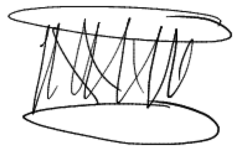
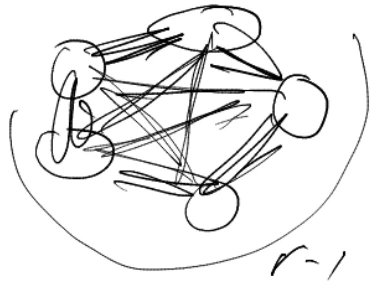
Forcing a clique

Turán thm If $G \not\supseteq \triangle$, then G has max. # of edges $= \Theta(n^2)$

if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$



If $G \not\supseteq K_r$ & G has max. # of edges $\Rightarrow$



$r-1$

$$\boxed{\text{Thm} \quad d_{avg}(G) \geq 2^{r-2} \Rightarrow G \geq_m K_r}$$

$\|G\| \geq 2^{r-2} \cdot |G|$

# of edges, # of vert.

we found K_{r-1} minor here

stronger version (Kostochka): $\exists c \in \mathbb{R}\ \forall r \in \mathbb{N}\quad d_{avg}(G) \geq c r \sqrt{\log r} \Rightarrow G \geq_m K_r$

& this is best poss. (up to a value of c)

motivation: Hadwiger conj.

$\chi(G) \geq r \Rightarrow G \geq_m K_r$

... weaker version of Hadwiger is true

$\chi(G) \geq c r \sqrt{\log r} \Rightarrow G \geq_m K_r$

$\Downarrow$ $\Uparrow$ (?) Not poss.

$\exists G' \subseteq G : \delta(G') \geq r-1 \Rightarrow d_{avg}(G') \geq r-1$

GW $\not\exists$ ... $< r$ — smallest deg.

$\forall G' \subseteq G : \delta(G') < r-1$

$\Rightarrow \chi(G) \leq r-1$

Hajós conj. $\chi(G) \geq r \Rightarrow G \geq_t K_r$

False but true if G has no short cycles

$(C_3, C_4, C_5, \ldots, C_\ell)$

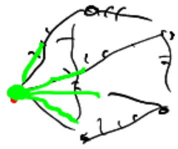
naive conj. $\chi(G) \geq r \Rightarrow G \supseteq K_r$

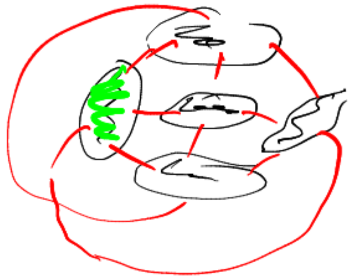
very false, even $G \supseteq K_3$ is not necess. true

We will prove: $d_\emptyset(G) \geq f(r) \Rightarrow \underline{G \geq_t K_r} \overset{}{\underset{(*)}{\Rightarrow}} G \geq_m K_r$

Q: $f(r) < 2^{r-1}$?

(read later pages: No.)





Prop. $\forall G$ with ≥ 1 edge $\exists H \subseteq G$ $\boxed{\delta(H) > \varepsilon(H)} \geq \varepsilon(G)$.

finite

min. degree

$$d_\emptyset(H) \geq \delta(H) > \frac{1}{2} d_\emptyset(H)$$

density $\frac{\|G\|}{|G|} = \frac{1}{2}\frac{2\|G\|}{|G|} = \frac{1}{2}\frac{\sum_v d_G(v)}{|G|} = \frac{1}{2} d_\emptyset(G)$

Proof If $\exists v \in V(G)$, then delete it.

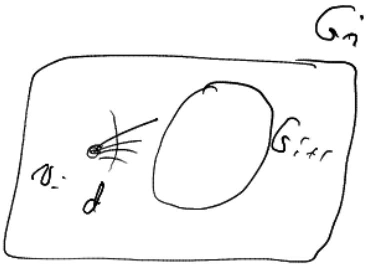
$G = G_0 \supset G_1 \supset G_2 \supset \dots$

If G_i has a vertex v_i: $\deg_{G_i}(v_i) \leq \varepsilon(G_i)$: then $G_{i+1} = G_i - v_i$.

Otherwise we end, $H := G_i$.

