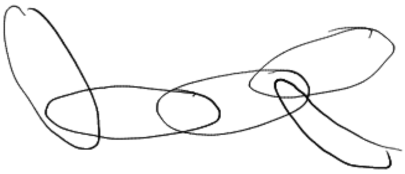
Tree width – brambles



$tw(G) = \min_{(T,v)} \max \{|V_t|-1, t\in T\}$

~~Free~~ guess (T,v): $tw(G) \le \dots$

$G \ge_m H \Rightarrow tw(G) \ge tw(H)$

$tw(K_n) = n-1$

$A,B \subseteq V(G)$ .... A,B touch $\iff$ 1) $A\cap B \neq \emptyset$ OR
2) $\exists a\in A, b\in B: ab \in E(G)$

Df **Bramble** is a coll. of mutually touching connected vertex sets.

$\mathcal{B} \subseteq P(V(G))$: 1) $B\in\mathcal{B} \to G[B]$ is connected
2) $B_1, B_2 \in \mathcal{B} \to B_1$ & B_2 touch

Df $C \subseteq V(G)$ is a cover of a bramble $\mathcal{B}$
if $\forall B\in\mathcal{B}$: $B\cap C \neq \emptyset$

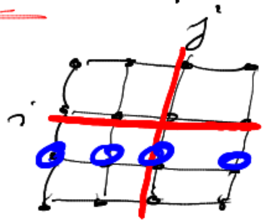
Order of $\mathcal{B}$ = $\min |C|$: C covers $\mathcal{B}$

k-bramble = bramble of order k

Ex. 1 $G = K_n$ $\mathcal{B} = \{\{v\}, v\in V(K_n)\}$

n-bramble (also has ~~n+1~~ k-bramble $\forall k<n$)

Ex. 2 $G = Grid_{k\times k}$



$(k=4)$

$\forall i,j \in \{1,\dots,k\}$ $cross_{i,j}$

$\mathcal{B} = \{ cross_{i,j} : i,j \in [k]\}$

a full row meets $\forall B\in\mathcal{B}$ | $|C|<k \to C$ misses a row & a col.
cover of size k | $\Rightarrow$ k-bramble

Thm (Seymour, Thomas, 1993) (duality tw & brambles)

$k \geq 0$, $k \in \mathbb{N}$, G a graph

$$tw(G) < k \iff G \text{ has no bramble of order } > k$$

$G = K_n$ $\quad k = n \quad tw(G) < n$

$\quad k = n-1 \quad tw(G) < n-1 \iff G$ has no br. $> n-1$

False

$\Rightarrow tw(K_n) = n-1$

$tw(G) =$ bramble

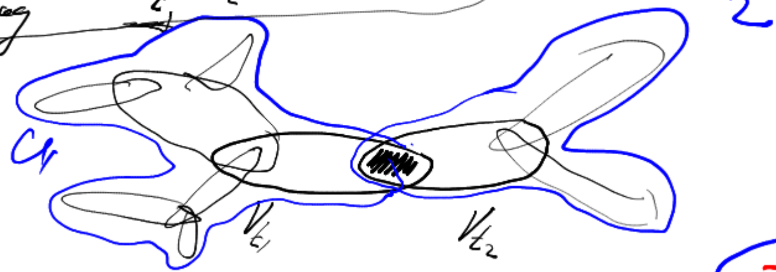
$G = Grid_{l,l}$ has l-bramble : $k = l-1 \quad tw(G) \geq k$

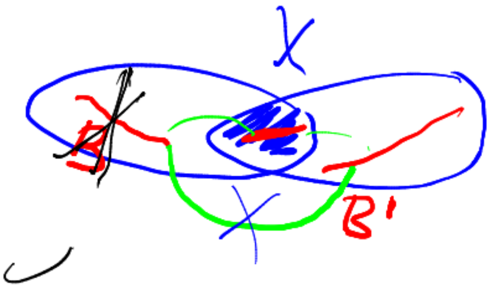
$l \geq tw(Grid_{l,l}) \geq l-1$

$\boxed{\text{Proof } \Rightarrow}$ G, $(T, \mathcal{V})$ t.d. of G $\Rightarrow$ order $\mathcal{B} \leq |V_t| \leq tw(G)+1$

