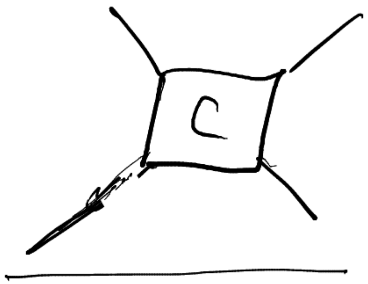
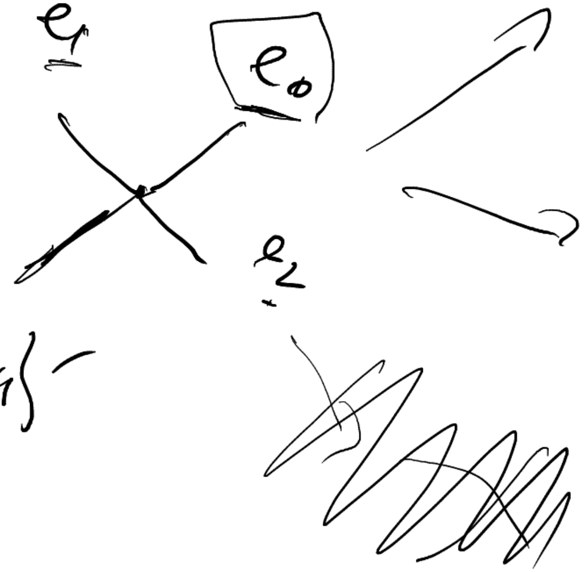
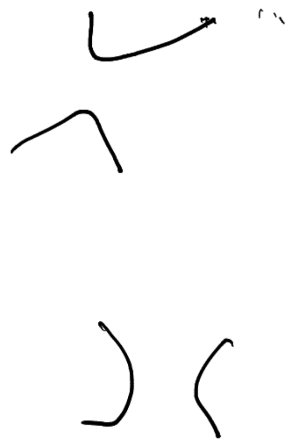
(Důsledek Spl. lemma)

