Graph classes characterization

Thm (Kuratowski)

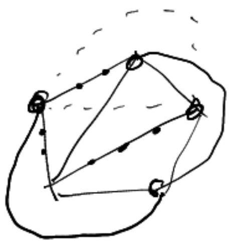
G is planar $\iff$ G does not contain K_5 or $K_{3,3}$ as a topological minor

$\Rightarrow$ easy

$\Leftarrow$ hard

$\Updownarrow$ def.

G has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$



K_5 not pl.

$K_{3,3}$ not planar

Thm G is bipartite $\iff$ G has no odd cycle (as a subgraph)

$C_3, C_5, C_7, \dots$

Df G, H – graphs $\qquad H \overset{ind.}{\subseteq} G$

- H is an induced subgraph of G $\quad H \sqsubseteq G \quad$: $V(H) \subseteq V(G)$, + "all edges"

$\Downarrow$

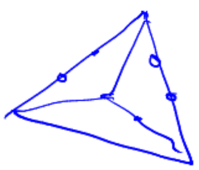
- H is a subgraph of G $\quad \boxed{H \subseteq G} \quad$ $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$

$\Downarrow$

- H is a topol. minor of G $\quad H \le_t G \quad$ $H' \subseteq G$ & H' is a subdiv. of H

$\Downarrow$

- H is a minor of G $\quad H \le_m G \quad$ H can be obtained from $H' \subseteq G$ by contracting edges



contr. edges $\quad K_4$

Df $\preceq$ ... one of the above, $\mathcal{G}$ ... graph class

$\mathcal{G}$ is $\preceq$-closed (closed with respect to $\preceq$)

if $H \preceq G$ & $G \in \mathcal{G} \implies H \in \mathcal{G}$

Ex. (1) $\mathcal{P} = \{\text{planar graphs}\}$ is $\preceq_m$ closed

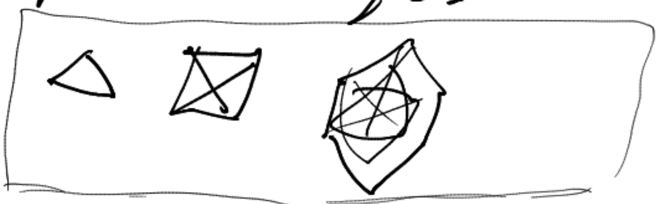
$\Rightarrow$ is also $\preceq_t$ closed, $\subseteq$-closed, $\sqsubseteq$-closed

(2) $\mathcal{B} = \{\text{bip. graphs}\}$ is $\subseteq$-closed ($\Rightarrow$ $\sqsubseteq$-closed)

is not $\preceq_t$-closed

(3) $\mathcal{C} = \{\text{graph where every component is a clique}\}$

- is $\sqsubseteq$-closed
- is not $\subseteq$-closed



$K_n \in \mathcal{C}$

Def $\mathcal{F}$ — a set of graphs $\qquad \neg(F \preceq G)$

$$\mathrm{Forb}_{\preceq}(\mathcal{F}) = \{ G : \forall F \in \mathcal{F} : F \npreceq G \}$$

forbidden

Useful characterization

Ex (1) $\mathrm{Forb}_{\preceq_t}(K_5, K_{3,3}) = \mathcal{P}$

(2) $\mathrm{Forb}_{\subseteq}(C_3, C_5, C_7, C_9, \ldots) = \mathcal{B}$

$NP \cap co\text{-}NP$

(3) $\mathcal{C} = \mathrm{Forb}_{\sqsubseteq}(K_{1,2})$

Observation $\mathrm{Forb}_{\preceq}(\mathcal{F})$ is $\preceq$-closed

Proof $G \in \mathrm{Forb}_{\preceq}(\mathcal{F})$, $H \preceq G$

Suppose $H \notin \mathrm{Forb}_{\preceq}(\mathcal{F}) \Rightarrow \exists F \in \mathcal{F} \quad F \preceq H \preceq G \Rightarrow F \preceq G$ ↯

