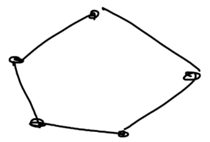
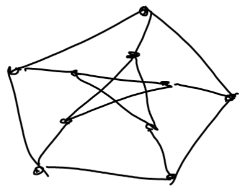
Large graphs – Graph Limits

What we study

- graph of www / computer network
- social networks
- biological networks (animals & interactions)
- stat. physics ~ particles / links



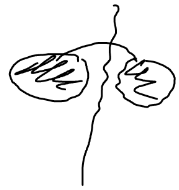


What we ask

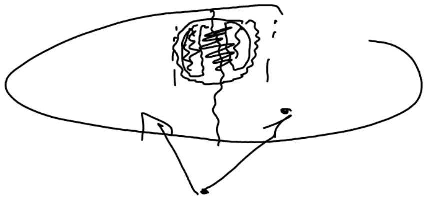
Q1: Does the graph have an odd or even # of nodes? --- BAD Q

Q2: Average degree = ? --- Good Q --- we need to allow an a prob. of error

Q3: Is the graph connected?



Q4: MAX-CUT



How we can do it?

sampling $|G| = n$

take random subset $S \in \binom{V(G)}{k}$ } poss. repeat

look at $G[S]$

easy if we have a list of all nodes

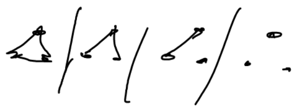
o/wise .... Markov chains

$\boxed{\text{works well if } \|G\| \approx c|G|^2}$

$\boxed{\text{dense graphs}}$

$k=2$

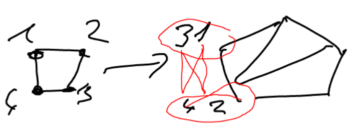
$k=3$



→ if G is rather sparse ($d_{avg} = \text{const}$) --- random sample is a.s. empty graph

random node & its complete neighborhood up to distance r $\{v : d(u,v) \le r\}$

homomorphism $f: F \to G$ is a map $f: V(F) \to V(G)$ s.t. $\forall uv \in E(F)$ $f(u)f(v) \in E(G)$



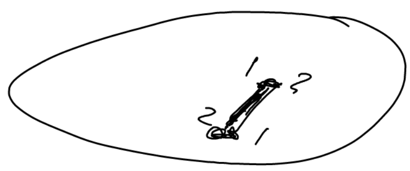
$\hom(F,G) = \#$ of hom $F \to G$

homomorph. to K_k $\iff$ k-coloring, ...

f is 1-1 then G contains a copy of F

Q: what if $F = \,:$? Find # of subgraphs of G isom. to F using homom.?

$$\hom(\text{edge}_{1,2}, G) = \sum_{v \in V(G)} \deg(v)$$



$$= 2\|G\|$$

$$\hom(:, G) = |G|^2$$

isom. copies of $\overline{K_2}$ in G $= \binom{|G|}{2} - \#$ copies of K_2

induced subgr.

$$= \binom{|G|}{2} - \frac{1}{2}\hom(\text{edge}, G)$$

similarly for $k \geq 2$ — exercise

$$t(F,G) := \frac{\hom(F,G)}{|G|^{|F|}} = P(\text{random mapping } V(F) \to V(G) \text{ is a homom.})$$

→ $\in [0,1]$:)

→ homom. are well behaved (algebr. theory ... theory of categories)

How to model large graphs?

p = const.

→ Erdős-Rényi (1959) -- $G(n,p)$ $V = \{1, \dots, n\}$, each edge w. prob. p

(variants: $G(n,m)$ --- $|V| = n$, $|E| = m$ $E \in \binom{\binom{V}{2}}{m}$ uniformly ...)

all vertices are almost the same

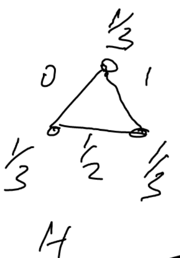
→ multitype random graph $G(n,H)$ (H – weighted graph, $|H| = q$; nodes have weights $\alpha_1 \dots \alpha_q > 0$, $\sum \alpha_i = 1$; edges have weights $\beta_{ij} \in [0,1]$)

template

Take n vertices, each chooses type 1 w. prob. α_1
2 α_2
⋮

β_{ji}

$\forall\, uv \in \binom{[n]}{2}$ we decide if $uv \in E(G)$ w. prob. β_{ij} i = type of u, j ... v



random half of the edges

More generally: template can be a graphon: $W: [0,1]^2 \to [0,1]$ symmetric ftion