$\therefore \underline{\varepsilon(G_{i+1}) \geq \varepsilon(G_i)}$ $(\forall i)$ $\Rightarrow$ $\boxed{\varepsilon(G_{i+1}) \geq \varepsilon(G)}$ $\Rightarrow$ can't go on $> |G|$ steps

So if we finish, $\varepsilon(H) \geq \varepsilon(G)$.



$\|G_i\| = \|G_{i+1}\| + d$ $\qquad d \leq \varepsilon(G_i)$

$|G_i| = |G_{i+1}| + 1$

$$\varepsilon(G_i) = \frac{\|G_i\|}{|G_i|} = \frac{\|G_{i+1}\| + d}{|G_{i+1}| + 1} \leq \frac{\|G_{i+1}\| + \varepsilon(G_i)}{|G_{i+1}| + 1}$$ easy to see

$$\varepsilon(G_i)|G_{i+1}| + \cancel{\varepsilon(G_i)} \leq \|G_{i+1}\| + \cancel{\varepsilon(G_i)}$$

$$\varepsilon(G_i) \leq \frac{\|G_{i+1}\|}{|G_{i+1}|} = \varepsilon(G_{i+1})$$ ✓

$\boxed{\text{Thm}}$ $d_\varnothing(G) \geq 2^{\binom{r}{2}} \Rightarrow G \geq_t K_r$ $(\forall r \geq 2)$

Note: $d_\varnothing \geq c r^2$ suffices for some c.

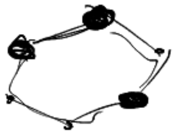
$\boxed{\text{Proof}}$ if $r = 2$ .... easy. Assume $r \geq 3$.

We show by ind. on $m = r, \dots, \binom{r}{2}$: $d_\varnothing(G) \geq 2^m \Rightarrow G \geq_t X$

$|X| = r, \ \|X\| = m$

$\boxed{m = r}$ We show $X = C_r$.

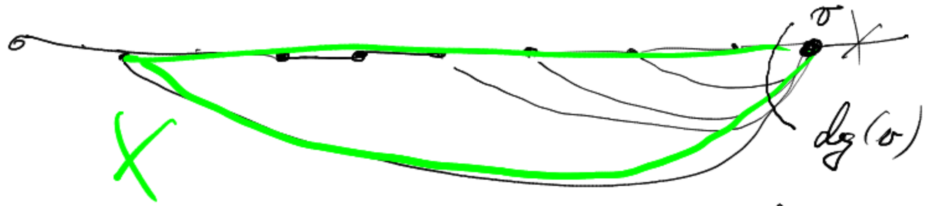
We need G to contain a cycle of length $\geq r$.



We know $d_\varnothing(G) \geq 2^r$

$\Rightarrow \exists H \subseteq G$: $\delta(H) \geq \frac{1}{2} d_\varnothing(H) \geq \frac{1}{2} d_\varnothing(G) = 2^{r-1} \geq r+1$

$(r \geq 3)$



$d_G(v) \geq \delta(H) \geq r+2$

$\Rightarrow$ green cycle has $\geq r$ vertices.

$\boxed{r < m \leq \binom{r}{2}}$ assume statement for smaller m's.

G : $d_\varnothing(G) \geq 2^m \Rightarrow \varepsilon(G) \geq 2^{m-1}$. WLOG G is connected.

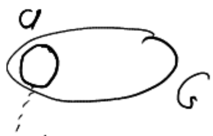
(otherwise: pass to a component with largest ε).

• Consider $\boxed{\text{max. set } U \subseteq V(G)}$ s.t.

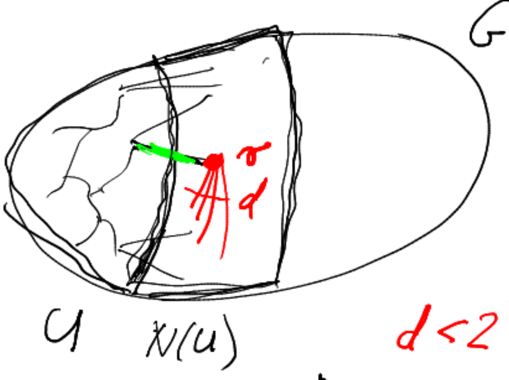
U = single vertex

1) $G[U]$ is connected &

2) $\varepsilon(G/U) \geq 2^{m-1}$



contr. U to 1 vertex to get G/U

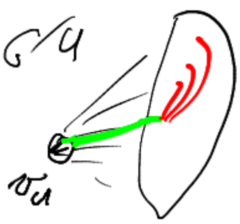


$H = G[N(U)] \neq \varnothing$

If $\delta(H) < 2^{m-1}$ we have wrong U.

