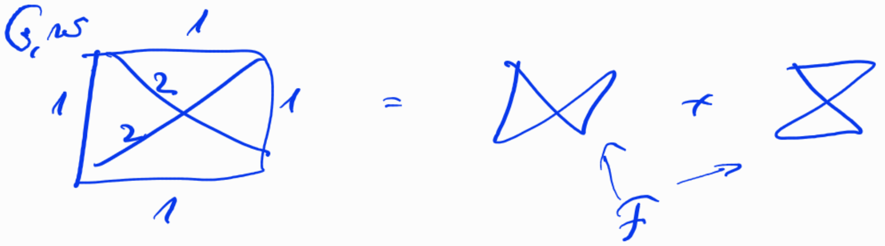
(Faithful) circuit covers (Chap. 2 of book CQ Zhang - CDC of graphs)

G -- graphs

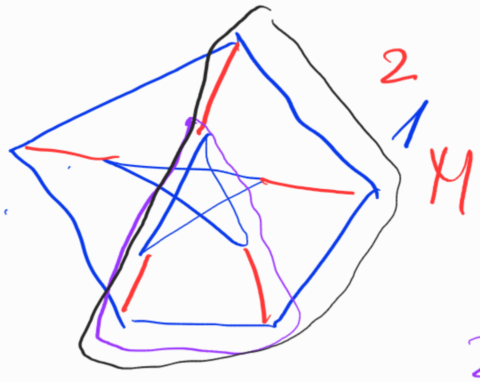
$w : E(G) \to \{1, 2, 3, \dots\}$

family $\mathcal{F}$ of circuits / cycles is <u>w-faithful circuit cover</u>

iff $\forall e$: e is contained in $w(e)$ circuits/cycles of $\mathcal{F}$



CDC conj: $\forall G$, $w \equiv 2$ $\exists$ w-f.c.c.



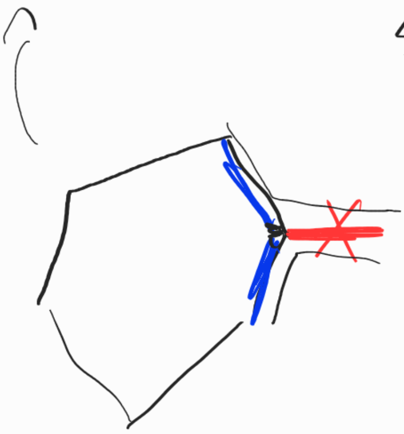
does not have w-fcc ✓

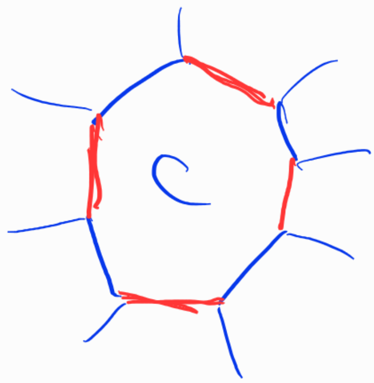
20 edges

3 blue 2 red

4 blue 2 red

$|C \cap M|$ is even





$|C| = 4k$

$|C| = \cancel{4}, 8, \cancel{12}, \dots$

w is eulerian $\iff$ $\forall v$ $\sum_{e \ni v} w(e)$ is even

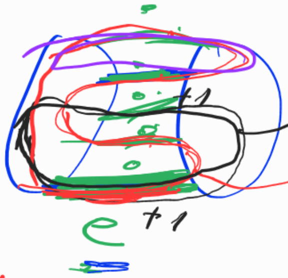
$\boxed{\exists\ w\text{-f.c.c.}}$ $\Uparrow$ F

$\Downarrow$

w is admissible $\iff$ $\forall$ edge-cut T $\forall e \in T$ $w(e) \le \frac{w(T)}{2}$, $\quad \sum_{e' \in T} w(e')$

$\boxed{2w(e) \le w(T)}$

contr. of C to $w(e)$



$+2$	$+2$
$+2$	$+4$
0	$+2$

$$2w(e) = \sum_{C \in F} 2[e \in C]$$

$$\le$$

$$w(T) = \sum_{e' \in T} \sum_{C \in F} [e' \in C] \quad \begin{cases} 1 & \text{if } e \in C \\ 0 & \text{o/wise} \end{cases}$$

$$= \sum_{C \in F} \sum_{e' \in T} [e' \in C]$$

contribution of C to $w(T)$

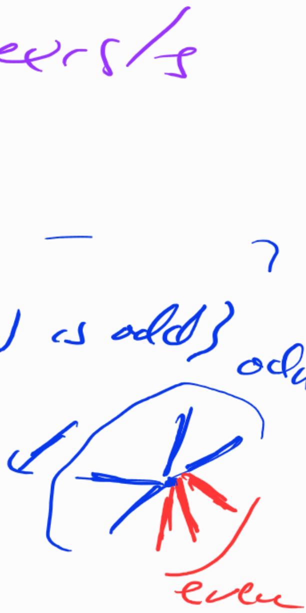
Q: If w is admissible & eulerian, is there w-f.c.c.?