G, 2-souvislý, kubický, $G \cong C_4$



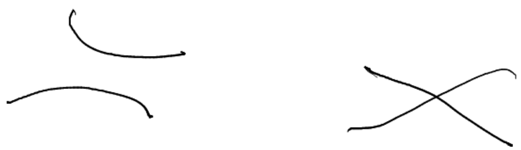
G/C





2-souvislý

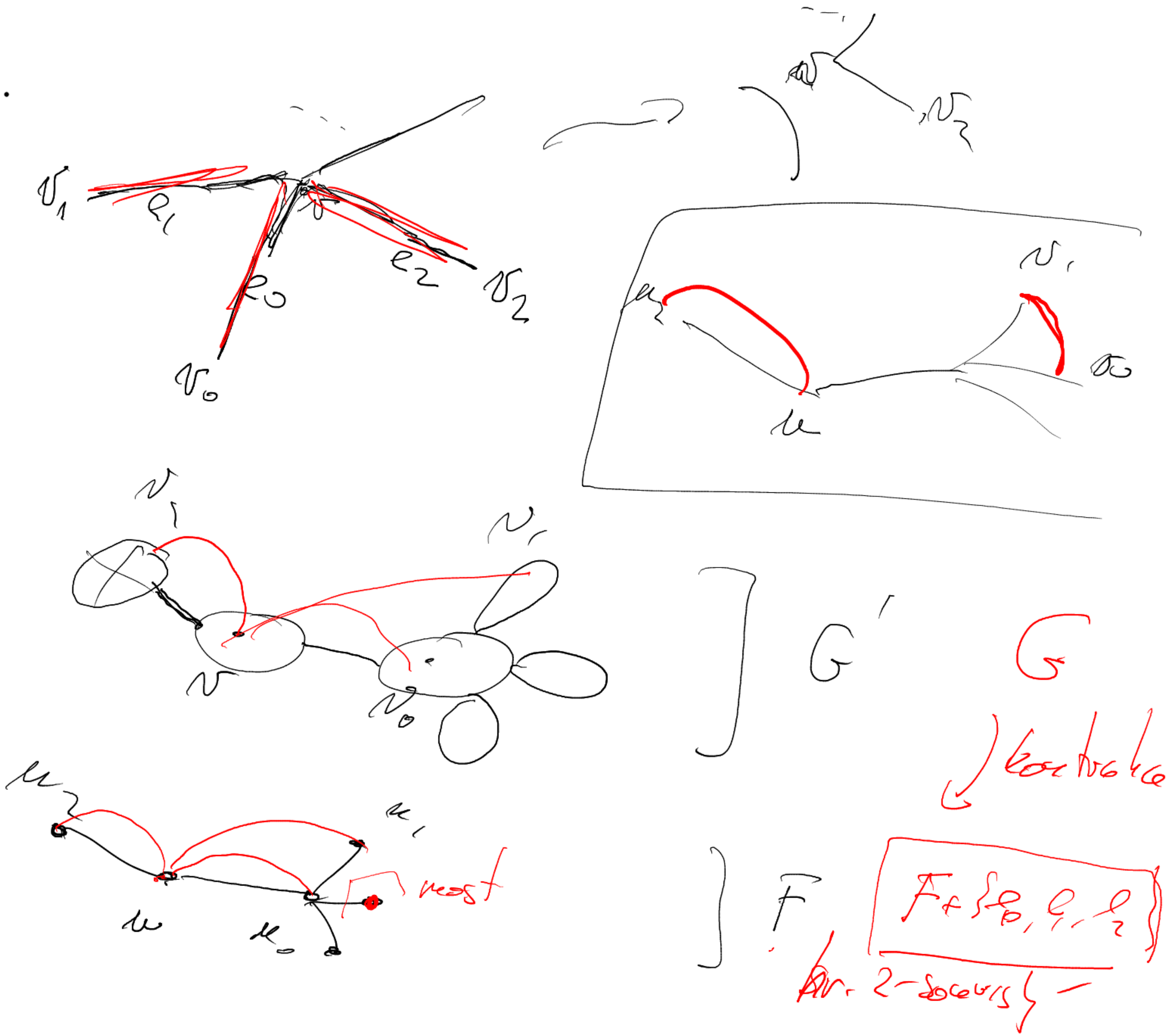
2-souvislý



Splitting lemma

Lemma 32 (Splitting lemma, Fleischner/Mader)**.** *Let G be a connected bridgeless graph, v a vertex with $\deg v \geq 4$, and e_0, e_1, e_2 three of its incident edges. Suppose that $G_{[v:e_0,e_1,e_2]}$ is connected. (This in particular holds whenever G is 2-connected.) Then at least one of $G_{[v:e_0,e_1]}$, $G_{[v:e_0,e_2]}$, is bridgeless connected.*

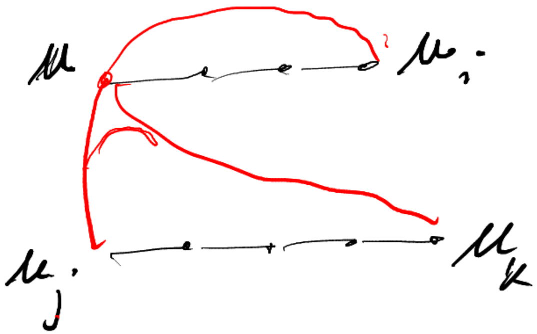
Proof. Let the edge e_i connect v with v_i. Let $G' = G - \{e_0, e_1, e_2\}$ and consider decomposition of G' into edge 2-connected blocks. Next contract each block to a vertex, what we get is a forest, say, F. Let u and u_i be the vertices of F corresponding to v and v_i ($i = 0, 1, 2$). As G is connected and bridgeless, the same is true for $F + \{uu_i : i = 0, 1, 2\}$. (In particular, the only leaves of F are among u, u_0, u_1, u_2.)

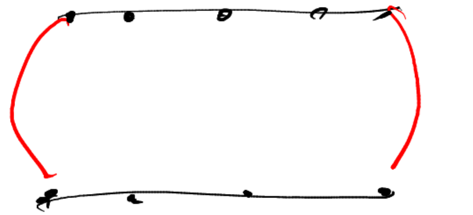


Also splitting, say, e_0, e_1 away from v corresponds to adding $F + \{uu_2, u_0u_1\}$ – for such graphs we need to check edge 2-connectivity. We have just two possibilities:

F is disconnected. As $G_{[v:e_0,e_1,e_2]}$ is connected and G bridgeless, the component containing u contains also (exactly) one u_i. Moreover, this component is a path connecting u with u_i. The other important vertices (say u_j, u_k, where $\{i,j,k\} = \{0,1,2\}$) are in the other component, this component is a $u_j - u_k$ path. In this case, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity. Easily, on of these includes the desired cases (as $0 \in \{i,j,k\}$). See the first case in Figure **??**.