$\forall \mathcal{B}\ \exists$ a part/bag V_t : V_t covers a $\mathcal{B}$

$t_1 t_2 \in E(T)$





$X = V_{t_1} \cap V_{t_2}$ separates U_1 from U_2

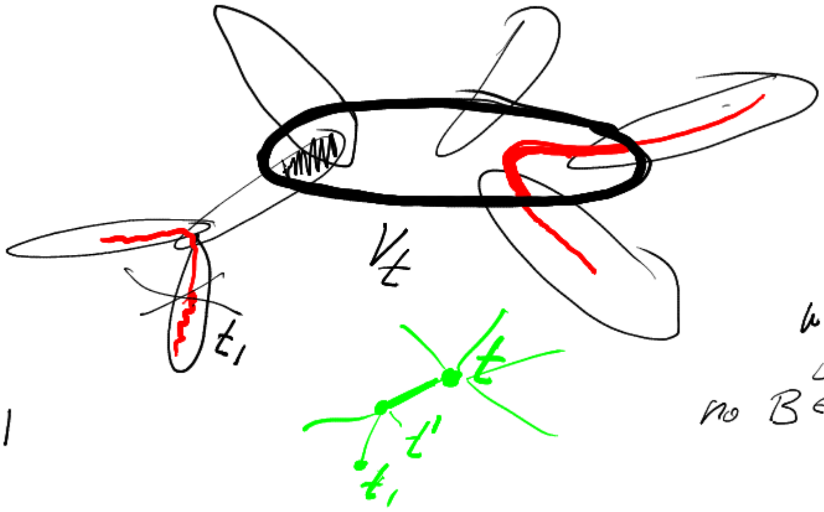
X covers $\mathcal{B}$ $\Rightarrow$ V_{t_1} covers $\mathcal{B}$ ✓

$\nexists B \in \mathcal{B}$ $\quad B \subseteq U_1 \setminus U_2$ or $B \subseteq U_2 \setminus U_1$

wlog: no $B \in \mathcal{B}$ s.t. $B \in U_1 \setminus U_2$

$t_1 \longrightarrow t_2$ ... we orient the tree edge

we find $t \in T$ with no outgoing edge $\Rightarrow V_t$ covers $\mathcal{B}$



Suppose $\exists B \in \mathcal{B}$

$B \cap V_t = \emptyset$

$B \cap V_{t_1} \neq \emptyset$

we know $t' \to t$

$\Downarrow$

no $B \in \mathcal{B} : B \in U_1 \setminus U_2$

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why t.d. are useful — gr. theory .... describe structure of graphs

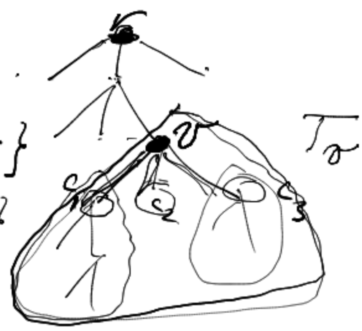
— $\boxed{\text{algorithms}}$ .... find efficient alg. for graphs of small t.w.

① $\underline{\alpha(G)}$ of $G = T$ is tree

→ $r \in T$ .... root

$\forall v : \; f^1(v) = \max\{ |U| : U \subseteq V(T_v) \;\&\; U \text{ indep. } \& \; U \ni v \}$

