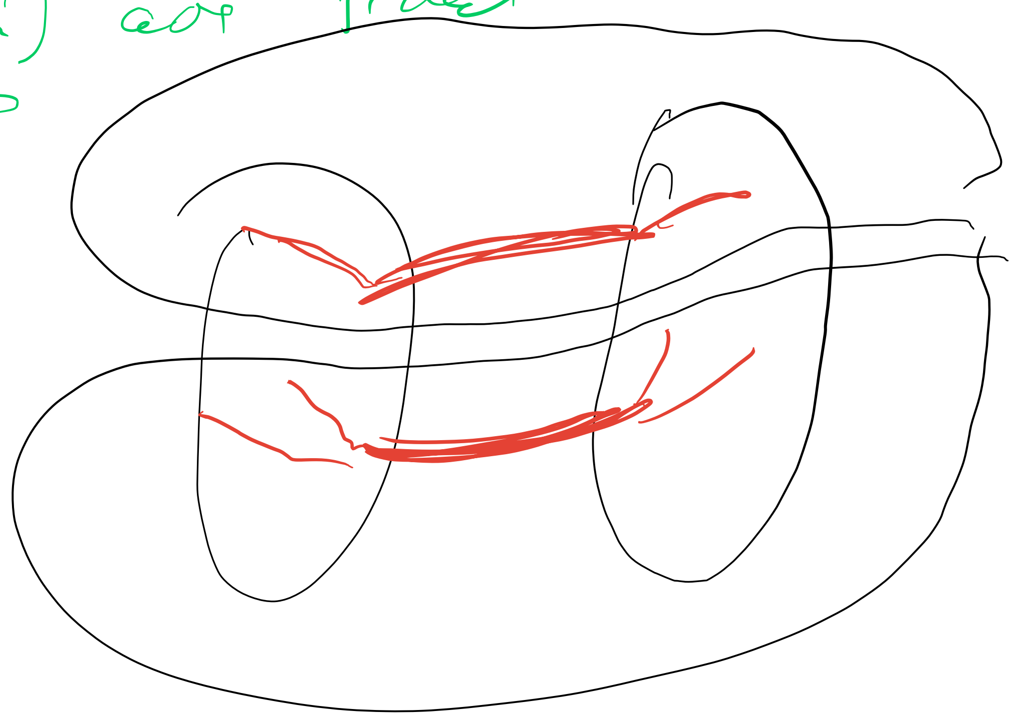
black components $\longrightarrow P$

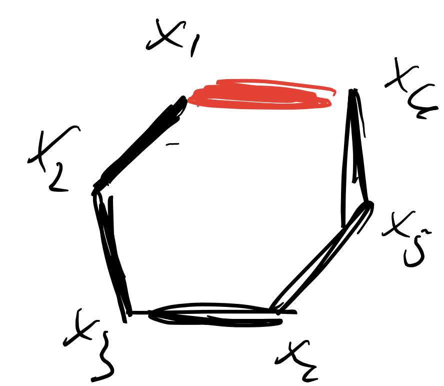
$\circledast$

$$|E(G/P)| \geq k(|V(G/P)|-1)$$

$\forall F_1 \ldots F_k$ all edges of $E(G/P)$ are used by $E(F_1) \cup \ldots \cup E(F_k)$