F is connected. Let T be the minimal subtree containing u_0, u_1, u_2. Let $w \in T$ be such that F is T plus a $w - u$ path. There is (at least one) i such that w is in a $u_i -$

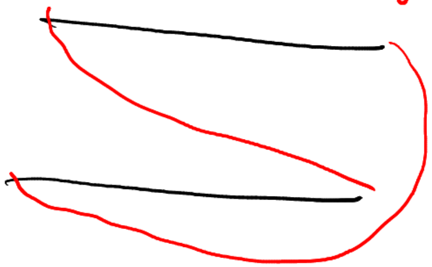


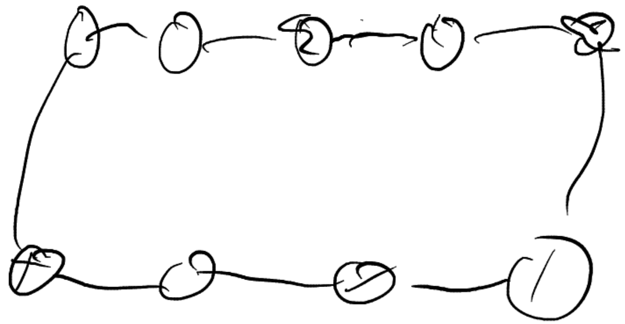


split. e_i, e_k

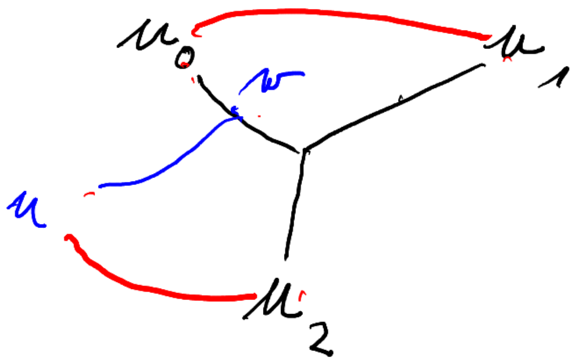
funguje pokud $i=0$, nebo $k=0$

split. e_i, e_j



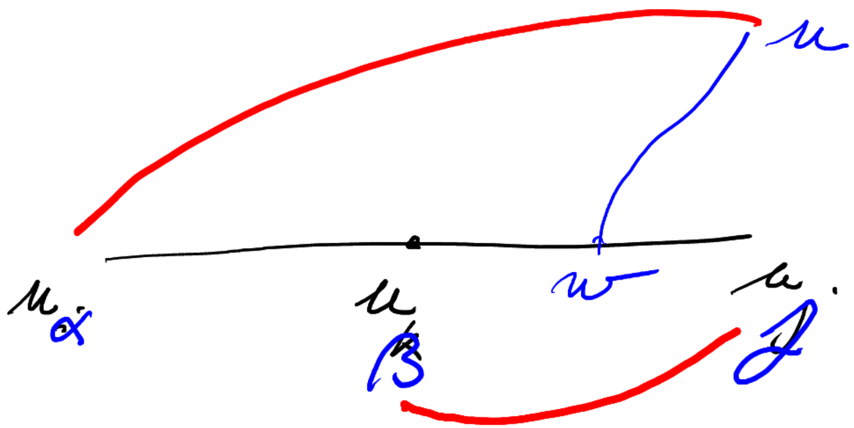


u_j and in a $u_i - u_k$ path (again, $\{i, j, k\} = \{0, 1, 2\}$). Again, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity, and at least one of these is what we search for. See the second case in Figure **??**. $\square$



$\exists i$: w je v $u_i - u_j$ cestě

& v $u_i - u_k$ cestě



Matching polytope and applications

We will look at various sets of edges geometrically. That is, we consider $\mathbb{R}^{E(G)}$ as a euclidean vector space (which it is) and study various polytopes in it. For a set $M \subseteq E(G)$, we define c_M – the *characteristic vector of* M – by $c_M(e) = 1$ if $e \in M$, and $c_M(e) = 0$ otherwise.

Definition 33. *The* matching polytope *of a (multi)graph G is defined by*

$$MP(G) = \mathrm{conv}\{c_M : M \text{ is a matching in } G\}.$$

It is not hard to see that all points c_M (for a matching M) are in fact vertices of $MP(G)$. Note that we consider non-perfect matchings too, so the zero vector is a vertex of every matching polytope.

