Combinatorics and graph theory 3 – 2020/21
Series 2

1. Consider $k = 0$, 1, 2, and 3. Is the k-sum of two planar graphs a planar graph?

2. Let G be a connected graph with no $K_{1,k}$-minor. Show that G has at most $10k$ vertices of degree more than 2.

3. Show that if a graph G has $n \geq 4$ vertices and at least $2n - 2$ edges, than G contains K_4 as a minor.

4. Show that if a graph G has $n \geq 4$ vertices and at least $3n - 5$ edges, than G contains K_5 as a minor.

5. Show that if G is a 3-connected graph containing K_5 as a topological minor, then either $G \cong K_5$ or G contains $K_{3,3}$ as a topological minor.

6. Using the statement of the previous exercise and Kuratowski's theorem show, that G has no $K_{3,3}$ minor if and only if G is a (≤ 2)-sum of planar graphs and copies of K_5.

7. Using the statement of the previous exercise show, that every graph of minimal degree at least 6 has a $K_{3,3}$ minor.

Hint 2: consider a spanning tree of G with maximal number of leaves.

Hint 5: let H be a subdivision of K_5 containing path $xv_1 \dots v_t y$, where $\deg(x) = \deg(y) = 4$, $\deg(v_1) = \dots = \deg(v_t) = 2$ *and $t \geq 1$. If H is a G, than as $\{x, y\}$ is not a cut in G, G must have a path P from $\{v_1, \dots, v_t\}$ to the rest of H. Then $H \cup P$ contains a subdivision of $K_{3,3}$.*