$E(G)$ are fixed





$\forall i$ $x_i x_{i+1}$ is black $\Rightarrow x_1 x_6$ is black

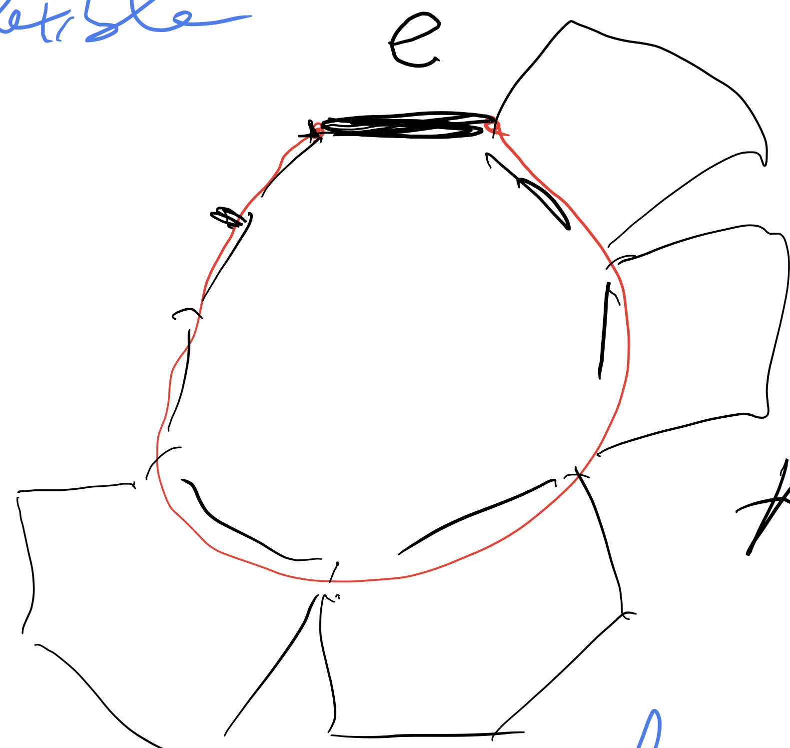
Block $K = \{ e \in E(G) ;\ \exists P_1 \ldots P_k$ meas. s.t. $e \notin \bigcup E(P_i) \}$

~ disj. paths in P

flexible

$$K = \bigcup_{(P_1 \ldots P_k)} \Big(E(G) \setminus \bigcup E(P_i) \Big)$$

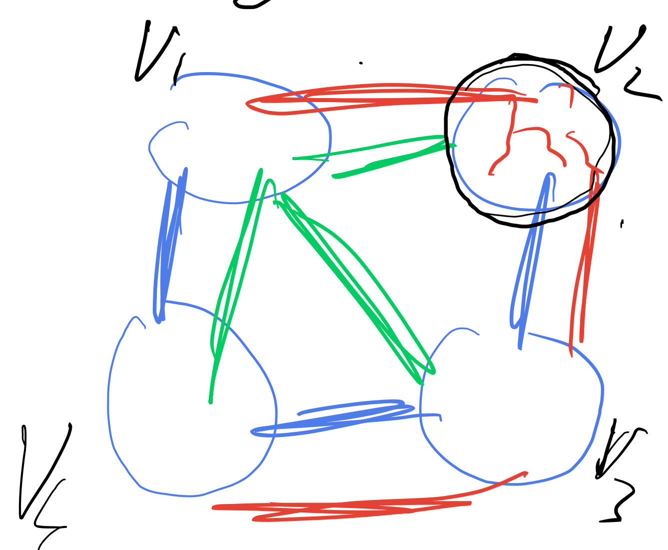
$(e \in P \neq \{V_i\})$ (more cond's)



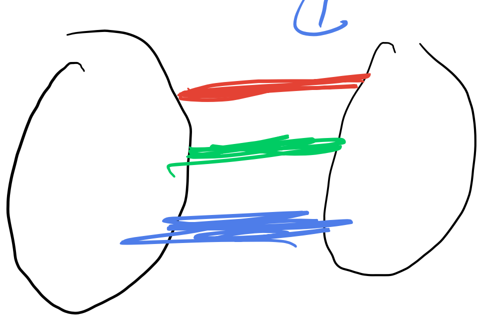
$\forall e \in K\ \exists$ cycle in K containing e — tri. [illegible]

$G[V_i]$ — [illegible] ind.

