Polynomial method

Lemma p ... prime, $0 \le j \le p-2$ $\Rightarrow$ $\sum_{x \in \mathbb{Z}_p} x^j = 0$ (0 in $\mathbb{Z}_p$)

ex: $p=5$, $j=1$ $\quad 0+1+2+3+4 = 0 \pmod 5$

$j=2 \quad 0+1+4+4+1 = 0$

$j=4 \quad 0+1+1+1+1 \neq 0 \Rightarrow p-1 \mod p$

Little Fermat theorem: $\forall p$ prime $\forall x \neq 0 \pmod p$ $x^{p-1} = 1 \pmod p$

Proof choose $g \in \mathbb{Z}_p$: $g \neq 0$ & $g^j \neq 1 \pmod p$ (Possible, as $g^j - 1$ has $\le j$ roots & $j+1 < p$.)

Observe that $x \mapsto gx$ is a bijection

$\sum_x x^j = \sum_x (gx)^j = g^j \sum x^j$ $\quad$ ($x, x' \Rightarrow gx = gx'$, $g \cdot (x-x')$ divis. by p $\Rightarrow x = x'$)

$\Rightarrow \sum x^j = 0$

∎

Multivariate polynomials $f \in \mathbb{Z}_p[x_1, \dots, x_n]$

$x_1 + \underbrace{x_2 x_3^4 x_4}_{6} + 11 \cdot x_1^{102} + \dots$ $\quad$ (degree 102)

$f: \sum c_\alpha x^\alpha$, $c_\alpha \in \mathbb{Z}_p$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$

total degree $\deg f = \max_{\alpha : c_\alpha \neq 0} (\alpha_1 + \dots + \alpha_n)$

Thm (Chevalley-Warning)

p is a prime; $f_1, \dots, f_n \in \mathbb{Z}_p[x_1, \dots, x_n]$

$\deg f_i = d_i$

$\sum_{i=1}^{n} d_i < n$

Then the number of solutions to $f_1(x) = f_2(x) = \dots = f_n(x) = 0$ (with $x = (x_1 \dots x_n)$) is divisible by p. In particular, if there is ≥ 1 solution then there are ≥ 2.

Proof $f(x) = \prod_{i=1}^{r} \left(1 - f_i(x)^{p-1}\right)$ if $f_i(x) \neq 0$ then $f_i(x)^{p-1} = 1 \Rightarrow f(x) = 0$

$\#$ of solutions $= \sum_{x \in \mathbb{Z}_p^n} f(x)$ if $f_1(x) = 0 \dots f_r(x) = 0$ then $f(x) = 1$

$$\deg f = \sum_{i=1}^{r} \deg f_i^{p-1} = \sum (p-1) \deg f_i = (p-1) \sum d_i < n(p-1)$$

$\boxed{f = \sum c_\alpha x^\alpha}$ $\forall \alpha,\ c_\alpha \neq 0$ $\deg x^\alpha = \alpha_1 + \dots + \alpha_n < n(p-1)$

$\Rightarrow \boxed{\exists i : \alpha_i \le p-2}$

so there are pol. g_{ij} ($i = 1, \dots, n$, $j = 0, \dots, p-2$, g_{ij} does not contain x_i)

s.t.

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{p-2} \underbrace{g_{ij}(x)}_{c_\alpha x^\alpha / x_i^j} \cdot x_i^j \qquad \text{(key step)}$$

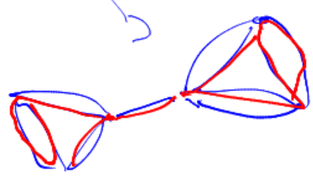
$$\sum_{x \in \mathbb{Z}_p^n} g_{ij}(x)\, x_i^j = \sum_{\substack{x_1, x_2 \dots x_{i-1}, x_{i+1} \\ \dots x_n \in \mathbb{Z}_p}} \sum_{x_i \in \mathbb{Z}_p} g_{ij}(x)\, x_i^j$$

$$= \sum_{\hat{x}_i} \left(g_{ij}(x) \underbrace{\sum_{x_i \in \mathbb{Z}_p} x_i^j}_{=0} \right) = 0$$

$$\sum_{x \in \mathbb{Z}_p^n} f(x) = \sum_x \sum_i \sum_j g_{ij}(x)\, x_i^j = \sum_i \sum_j \overbrace{\left(\sum_x g_{ij}(x)\, x_i^j \right)}^{0} = 0 \qquad \blacksquare$$

Thm G is a multigraph, all degrees are 4 or 5, G is not 4-regular.

Then G has 3-regular submultigraph.



Proof $r = |V(G)|$ $p = 3$ $x_e = 0, 1, 2 \Rightarrow x_e^2 = 0, 1, 1$

$\forall v \in V(G)$ $f_v = \sum_{e \text{ inc. with } v} 1 \cdot x_e^2$ $\deg f_v = 2$, $\sum \deg f_v = 2r < n$

$\#$ sol. to $[f_v(x) = 0\ \forall v]$ is divis. by 3

$\#$ var. $= n = |E(G)| > 2r$ 1 solution: $x_e = 0\ \forall e$ ✓ $\boxed{F = \{e \in E(G) : y_e \neq 0\}}$

$\frac{1}{2} \sum \deg > \frac{1}{2} \cdot 4r$ $\Rightarrow$ another sol. $y = (y_e)$ $f_v(y) = \deg_F v = 0 \pmod 3$

Nullstellensatz

zero values — theorem

Hilbert's Nullstellensatz (alg. geometry)

if $[g_1(x)=g_2(x),\dots,g_m(x)=0 \Rightarrow f(x)=0]$

then $[\exists k\ \exists$ pol. $h_1\dots h_m : f^k = \sum h_i g_i]$ $\Uparrow$ trivial

Lemma (easy by ind.)