$d < 2^{m-1}$

$U' := U \cup \{v\}$ is connected

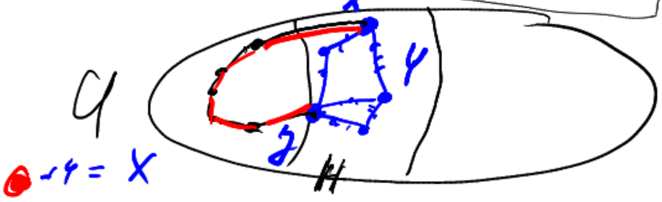


$\varepsilon(G/U')$: compared to G/U we lose 1 vertex & $1+d$ edges

$\varepsilon(G/U') \geq 2^{m-1} \Leftarrow$ $d+1 \leq 2^{m-1}$.

So : $\boxed{\delta(H) \geq 2^{m-1}} \Rightarrow d_\varnothing(H) \geq 2^{m-1} \Rightarrow \exists Y : |Y| = r$

$\|Y\| = m-1$



&

x, y ... branch vert. s.t. x-y path is missing in H

$H \geq_t Y$

$\boxed{\text{add it using } U}$

Def G is *k-linked* iff $\forall\, x_1 \dots x_k, y_1 \dots y_k$ distinct vertices

there are pairwise disj. paths $x_1 \text{---} y_1$
$x_2 \text{---} y_2$
$\vdots$
$x_k \text{---} y_k$

obviously (by Menger's thm) k-linked $\Rightarrow$ k-connected
$\not\Leftarrow$

[Jung 1970]

$\boxed{\text{Thm}}$ $\exists f: \mathbb{N} \to \mathbb{N}$ with: $f(k)$-connected $\Rightarrow$ k-linked

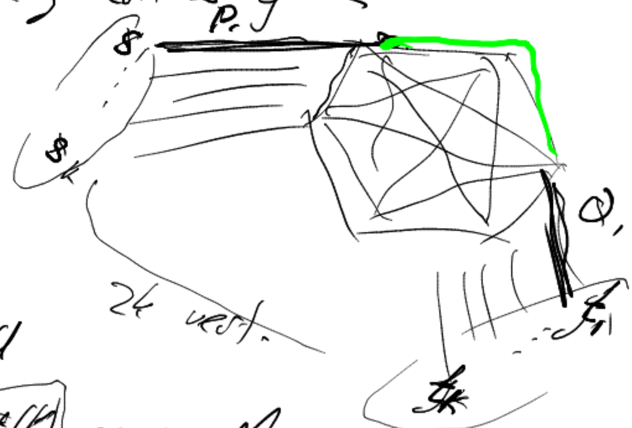
(best: [Thomas, Wollan 2005] G is $2k$-conn. & $\varepsilon(G) \geq 8k$
$\Rightarrow$ G is k-linked)

$$f(k) = 2k + 2^{\binom{3k}{2}}$$

Proof $f(k) \geq 2^{\binom{3k}{2}} \Rightarrow G \geq_t K_{3k}$
$f(k) \geq 2k \Rightarrow G$ is 2k-connected

$\Big\} \Rightarrow G$ is k-linked

K ... subdivision of K_{3k} in G, U = branching vertices of K $\quad |U| = 3k$



disj. paths $P_1 \dots P_k$, $Q_1 \dots Q_k$ --- ex. by Menger's thm

P_i: starts @ s_i, ends in U,
no inner vert. in U

Q_i: ends at t_i, starts in U, no inn. vert. in U

choose P_i, Q_i s.t. total # of edges outside $E(K)$ as small as poss.

$u_1 \dots u_k \in U$, not ends of P_i, Q_i

$\forall i$ L_i be the path in K from u_i to the end of P_i in U

v_i: first vertex of L_i on any path $P_1 \dots Q_k$

$P \rightsquigarrow P v_i L u_i$ $\;\to L_i \cup M_i$ intersects only P_i & Q_i.
So, we can use it to glue P_i with Q_i to connect s_i to t_i.

similarly: M_i, w_i