(using (*) for $G[V_i]$)



frozen



→ def. K

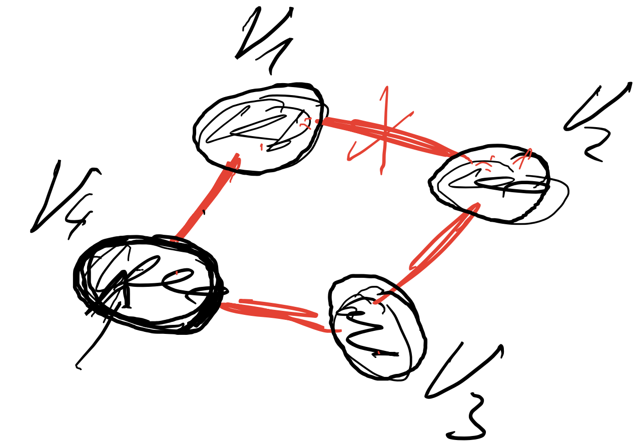
→ comp. of (G, K) — $\underbrace{V_1, \dots, V_t}_{P}$ — $\boxed{\circledast \text{ is tight}}$

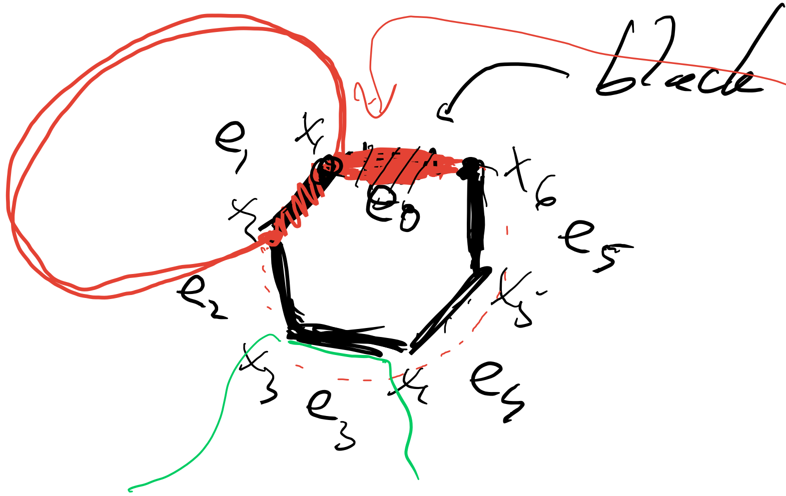
$$E(G/P) = k\,(|V(G/P)| - 1)$$

$\Uparrow$

no red cycle in G/P

— $\forall i$ use ind. on $G[V_i]$





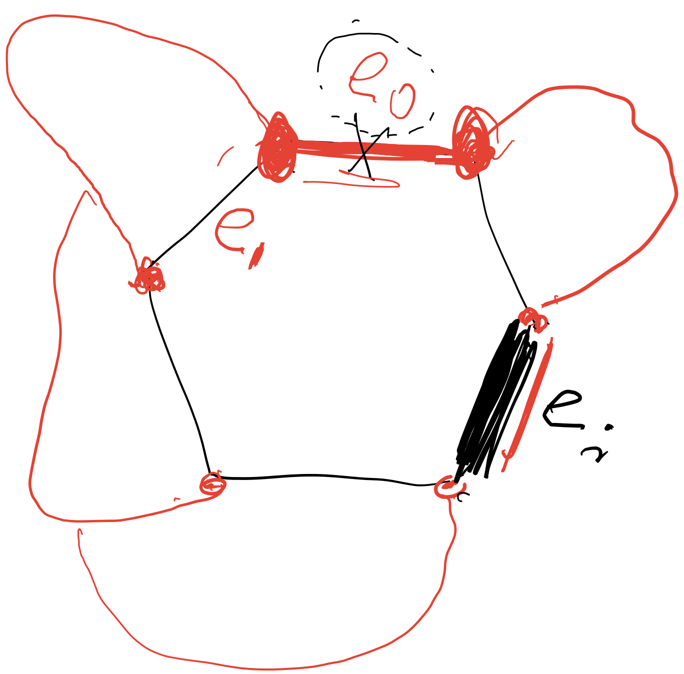
$F_1^1 \ldots F_k^1$ that don't use e_1

$F_1^2 \ldots F_k^2$ — " — e_2

⋮

e_i is black ... $\exists F_1^i \ldots F_k^i \not\ni e_i$

$\Rightarrow \forall j$ F_1^j connects both ends of e_i



$x_1 \sim x_2$ in red

$x_2 \sim x_3$ — " —

$x_3 \sim x_4$ — " —

⋮

$x_5 \sim x_6$