$f^0(v) = \quad$ — // — $\quad U \not\ni v \}$



→ post-order

→ v is a leaf ..... $T_v = \bullet$, $\quad f^1(v) = 1, \; f^0(v) = 0$

→ v has children $c_1 \dots c_\ell$

$$f^1(v) = 1 + \sum_{i=1}^{\ell} f^0(c_i)$$

$$f^0(v) = 0 + \sum_{i=1}^{\ell} \max\{ f^0(c_i), \underline{f^1(c_i)} \}$$

Output $\quad \max\{ f^0(r), f^1(r) \}$

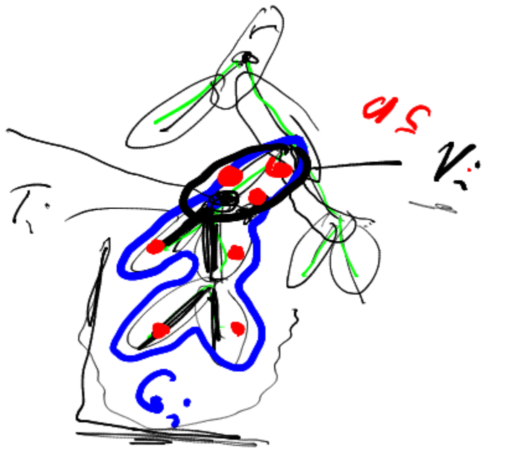
$\Rightarrow$ Correct algo for $\alpha(T)$, linear-time

(2) G with a t.d. (T, V) of width $\le k$

Algo in time $O(k^2 \cdot 4^k \cdot |V(G)|)$ $\boxed{|T| \le |G|}$

fix $r \in T$, T_i .... subtree rooted at $i \in T$

$$S_i = \bigcup_{t \in T_i} V_t \qquad G_i = G[S_i]$$



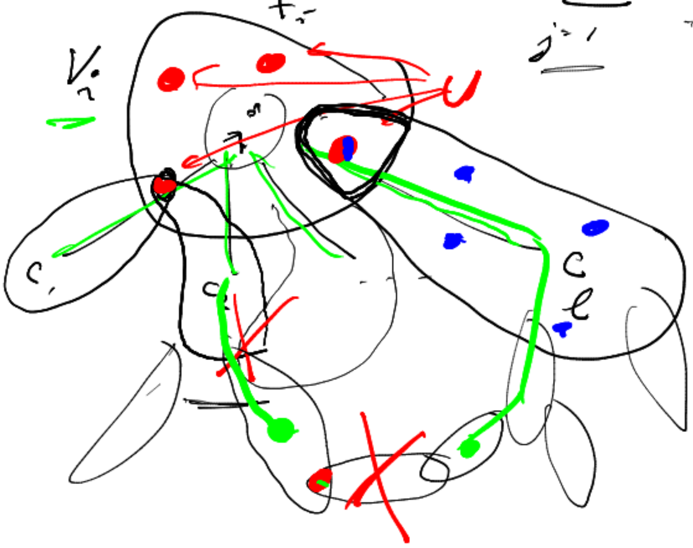
$\forall i\ \forall U \subseteq V_i$

$$f_i^U = \max\{|S| : S \text{ ind. set in } G_i \ \& \ S \cap V_i = U\}$$

i is a leaf $\Longrightarrow$ $U \subseteq V_i$: $f_i^U = \begin{cases} |U| & \text{if } U \text{ is indep.} \\ -\infty & \text{otherwise} \end{cases}$ $\qquad 2^{|V_i|} \cdot 2 \cdot 2^k$

i has children $c_1 \ldots c_\ell$

$$f_i^U = |U| + \sum_{j=1}^{\ell} \max\{ f_{c_j}^W - |U \cap W| : W \subseteq V_{c_j} \ \& \ W \cap V_i = U \cap V_{c_j} \ \& \ W \cup U \text{ is indep.} \}$$



W

$j \cdot \ell$

$\Leftrightarrow$ $\boxed{W \text{ is indep.}}$

$\Rightarrow$ correct algo for $\alpha(G)$

Output $\max\{f_r^U : U \subseteq V_r\}$

$G_{c_1}, G_{c_2}, \ldots$ don't intersect outside of V_i

$\boxed{\text{for one } U:}$ 2^{k+1} for $\forall$edge

for all U of i : $2^k \cdot 2^k \cdot \#$edges

$\rightarrow$ $2^{2k} \cdot |T| \le 2^{2k} \cdot |G|$