~~Thm~~ (Combinatorial Nullstellensatz, Alon–Tarsi)

$p \in \mathbb{C}[x_1,\dots,x_n]$, every occurence of x_i in p has deg. $\le d_i$

suppose $S_1,\dots,S_n \subseteq \mathbb{C}$; $|S_i| \ge d_i+1$

If $p \ne 0$ then $\exists s_1 \in S_1,\dots, s_n \in S_n$ $p(s_1,\dots,s_n)\ne 0$

Thm (Comb. Nullstellensatz) (works over any field)

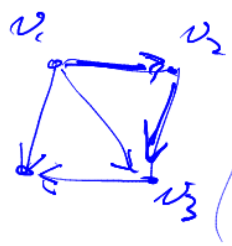
$p \in \mathbb{C}[x_1,\dots,x_n]$, $\deg p = \sum_{i=1}^{n} d_i$

$[x_1^{d_1}\dots x_n^{d_n}]p \ne 0$, $|S_i| \ge d_i+1$ $\Longrightarrow$ $\exists s_i \in S_i \dots$ $p(s_1,\dots,s_n)\ne 0$

G … undir. graph, $V(G)=\{v_1,\dots,v_n\}$; $\vec{G}$ any orient. ($(v_i,v_j) \Rightarrow i<j$)

$$P_{\vec G}(x_1,\dots,x_n) = \prod_{(v_i,v_j)\in E(\vec G)} (x_j - x_i)$$

$P_{\vec G}(x) \ne 0 \iff v_i \mapsto x_i$ is a good coloring



$P_{\vec G}(x_1\dots x_4) = (x_2-x_1)(x_3-x_2)\cdot(x_3-x_1)(x_4-x_1)\cdot(x_4-x_3)$

Thm G undir. gr. $\vec G$ any or. of G,

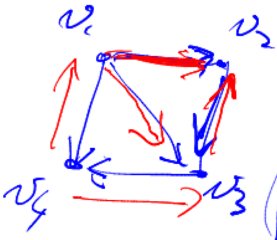
$d_1\dots d_n \in \mathbb{N}$, L: list assignment s.t. $\forall i$: $|L(v_i)| > d_i$

If $[x_1^{d_1}\dots x_n^{d_n}]P_{\vec G} \ne 0$ then G is L-colorable

Proof $S_i := L(v_i)$

$\sum d_i = |E(G)|$

observe: all monomials have the same degree



$P_{\vec{G}}(x_1 \dots x_4) = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1)(x_4 - x_1)(x_4 - x_3)$

$\dots\dots + x_2(-x_2)x_3(-x_1)(-x_3)$

$(-1)^3 x_1 x_2^2 x_3^2$

$\dots\dots \vec{G}' \dots\dots (-1)^s x_1^{d_1} \dots x_n^{d_n}$ $\qquad d_i = \deg^-_{\vec{G}'} v_i$

$s = \#$ of edges where $\vec{G}$ & $\vec{G}'$ differ $=: \operatorname{sign}(\vec{G}')$

$$[x_1^{d_1} \dots x_n^{d_n}] = \sum_{\vec{G}' \in \mathcal{O}_{d_1 \dots d_n}} \operatorname{sign}(\vec{G}') \longrightarrow \deg^-(v_i) = d_i \quad \forall i$$

Corollary G undir. gr. on $\{v_1 \dots v_n\}$

$\vec{G} \in \mathcal{O}_{d_1 \dots d_n}$; $\mathcal{E} = \{F \subseteq E(\vec{G}) : F$ is Eulerian$\}$

$$[x_1^{d_1} \dots x_n^{d_n}] = (\pm) \sum_{F \in \mathcal{E}} (-1)^{|F|}$$

$\deg^+_F v_i = \deg^-_F v_i \quad \forall i$

$\longrightarrow \vec{G}' \rightsquigarrow F = \{e : e$ is or. diff. in $\vec{G}$ & $\vec{G}'\}$

$\vec{G}' \in \mathcal{O}_{d_1 \dots d_n} \iff F$ is eulerian

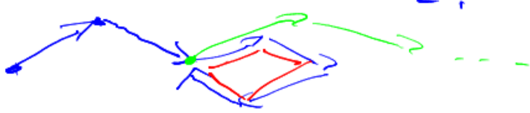
Corollary G undir, bipartite on $\{v_1 \dots v_n\}$

$\vec{G} \in \mathcal{O}_{d_1 \dots d_n}$. Then $[x_1^{d_1} \dots x_n^{d_n}] P_G \neq 0$,

therefore G is L-colorable whenever $|L(v_i)| > d_i \quad \forall i$.

observe: F is eulerian & G bipartite then $|F|$ is even.

decompose F to disjoint or. cycles



\# cycle in bip. graph is even $\Longrightarrow$ $|F|$ is even

$\Rightarrow$ $\left[x_1^{d_1} \cdots x_n^{d_n}\right] P_{\vec{G}} = \pm \sum (-1)^0 = \pm$ \# of euler. subgr. $\neq 0$

$\emptyset \in \mathcal{E}$

not hard using $|E| \le 2|V| - 4$ & Hall's Thm

Corollary G is planar, bipartite

$\Longrightarrow$ G has orient. with all in-degrees ≤ 2.

Therefore G is 3-choosable.