$F_1^1 - e_0$

$\Rightarrow \forall i$ $x_1, \ldots, x_6$ in the same comp of F_1^i

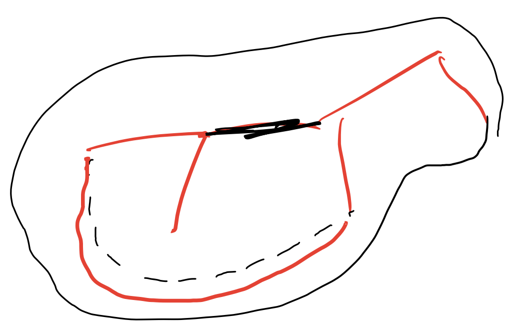
Claim $\forall e^\circ \notin K \;\exists U \subseteq V$: $U \ni e_0$

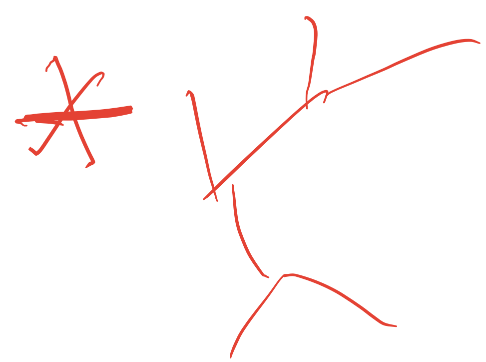
& U is connected in $\forall F_i^\circ$

same color $\forall F \in \mathcal{F}$

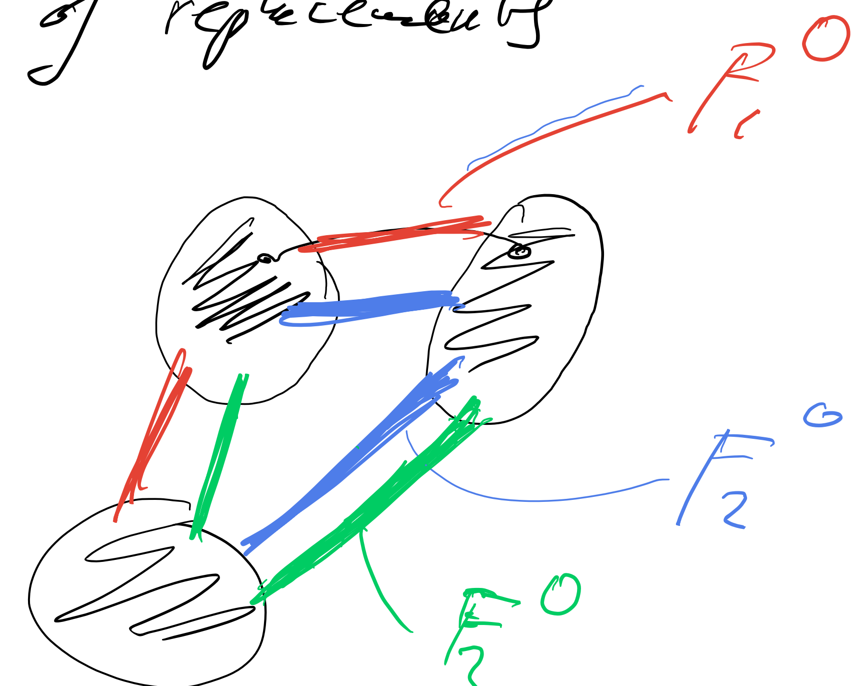
$F^\circ = (F_1^\circ, \ldots, F_k^\circ)$ — a max. k-tuple of forests

$\mathcal{F}$ = all k-tuples obtainable from F° by a seq. of replacements





red component stays the same



that contains k DST?

A: Yep! And this means we are done, aren't we?

B: But we wanted DST in the whole graph.