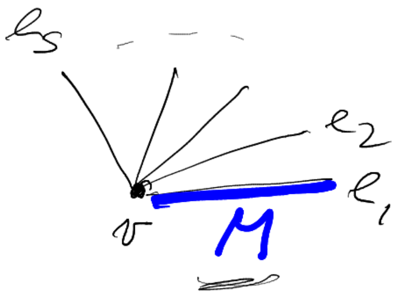
For many application it is desirable to obtain description of the matching polytope as

$c_M = \sum_{i=1}^{k} \alpha_i c_{M_i}$

$\alpha_i \geq 0$

$\sum \alpha_i = 1$

$M_1, \dots, M_k \neq M$



$e_1\ e_2\ \dots\ e_s\ \dots$

c_M 1 0 ... 0 0 1 1 0

c_{M_i} 1 0 0 ... 0

$\forall v$: v je pokryt M → v je pokryt M_i $\forall i$ & ve stejné hraně

c_v 0 0 ... 0

c_{M_i} 0 ... 0

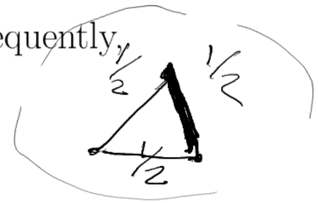
⟹ na $\forall\ \delta(v)$ je $M = M_1 = \dots = M_k$

⟹ $M = M_1 = \dots = M_k$

an intersection of halfspaces. An application for a problem related to Berge–Fulkerson conjecture will follow shortly, for an (original) application in combinatorial optimization consider the task to find a maximal matching in a graph with weighted edges. This is the same as solving a linear program over the matching polytope, and we can do this using ellipsoid method. (We only need to provide an efficient representation of the matching polytope, for details see XXX.)

For each $f \in MP(G)$ and $v \in V(G)$ we have $\sum_{e \in \delta(v)} f(e) \le 1$ (we sum over all edges incident with one vertex), as this inequality holds for all vectors c_M. This, however does not describe $MP(G)$ completely (Exercise!).

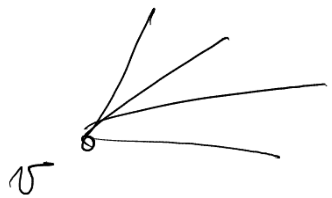
Next, we observe that for each vertex set X of odd size, each matching uses at most $(|X|-1)/2$ edges induced by X. Consequently,



vrch. popis $\rightsquigarrow$ stěn. popis

$\max \langle w, c \rangle$: $c \in MP(G)$

→ max. je ve vrcholu ... $c = c_M$ pro nějaké M



$\forall M$ ... pak $|M \cap \delta(v)| \le 1$

$\Rightarrow c_M(\delta(v)) \le 1$

$= \sum_{e \in \delta(v)} c_M(e)$

$f(\delta(v)) \le 1$

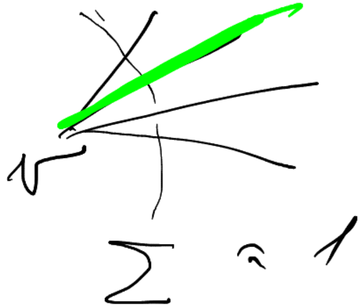
Theorem 35. *Let $PMP(G)$ be the polytope of perfect matchings that is $PMP = \mathrm{conv}\{c_M : M$ is a perfect matching in $G\}$.*

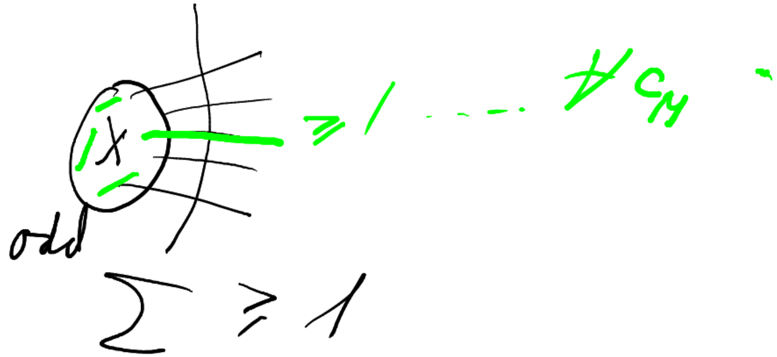
Then $PMP(G) \stackrel{\subseteq}{=}$

$$\{f \in \mathbb{R}^{E(G)} : \; f(\delta(v)) = 1 \;\forall v \in V(G)$$
$$f(\delta(X)) \geq 1$$
$$\forall X \subseteq E(G) \text{ of odd size}\}\,.$$

platí $\forall c_M$

$*$





Now we give the postponed proof of Theorem 35.