$\forall$ vertex chooses a type $\sim$ Unif.$(0,1)$

$$\begin{array}{cccc} 0 & & & 1 \\ & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \frac12 \\ 0 & \frac12 & 0 \end{bmatrix} & & \\ 1 & & & 1 \end{array}$$

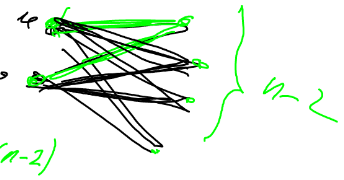
Quasirandom graphs (Chung, Graham, Wilson 1989)

(G_n) -- seq. of graphs, $|G_n| \to \infty$ $\quad 0<p<1$

(QR1) Almost all degrees are $pn\,(1+o(1))$

— " — codegrees $p^2 n\,(1+o(1))$

$E \deg 1 = p(n-1)$
r.v.
$E \operatorname{codeg}(1,2) = p^2(n-2)$



(QR2) $\forall$ fixed graph F, $\hom(F,G) = p^{\|F\|} \cdot n^{|F|}\,(1+o(1))$

$n = |G|$ $\qquad E \hom(F, G(n,p))$

$\Downarrow$

(QR3) $\|G\| = p\frac{n^2}{2}\,(1+o(1))$ $\qquad \|G\| = \frac12 \hom(K_2, G)$

$\#\, C_4$-subgraphs $= \frac{p^4 n^4}{8}\,(1+o(1))$ $\qquad \#\, C_4\text{'s} = \frac18 \hom(C_4, G)$

(QR4) $\qquad \forall S$ of $\frac{n}{2}$ nodes of G_n induces $\sim \frac{pn^2}{8}\,(1+o(1))$

(QR5) $\forall X, Y \subset V(G_n)$ disjoint : $|E_G(X,Y)| = p|X|\cdot|Y|\,(1+o(1))$

$\hookrightarrow$ Szemerédi reg. lem. ---- ε-reg. pair

Theorem QR1 $\Leftrightarrow$ QR2 $\Leftrightarrow$ QR3 $\Leftrightarrow$ QR4 $\Leftrightarrow$ QR5

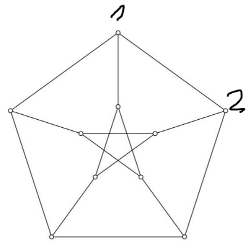
Example Paley graphs $\quad q \equiv 1 \pmod 4$ & q prime

$V = \{0,1,\dots,q-1\}$, $\quad ij \in E \Leftrightarrow i-j \equiv k^2 \pmod q$ for some $k = 1,\dots,q-1$

Claim: Paley graphs are QR with $p = \frac12$.

for all primes $4k+1$

How to approximate large graphs ?)



$$\begin{matrix}0&1&0&0&1&1&0&0&0&0\\1&0&1&0&0&0&1&0&0&0\\0&1&0&1&0&0&0&1&0&0\\0&0&1&0&1&0&0&0&1&0\\1&0&0&1&0&0&0&0&0&1\\1&0&0&0&0&0&0&1&1&0\\0&1&0&0&0&0&0&0&1&1\\0&0&1&0&0&1&0&0&0&1\\0&0&0&1&0&1&1&0&0&0\\0&0&0&0&1&0&1&1&0&0\end{matrix}$$