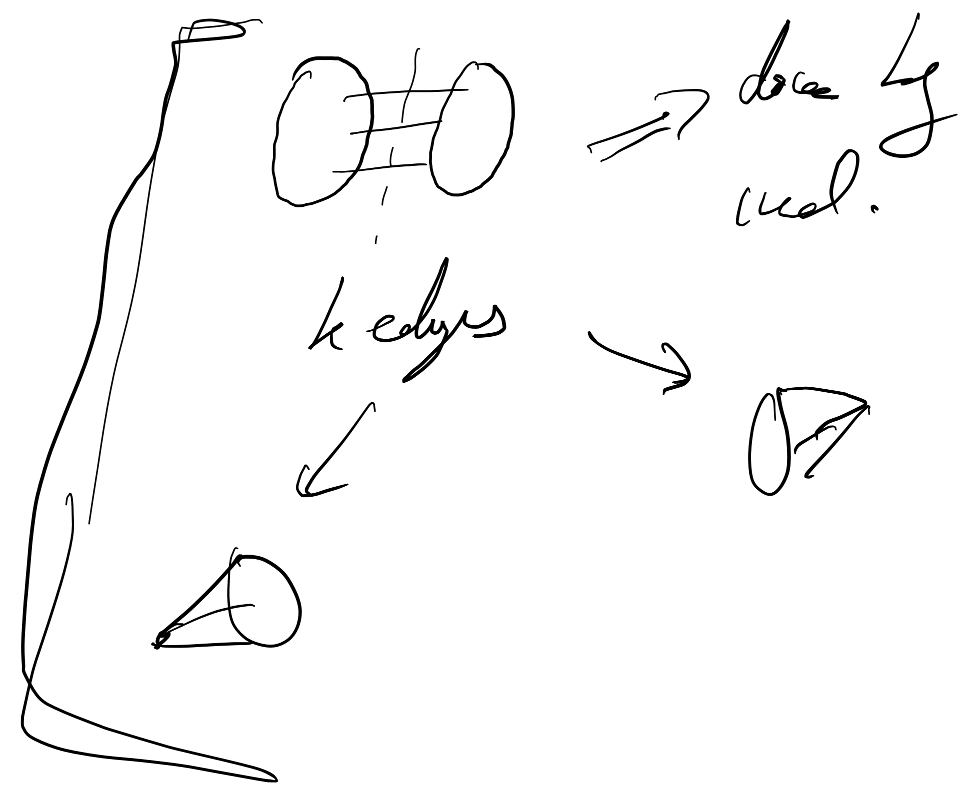
A: That's what induction is for ...

Proof by ind. on $|V(G)| + |E(G)|$

Theorem 40 (Nash-Williams, Tutte). *A multi-graph G contains k edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$ the contracted graph $G/\mathcal{P}$ satisfies*

$$|E(G/\mathcal{P})| \geq k(|V(G/\mathcal{P})| - 1).$$

Corollary 41. *Every $2k$-connected graph contains k edge-disjoint spanning trees.*



Exercises: 7. Based on the previous dialogue, write down a complete (but short :-)) proof of the theorem.

8. Using the theorem, show that every $2k$-connected graph contains k edge-disjoint spanning trees (Corollary 41).

9. In analogy with the above guess the characterization of graphs G, such that $E(G)$ can be decomposed into k forests. If you are brave enough, you can prove it along the same lines.

ADD:

* snarks – 2-sum, 3-sum, Isaacs dot product * Flower snarks * Construction of snarks by superposition * high-girth snarks * CDC – properties of a minimal counterexample – v dukazu, ze neobsahuje 4-cyklus jsem asi udelal chybu, napsat a spravne!!! * tensions, flows as orthogonal vectors * XY mappings, Petersen-flow conjecture [Jaeger]

prep Toky!! – rychle zopak: tenze kolme na toky, tenze v dualu jsou toky, defce (chybi dk. dvoji char. tenzi) – nad $\mathbb{Z}_2$: rezy, cykly – defce TT, FF – homo $\Rightarrow$ TT – hypo: kazdy graf bez mostu ma FF_2 zobr.

do Pt – dualni defce – Pt coloring – normalni 5-obarveni

– FT_2 odpovida CDC (s omezenim, ktere cykly se smeji protinat) – aplikace: 4-CDC $\iff$ 3-CDC

– zacni 3-tok? – prehled co se vi a co je hypo – motivace: 4CT, Grotzsch thm – prehled dukazu