Proof. Let P_G be the polytope defined by the inequalities (*). Easily $PMP_G \subseteq P_G$, as all vertices of the perfect matching polytope (i.e., all c_M for a perfect matching M) satisfy the inequalities (*). For the other inclusion, we proceed by contradiction: we take the graph G with smallest $|V(G)|+|E(G)|$, and one vertex f of P_G such that $f \notin PMP_G$.

We have $0 < f(e) < 1$ **for each edge** e **of** G**.** If $f(e) = 0$ for some edge e, we let $G' = G - e$ and f' to be the restriction of f to $E(G')$. It is easy to check that $f' \in P_{G'}$, and as G' is smaller than G, we have $P_{G'} = PMP_{G'}$ and f' is a convex combination of characteristic vectors of perfect matchings of G'. When we take these matchings

$$P_G := \{f \in \mathbb{R}^{E(G)} : \ f(\delta(v)) = 1 \ \forall v \in V(G)$$
$$f(\delta(X)) \geq 1 \ \ \forall X \subseteq E(G) \ \textit{of odd size}\}\,.$$

chceme: $PMP_G \supseteq P_G$

$\exists\, G : PMP_G \not\supseteq P_G$

$f \in P_G$, $f \notin PMP_G$, f je vrchol P_G

$G' = G - e$ $\quad f(e) = 0$

$f' = f|_{E(G')}$ $\quad f' \in P_{G'} = PMP_{G'}$

$\Rightarrow f' = \sum \alpha_i\, c_{M_i}$

M_i ... PP v G'

$\Downarrow$

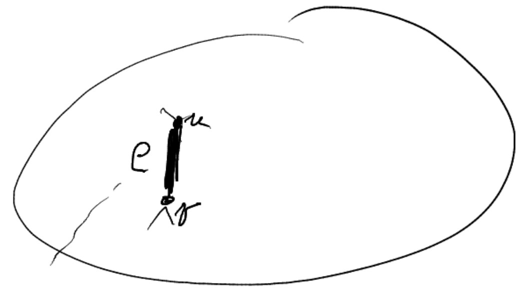
M_i je PP v G

as perfect matchings of G (by extending the characteristic vector by a 0 in the coordinate indexed by e), we get $f \in PMP_G$, a contradiction.

On the other hand, if $\boxed{f(e) = 1}$ for some edge $e = uv$, then we put $G' = G - u - v$. Again, we let $f' = f|_{E(G')}$ and we check that $f' \in P_{G'} = PMP_{G'}$. By extending all the perfect matchings that occur in the convex combination for f' by the edge e we get perfect matchings whose convex combination is f, again a contradiction.

G has no vertices of degree ≤ 1. G certainly does not have isolated vertices (~~by inequality (??)~~), and if v is a vertex incident only with an edge e, then $f(e) = 1$, which we already disproved. Consequently, $|E(G)| \geq |V(G)|$.

Case 1. $|E(G)| = |V(G)|$ G is 2-regular,

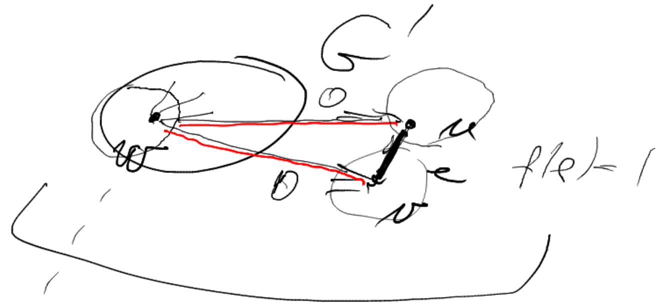


$f(e)=1$

$f' = f|_{E(G')}$

$f' \in P_{G'} = PMP_{G'}$

$G \rightsquigarrow$ $G' = G - u - v$



$f(e)=1$

$f'(\delta(w)) = 1$ che,

$f(\delta(w)) = 1$ v, ...