(No) ... Give characterization.

If G, w is adm., eul., no w-f.c.c. exists then G, w is called $\boxed{\text{contra pair}}$

Observations 1) w is eulerian $\iff$ $\{e : w(e) \text{ is odd}\}_{\text{odd}}$ is a cycle

\# of blue is even

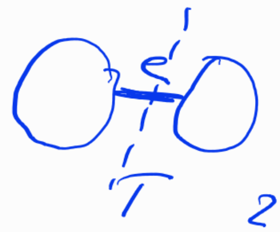


even sum of w

2) If $w : E \to \{1,2\}$ is eulerian then (TODO: ERROR in proof)

w is admissible $\iff$ G is bridgeless

$\Longrightarrow$



$\Longleftarrow$

$2w(e) \le w(T)$

$2w(e) \le w(T) = w(e)$

if $w(e)=1$ ✓

$w(e)=2$

✓

not admissible $2 \cdot 2 \not\le 2+1$

not eulerian

G is cubic bridgeless, w is eulerian

$\boxed{\text{Lemma}}$ (Seymour) G is cubic, w eulerian $\to \{1,2\}$

(1) $\exists$ w-fcc by 3 gebs $\iff$ (2) G is 3-edge-colourable

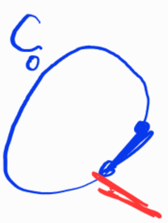
$\boxed{2 \Rightarrow 1}$ $c : E(G) \to \{1,2,3\}$

$C_{ij} = \{e : c(e) \in \{i,j\}\}$

C_{12}, C_{23}, C_{31} is a CDC

$C_0 = \{e : w(e)=1\}$ ... a geb

$C_0 \Delta C_{12}$, $C_0 \Delta C_{23}$, $C_0 \Delta C_{31}$ is a coll. of gebs (3)



$e \in C_0$ is covered $1\times$ ✓

$e \notin C_0$ — " — $2\times$ ✓

Thm 27

(1) $\Longrightarrow$ $\exists$4-CDC $\to$ 3-e-c. (2) ―――― was proved

$\boxed{\text{Corollary}}$ G is planar, bridgeless, w eul., $\to \{1,2\}$

$\Rightarrow \exists$ w-fcc.

G planar $\Rightarrow$ $\chi(G^*) \le 4$ $\Rightarrow$ G has $\mathbb{Z}_2^2$-NZF

$\Rightarrow G$ is 3-e-c $\Rightarrow$ G, w . ----

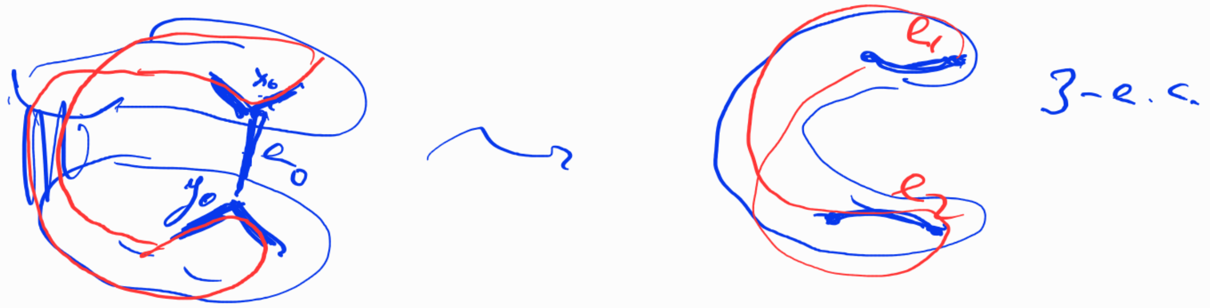
<u>Corollary</u> G bridgeless, cubic

$e \in E(G)$: $\overline{G-e}$ is 3-e-c.

suppress vertices of deg 2

$\Rightarrow$ G has 5-CDC

$\exists C' \ni e_1, e_2$



$\exists C_0$ containing x_0, y_0

$H = \overline{G-e_0}$

is 3-e-c $\Rightarrow$

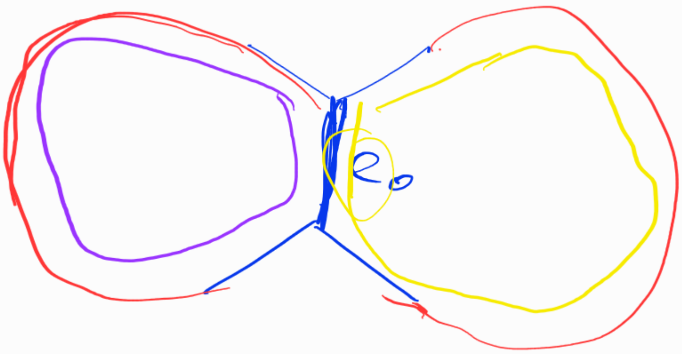
H is bridgeless & conn.