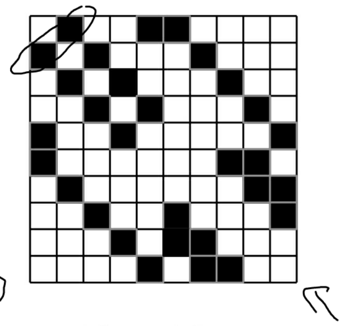


FIGURE 1.3. The Petersen graph, its adjacency matrix, and its pixel picture

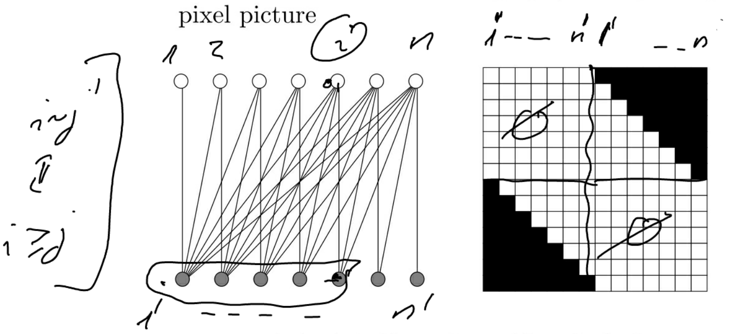
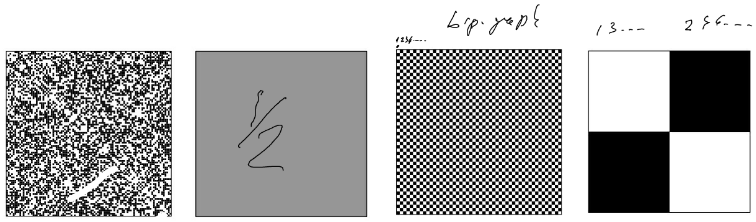
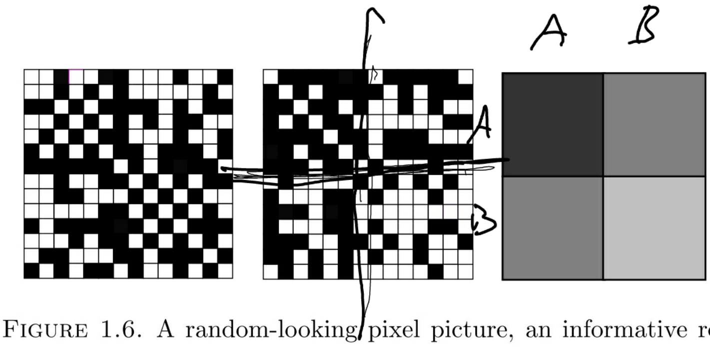


FIGURE 1.4. A half-graph and its pixel picture

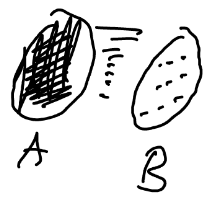


rearrange rows & cols
--- nicer picture

FIGURE 1.5. A random graph with 100 nodes and edge density 1/2, a random graph with very many nodes and edge density 1/2, a chessboard, and the pixel picture obtained by rearranging the rows and columns.



≈ reg. lemma



density in A
in B
between

FIGURE 1.6. A random-looking pixel picture, an informative re-arrangement, and its regularity partition

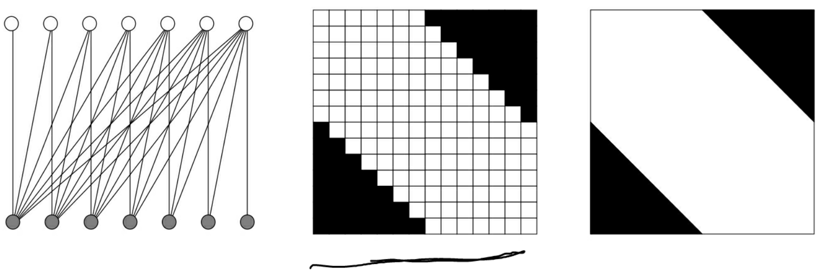


FIGURE 1.7. A half-graph, its pixel picture, and the limit function

graphons
symmetr. fctns
$w:[0,1]^2\to[0,1]$
measurable
$P(\{w(x,y)\le c\})$

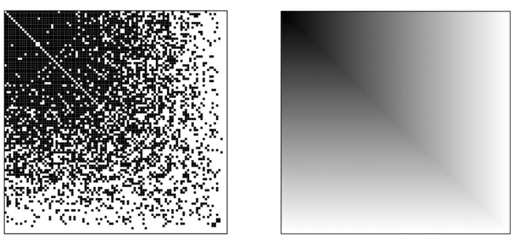


FIGURE 1.8. A randomly grown uniform attachment graph with 100 nodes, and a (continuous) function approximating it

