(Blow-up Lemma; Komlós-Sárközy, Szemerédi 1997)

Lemma $\forall d \in (0,1]\ \forall \Delta \geq 1\ \exists \varepsilon_0 > 0$:

$\forall G$ graph $\forall H$ graph $\Delta(H) \leq \Delta$ $\forall s \in \mathbb{N}$

if R ... regul. graph of G with $\varepsilon \leq \varepsilon_0$, $\ell \geq \frac{2s}{d^\Delta}$, d

then $\boxed{H \subseteq R_s \implies H \subseteq G}$

Turán thm: many edges of G $\implies$ compl. subgr. of G

Erdős-Stone — // — $\implies$ $H \subseteq G$

H bipartite .... separate problem

Proof of the blow-up lemma

given d, Δ, choose ε_0 s.t. 1) $\varepsilon_0 < d$

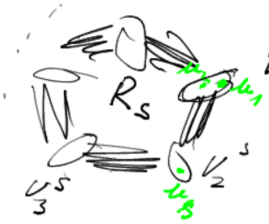
2) $(d-\varepsilon_0)^\Delta - \Delta \cdot \varepsilon_0 \geq \frac{1}{2} d^\Delta$

let G, H, s, R be given $\quad (\varepsilon_0 \to 0 \quad d^\Delta)$

$\{V_0, V_1, \dots, V_k\}$ ε-reg. part. that gives as R, $\varepsilon \leq \varepsilon_0$.

$H \subseteq R_s$, H has vert. $u_1, \dots, u_h$

$\boxed{\text{goal}}$ def. an embedding $u_i \mapsto v_i \in V_{\sigma(i)}$



$u_i \in V_j^s$

$j = \sigma(i)$

$\sigma : \{1, \dots, h\} \to \{1, \dots, k\}$

$H \hookrightarrow G$

injective & homom. $\implies$ gives $H \subseteq G$ $\implies$ $v_1, v_2, \dots, v_h$ distinct & $u_i u_j \in E(H) \implies v_i v_j \in E(G)$

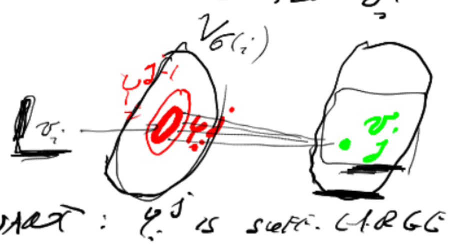
$\forall i$: "target set" $Y_i \subseteq V_{\sigma(i)}$ ... poss. candidates for v_i

$Y_i^0 := V_{\sigma(i)} \supseteq Y_i^1 \supseteq \dots \supseteq Y_i^i \ni \{v_i\}$

$(u_i u_j \notin E(H))$ ... $\boxed{Y_i^j = Y_i^{j-1}}$

$(u_i u_j \in E(H))$ ... $Y_i^j = Y_i^{j-1} \cap N(v_j)$

$\ell = |V_{\sigma(i)}|$

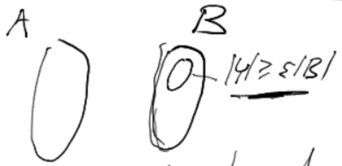


$j < i$

WANT: Y_i^j is suff. LARGE

WANT: $|Y_i^j| \geq (d-\varepsilon)|Y_i^{j-1}|$ (*)

$d \in (0,1)$
$d-\varepsilon \in (0,1)$



all but $\leq \varepsilon|A|$ vert. in A have $\geq (d-\varepsilon)|B|$ neighb. in Y

$(A,B) = (V_{\sigma(j)}, V_{\sigma(i)})$ .... this is ε-regular pair .... $u_i u_j \in E(H)$

$H \subseteq R_s$

& $d(A,B) \geq d$

need: $|Y_i^{j-1}| \geq \varepsilon \ell$

get: $\leq \varepsilon\ell$ bad vert. in $V_{\sigma(j)}$. all others have

$|Y_i^{j-1} \cap N(x)| \geq (d-\varepsilon)|Y_i^{j-1}|$ .... good for i

→ u_j has $\leq \Delta$ neighbors in H .... $\leq \varepsilon \ell \cdot \Delta$ bad vertices in $V_{\sigma(j)}$.

