Combinatorics and graph theory 3 – 2020/21
Series 6

1. Every two disjoint set of vertices of a graph form a 1-regular pair.

2. Suppose sets A, B are disjoint and for every $X \subseteq A$, $Y \subseteq B$, $X, Y \neq \emptyset$ we have
$$|d(X,Y) - d(A,B)| \leq 0$$
(something like 0-regularity). What can you say about the graph induced by $A \cup B$?

3. Suppose $\varepsilon_1 > \varepsilon_2 > 0$. Which property is stronger: ε_1-regularity or ε_2-regularity?

4. Show that ε-regular partition of a graph G is also ε-regular partition of the complement, $\bar{G}$.

5. Let $n(\varepsilon, m)$ be any function. Suppose the following version of regularity lemma holds: For every $\varepsilon > 0$ and every m there is M such that every graph with $n \geq n(\varepsilon, m)$ vertices has ε-regular partition with k parts, where $m \leq k \leq M$.

   Show that this implies the usual regularity lemma.

6. Let (G_n) be a sequence of graphs such that $|G_n| = n$ and $\|G_n\| = o(n^2)$. Show that regularity lemma holds for (G_n) – that is the weaker version is true, where we can only choose graphs from the sequence (G_n).

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7. Let (A, B) be ε-regular pair with $d(A,B) = d$ and $|A| = |B| = n$. Then there is $X \subseteq A$, $Y \subseteq B$, for which
   - $|X| = |Y| \geq (1 - \varepsilon)n$,
   - every vertex in X has at least $(d - 2\varepsilon)|B|$ neighbors in Y, and
   - every vertex in Y has at least $(d - 2\varepsilon)|A|$ neighbors in X.

8. "Restriction of a regular pair" Let (A, B) be an ε-regular pair (in some graph) with $d(A,B) = d$. Suppose further $\alpha > \varepsilon$ and let $X \subseteq A$, $Y \subseteq B$ satisfy $|X| \geq \alpha|A|$, $|Y| \geq \alpha|B|$. Then (X, Y) is ε'-regular pair, where $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$ and $d(X,Y) = d'$, while $|d - d'| \leq \varepsilon$.

9. * "common neighborhood" Let (A, B) be an ε-regular pair (in some graph) with $d(A,B) = d$, let $s > 0$ be an integer. For a tuple $\vec{a} = (a_1, \ldots, a_s) \in A^s$ we let $N(\vec{a}) = \cap_{i=1}^{s} N(a_i)$ be the common neighborhood of vertices in $\vec{a}$. Let the set $Y \subseteq B$ satisfy $(d - \varepsilon)^{s-1}|Y| \geq \varepsilon|B|$. Then
$$|\{\vec{a} \in A^s : |Y \cap N(\vec{a})| < (d - \varepsilon)^s |Y|\}| < s\varepsilon |A|^s .$$

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10. Let $|A| = |B| = |C| = n$, let (A, B), (B, C), (C, A) be three ε-regular pairs, for some $\varepsilon \in (0, 1/2]$. Let $t = t(A, B, C)$ the number of triangles with one vertex in A, another in B and the third in C. Then
$$|t - d(A,B)d(B,C)d(C,A)n^3| \leq 13\varepsilon n^3.$$

Hint for 4: Choose sufficiently large M.

Hint for 5: We can choose any partition into not too many parts. If n is large enough, and thus $\|G_n\|/n^2$ small enough, then for every tested pairs $X \subseteq A$, $Y \subseteq B$ we have $d(X,Y) \leq \varepsilon$, and thus $(A,B))$ is an ε-regular pair.

Hint for 6: Use twice the lemma from class.

Hint for 8: Use induction over s, for $s = 1$ we proved it in class.

Hint for 9: According to a theorem from class, majority of vertices in A has a typical number of neighbors in B and in C. For them use the definition of regular pair for (B,C).