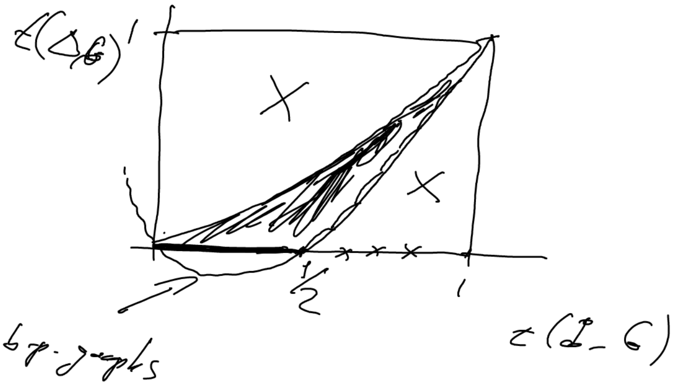
Application to extremal gr. th.

Mantel's thm / spec. typ. of Turán's thm:

$$\|G\| > \frac{n^2}{4} \Rightarrow G \supseteq K_3$$

$$t(K_2, G) > \frac{1}{2} \Rightarrow t(K_3, G) > 0$$

General Q: For what $x, y \in [0,1]^2$ is there G s.t. $t(K_2, G) = x$, $t(K_3, G) = y$?



Goodman's thm

$$\boxed{t(K_3, G) \geq t(K_2, G) \cdot (2t(K_2, G) - 1)}$$

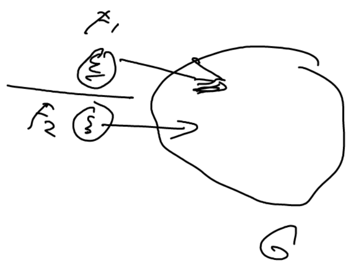
spec. case of Kruskal-Katona

$$t(K_3, G) \leq t(K_2, G)^{3/2}$$

Observations

$F_1F_2 := F_1 \cup F_2$ (disj.)

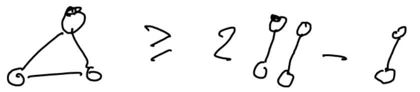
$$t(F_1F_2, G) = t(F_1, G) \cdot t(F_2, G)$$



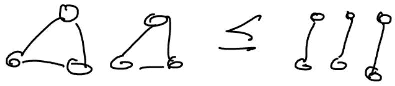
F will mean $t(F, G)$

$$F_1F_2 = F_1 \cdot F_2$$

Goodman's thm: $K_3 \geq K_2 \cdot (2K_2 - 1) = 2K_2^2 - K_2$



Kruskal-Katona



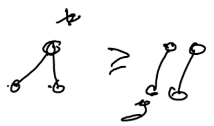
$$t(C_4, G) \geq t(K_2, G)^4 \quad \text{(Erdős)}$$

$$t(P_k, G) \geq t(K_2, G)^{k-1}$$

(P_k has k nodes)

(for reg. gr. we have =)

$k = 3$



$$n \cdot \hom(P_3, G) \geq \hom(K_2, G)^2$$

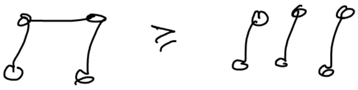
$$n \cdot \sum_v \deg(v)^2 \geq \left(\sum_v \deg v\right)^2$$