NEED: $|Y_j^{j-1}| > \varepsilon \ell \Delta + s$

in G

$Y_j^{j-1} \ni v_j$

$V_{\sigma(j)}$ 1) v_j not bad for each i: $u_i u_j \in E(H)$

2) $v_j \neq v_1, v_2, v_3, \dots$

$\leq s-1$ vertices among $v_1, v_2, \dots, v_{j-1}$ are $\in V_{\sigma(j)}$

$$|Y_j^{j-1}| - \varepsilon \ell \Delta \geq (d-\varepsilon)^\Delta \ell - \Delta \varepsilon \ell \geq \left((d-\varepsilon_0)^\Delta - \Delta\varepsilon_0\right)\ell \geq \frac{1}{2} d^\Delta \ell \geq s > 0$$

$\varepsilon \leq \varepsilon_0$

$|Y_j^{j-1}| \geq \varepsilon \ell$

$|Y_j^0|$

by assumption on ℓ

$\Rightarrow$ we can find v_j so that (*) holds for all i s.t. $u_i u_j$

$H \subseteq R_s$

in R_s V_i^s, $|V_i^s| = s$ ■

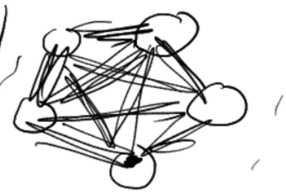
Turán's theorem: many edges $\Rightarrow K_r$ subgraph

$r=3$ reference case

$\lfloor \frac{n}{2} \rfloor$ $\lceil \frac{n}{2} \rceil$

$T_2(n)$

$T_{r-1}(n)$ Turán graph



$t_{r-1}(n) = \|T_{r-1}(n)\|$

$r-1$ parts as eq. as poss

Thm $|G| = n$, $\|G\| > t_{r-1}(n) \Rightarrow G \supseteq K_r$

$ex(n,H) = \max\{\|G\| : |G| = n \;\&\; G \not\supseteq H\}$

$ex(n, K_r) = t_{r-1}(n)$

$t_{r-1}(n) \le \frac{1}{2} n^2 \boxed{\frac{r-2}{r-1}}$

Thm (Erdős–Stone 1946) $\forall r \ge 2, \forall s \ge 1\ \forall \gamma > 0\ \exists n_0\ \forall n \ge n_0$
Every graph with n vertices & $\ge t_{r-1}(n) + \gamma n^2$ edges has K_r^s subgraph.

K_r blowed-up s-times

K_3^s

Corollary Erdős–Stone–Simonovits (proof … exercise)

$\forall H\ \ \|H\| > 0$

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \underbrace{\frac{\chi(H)-2}{\chi(H)-1}}$$

$\Rightarrow H \in \mathcal{B} \to ex(n,H) = o(n^2)$

Proof of Erdős–Stone $s=1$ … Turán's thm

Assume $s \ge 2$. Fix G : $|G| = n$, $\|G\| \ge t_{r-1}(n) + \gamma n^2$

want : $G \supseteq K_r^s$ if n is large enough. $\gamma < \frac{1}{2}$

By blow-up lemma : $R_s \supseteq K_r^s$ is suff if ……

$H = K_r^s \quad \Delta = \Delta(H) = (r-1)s$

$d = \gamma \leadsto \varepsilon_0 > 0, \quad m > \frac{1}{\gamma}, \quad \varepsilon > 0 : \quad \varepsilon < \gamma/2, \ \varepsilon \le \varepsilon_0$

$\varepsilon, m \xrightarrow{R.L.} M$

$\delta = 2\gamma - \varepsilon^2 - 4\varepsilon - d - \frac{1}{m} > 0$

assume : $n \ge \underbrace{\frac{2Ms}{d^\Delta(1-\varepsilon)}} \ge M \ge m \Rightarrow$ R.L. applies, ε groups $\{V_0, V_1, \dots, V_k\}, |V_i| = \ell$