$w(e) \mapsto$ 1 $e \in E(C_0)$ -- circuit

2 $e \notin E(C_0)$

--- eulerian & admissible

$\exists \mathcal{F}$ ... ω-f.c.c. of 3 gales

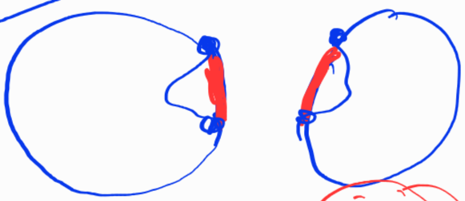


$\mathcal{F} \cup C_4 \cup C_5$ is a 5-CDC

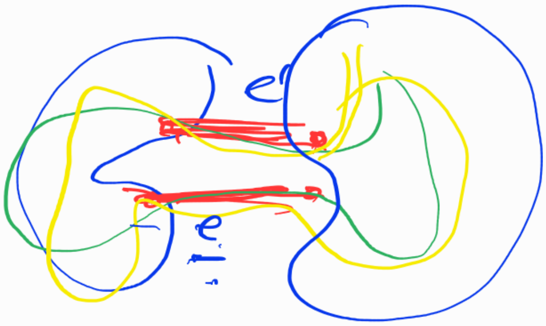
$\boxed{\text{Contrapairs}}$

– (Pf, ω) ... is a contrapair
⋮
2 on a PM

$\boxed{\text{Lemma}}$



$\rightsquigarrow$



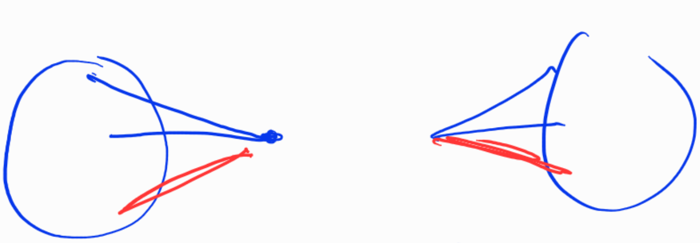
≥ one contrapair $\longrightarrow$ $G := G_1 \oplus_2 G_2$ is a CP

$\omega := \omega_1 \oplus_2 \omega_2$

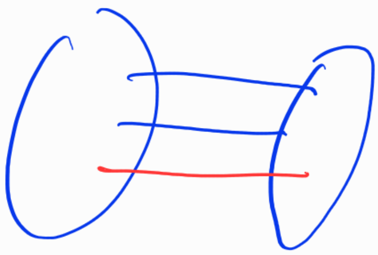
Suppose $\exists \mathcal{F}$ ... ω-fcc $\exists C_1, C_2 \in \mathcal{F}$: $C_i \supseteq \{e, e'\}$

$\rightsquigarrow \exists \mathcal{F}_1, \mathcal{F}_2$ ... ω_i-f.c.c. of G_i

$\boxed{\text{Lemma}}$



$\rightsquigarrow$

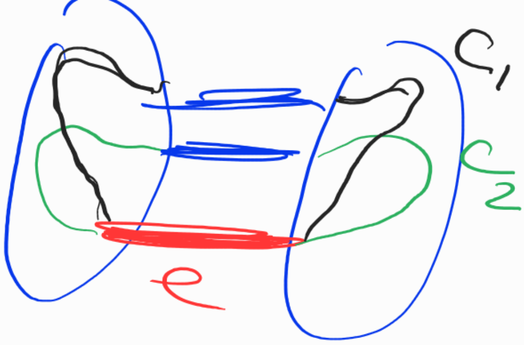


≥ one of (G_i, ω_i) is a CP then is a CP $\omega_1 \oplus_3 \omega_2 = \omega$

Pf Suppose ∃ $\mathcal{F}$ ... ω-fcc

⇐

∃ ω_i-f.c.c. of G_i for $i=1$
& for $i=2$. ∎



$\boxed{\text{Conj: (Seymour)}}$ ⟹ CDC ($\omega=2$)

$w: E(G) \to \{1,2,\dots\}$ admiss. colouring, G bridgeless s.t.
∀e $w(e)$ is even.
Then (G,w) has a f.c.c.

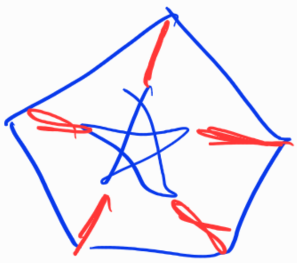
$\boxed{\text{Conj. (Goddyn)}}$ .... ∀e: $w(e) \geq 2$ .... ∃ fcc

$\boxed{\text{Conj: (Seymour, strong CDC)}}$? disproved

If $w: E(G) \to \{1,2\}$, G bridgeless cubic
$\{e: w(e)=1\}$ is a circuit ⟹ ∃ ω-fcc

(∀ circuit C ∃ CDC containing C)

not true if C = 2 cycles



Problem (Jackson)

Given: (G, w)
Q: $\exists$ nw-fcc

$\Bigg\}$ NP-complete?