Df F is $\preceq$-obstruction for $\mathcal{G}$ if $F \notin \mathcal{G}$

but $\forall F' \prec F : F' \in \mathcal{G}$

$\iff$ F is a minim. elem. of $\mathcal{G}^c$

Ex. ① K_5 is $\leq_t$-obstr. for $\mathcal{P}$

$K_{3,3}$ — " — —

② any odd cycle is $\subseteq$-obstr. for $\mathcal{B}$

Lemma $\mathcal{G}$ ... class, $\preceq$ is locally finite $\quad \{H : H \preceq G\}$ is finite $\forall G$

TFAE ① $\mathcal{G}$ is $\preceq$-closed

② $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathcal{F})$ for some $\mathcal{F}$

③ $\mathcal{G} = \mathrm{Forb}_{\preceq}(\mathcal{F})$ for $\mathcal{F} = \{\text{obstructions for } \mathcal{G}\}$

Proof ① $\Rightarrow$ ③

$\mathcal{G} \subseteq \mathrm{Forb}_{\preceq}(\mathcal{F})$ $\quad$ $G \in \mathcal{G}$, for contr: $\exists F \in \mathcal{F} : F \preceq G \Rightarrow F \in \mathcal{G}$ ↯

$\supseteq$ $\quad$ $G \notin \mathcal{G}$, $S = \{H \preceq G : H \notin \mathcal{G}\}$ is finite

$\Rightarrow S$ has minim. elements, $F \in \min_{\preceq}(S)$

$F \notin \mathcal{G}$, $\forall F' \prec F : F' \in \mathcal{G}$ (otherwise $F' \in S$, F is not min.)

$\Rightarrow F$ is obstruction $\Rightarrow F \in \mathcal{F}$

Found: $F \in \mathcal{F} : F \preceq G$, so $G \notin \mathrm{Forb}$...

③ $\Rightarrow$ ② trivi.

② $\Rightarrow$ ① above

$\mathrm{Forb}_{\subseteq}(tK_2) = \{G : G \not\supseteq tK_2\}$

$\Updownarrow$

$\underbrace{[\,[\,\cdots\,[}_{t}$ G has no match. of size t.

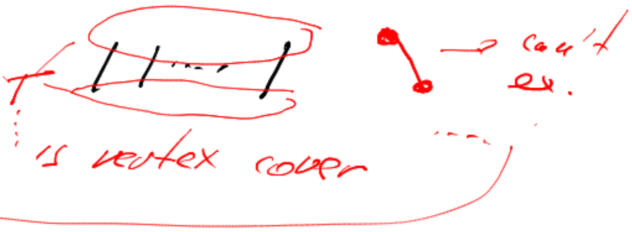
Berge–Tutte formula for the max. size of match.

Thm 1) $G \in \mathrm{Forb}_{\subseteq}(tK_2) \Rightarrow G$ has vertex cover of size $\leq 2(t-1)$

2) G has v.c. $\leq t \Rightarrow G \in \mathrm{Forb}_{\subseteq}((t+1)K_2)$

(rough character.)

Proof ① M … max. match. in G



② $|T| \leq t$ T … v.c.

suppose G has match. M, $|M| \geq t+1$ …… $\forall e \in M$ intersects T in a diff. vertex ↯

(Chudnovski, Robertson, Seymour, Thomas, 2002)

$$\boxed{\text{Thm} \quad \text{perfect graphs} = \mathrm{Forb}_{\subseteq}(C_5, C_7, C_9, \dots, \overline{C_7}, \overline{C_9}, \overline{C_{11}}, \dots)}$$

$\overline{C_7}$ = non edges in C_7

G is perfect if $\forall H \subseteq G$ $\boxed{\chi(H) = \omega(H)}$

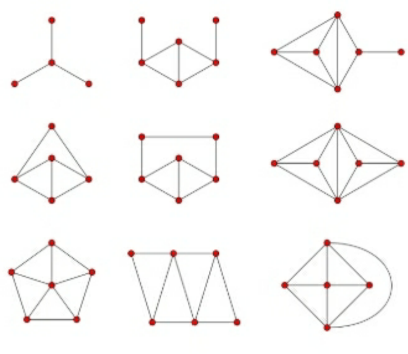
$\geq$ always

$\overline{C_5} \cong C_5$

Thm A graph G is a line-graph of another graph

$\Updownarrow$

$G \in \mathrm{Forb}_{\subseteq}($  $, \dots\dots)$



$L(G)$: $V = E(G)$

$E = \{\{e,f\} : e \,\&\, f$ share a vertex$\}$