$\ell = \frac{n - |V_0|}{k} \ge \frac{n - \varepsilon n}{M} = n\frac{1-\varepsilon}{M} \ge \frac{2s}{d^\Delta}$ … ass. of BL $\quad n \ge k\ell$

$\leadsto R$ … reg. graph ε, ℓ, d

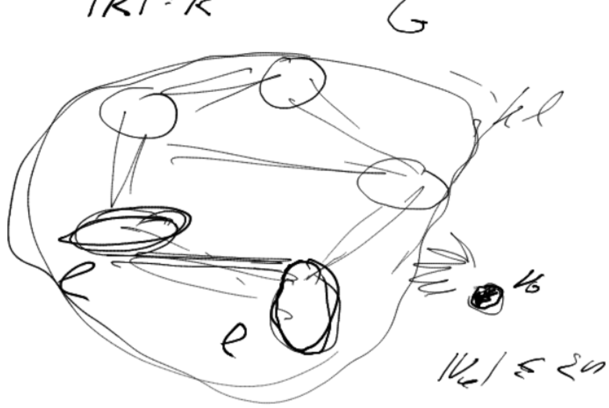
B.L. : $K_r^s \subseteq G \Leftarrow K_r^s \subseteq R_s \Leftrightarrow \boxed{K_r \subseteq R}$

 $\subseteq G \Longleftarrow$  $\subseteq R_S \Longleftrightarrow$  $\subseteq R$

$\boxed{\text{Plan}}$ Show $K_r \subseteq R$ by Turán's thm

We want: many pairs i,j: (V_i, V_j) ε-reg. pair & $d(V_i,V_j) \geq d$

We need $\|R\| \geq t_{r-1}(k)$ $\qquad |R| = k$



$|V_0| \leq \varepsilon n$

(in V_0) $\qquad$ (V_0 -- rest)

$$\boxed{t_{r-1}(n)+\gamma n^2} \leq \|G\| \leq \binom{|V_0|}{2} + |V_0| \cdot k\ell$$

(irr. pairs) $\qquad$ (low dens.)

$$+ \underline{\varepsilon k^2 \cdot \ell^2} + \underline{d\ell^2 \cdot \binom{k}{2}}$$

(good pairs) $\qquad$ (inside V_i)

$$+ \underline{\|R\| \cdot \ell^2} + \underline{k \cdot \binom{\ell}{2}}$$

$$\leq \tfrac{1}{2}\varepsilon^2 n^2 + \varepsilon n \cdot n + \varepsilon k^2\ell^2 + \tfrac{1}{2} d k^2 \ell^2 + \tfrac{1}{2}\ell^2 k + \|R\| \cdot \ell^2 \qquad \Big/ \cdot \frac{1}{\ell^2}$$

$$\|R\| \geq \tfrac{1}{2}k^2 \left(\frac{t_{r-1}(n)+\gamma n^2 - \frac{1}{2}\varepsilon^2 n^2 - \varepsilon n^2}{\frac{1}{2}k^2 \cdot \ell^2} - 2\varepsilon - d - \frac{1}{k} \right)$$

$n \geq k\ell$

$k \geq m$

$-\frac{1}{k} \geq -\frac{1}{m}$

$$\geq \tfrac{1}{2}k^2 \left(\frac{t_{r-1}(n)}{\frac{1}{2}n^2} + \underbrace{2\gamma - \varepsilon^2 - 4\varepsilon - d - \frac{1}{m}}_{\delta > 0} \right)$$

$$= \tfrac{1}{2}k^2 \cdot \left(\frac{t_{r-1}(n)}{\binom{n}{2}} \cdot \left(1 - \frac{1}{n}\right) + \delta \right)$$

$$> \tfrac{1}{2}k^2 \cdot \frac{r-2}{r-1} \geq t_{r-1}(k)$$

$$\boxed{t_{r-1}(n) \leq \frac{n^2}{2} \frac{r-2}{r-1}}$$

$$\frac{t_{r-1}(n)}{\binom{n}{2}} \longrightarrow \frac{r-2}{r-1}$$