$$\frac{\sum_v \deg(v)^2}{n} \geq \left(\frac{\sum_v \deg v}{n}\right)^2$$

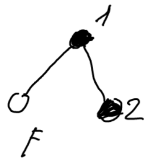
image of X

$f(y)$

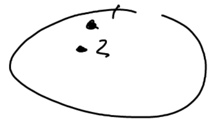
ineq. arithmetic & quadr. mean



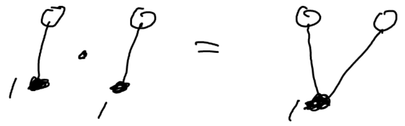
labeled graphs

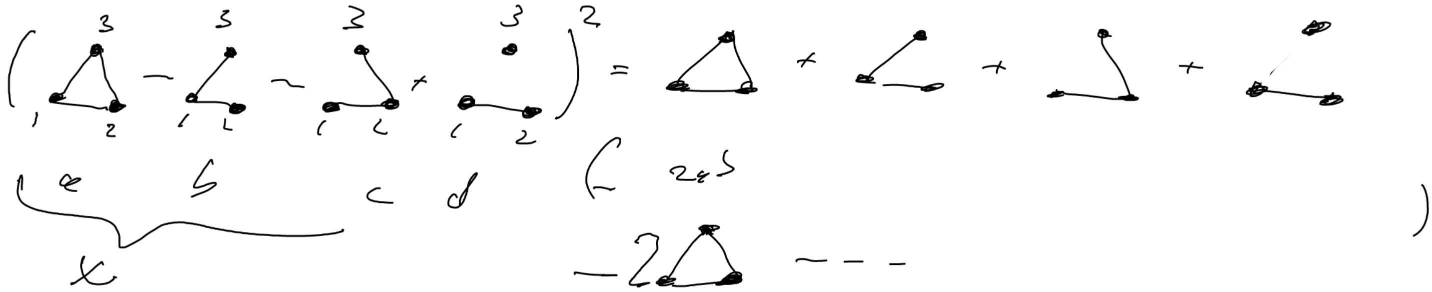


hom. must resp. labels

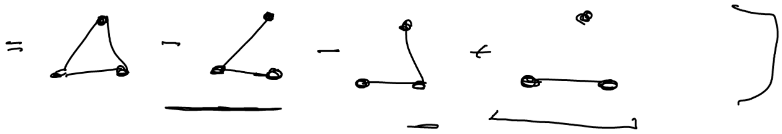
hom(F, ) = # hom. where $1 \to 1$ & $2 \to 2$

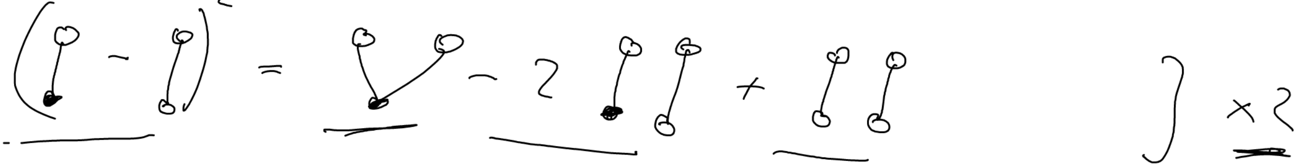
F_1, F_2 labeled : $F_1 \cdot F_2$ = disj. union + identif. of vertices with the same label

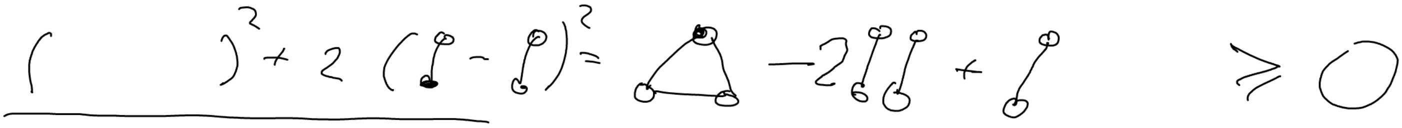


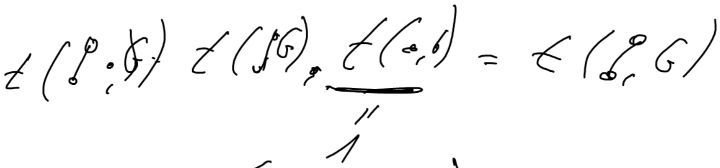


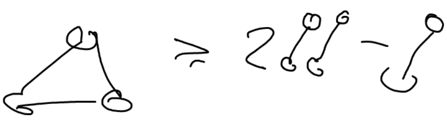
$x^2 = x$











Goodman's ineq. :)

quantum graph algebra

flag algebras, Razborov