$\|$

$f'(\delta(w)) + f(wu) + f(wv)$

G' $\quad 0 \quad 0$

convex $P_G = PMP_G$

$\emptyset$

$\Rightarrow f(e)=1$

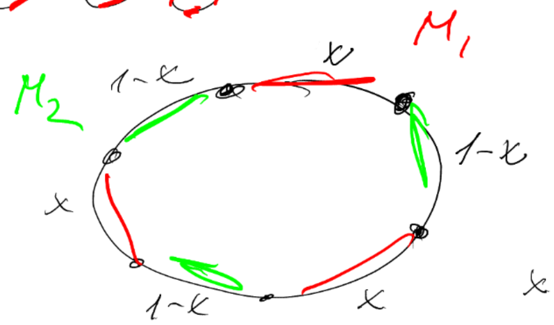
$f'(\delta(x)) = f(\delta(x)) \geq 1$ dle předp.

thus a disjoint union of circuits. None of these is odd (otherwise we let X be the set of vertices of an odd circuit and get a contradiction with inequality (**??**)). For even circuits it is easy to ... (Exercise!).

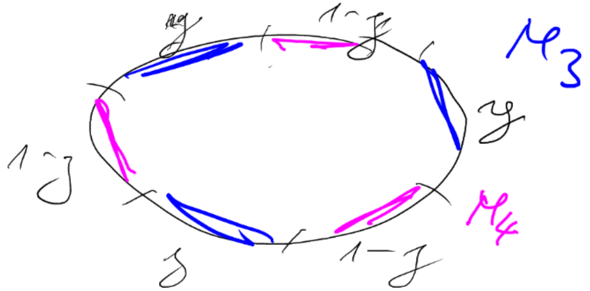
Case 2. $|E(G)| > |V(G)|$ As f is a vertex of a polytope in $\mathbb{R}^{E(G)}$, at least $|E(G)|$ of the inequalities are satisfied with an equality. (Exercise!) Thus, one of them must be (*) $\sum_{e\in\delta(X)} f(e) = 1$ for some $X \subseteq V(G)$, such that $1 < |X| < |V(G)|$ and $|X|$ is odd. As $|X|$ is odd, every perfect matching of G contains an edge of $\delta(X)$. This together with (*) implies that each of the sought-for matchings involved in the representation of f contain exactly one edge of $\delta(X)$. This suggest that we may want to treat X as a single vertex: if there is a representation for f, then this change of the graph will transform them in matchings.

$$P_G := \{f \in \mathbb{R}^{E(G)} : \ f(\delta(v)) = 1 \ \forall v \in V(G)$$
$$\underline{f(\delta(X))} \geq 1 \ \ \forall X \subseteq E(G) \ \textit{of odd size}\}\,.$$

$x \in (0,1)$



$x \cdot C_{M_1} + (1-x)\, C_{M_2}$



$$f = x\, C_{M_1 \cup M_3} + (y-x)\, C_{M_2 \cup M_3} + (1-y)\, C_{M_2 \cup M_4}$$

$x < y$

$1-x = y-x + 1-y$ ✓

Case 2. $|E(G)| > |V(G)|$ As f is a vertex of a polytope in $\mathbb{R}^{E(G)}$, at least $|E(G)|$ of the inequalities are satisfied with an equality. (Exercise!) Thus, one of them must be (*) $\sum_{e\in\delta(X)} f(e) = 1$ for some $X \subseteq V(G)$, such that $1 < |X| < |V(G)|$ and $|X|$ is odd. As $|X|$ is odd, every perfect matching of G contains an edge of $\delta(X)$. This together with (*) implies that each of the sought-for matchings involved in the representation of f contain exactly one edge of $\delta(X)$. This suggest that we may want to treat X as a single vertex: if there is a representation for f, then this change of the graph will transform them in matchings.

$P_G := \{f \in \mathbb{R}^{E(G)} :\ f(\delta(v)) = 1\ \forall v \in V(G)$ ... $|V(G)|$

$f(\delta(X)) \geq 1\ \forall X \subseteq E(G)$ *of odd size*$\}$. $|X|>1$

$\subseteq \mathbb{R}^{E(G)}$

... #vrchol je průnik $\geq |E(G)|$ stěn

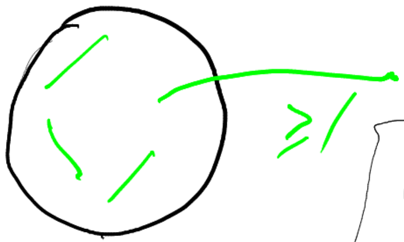
vrchol f leží na stěně typu 2

$\exists X$ lichá $\quad f(\delta(X)) = 1$

tedy je =

pokud $f = \sum_i \alpha_i c_{\pi_i}$ $\Rightarrow f(\delta(X)) \geq \sum \alpha_i \cdot 1$

$= 1$



≥ 1

$c_{\pi_i}(\delta(X)) \geq 1$

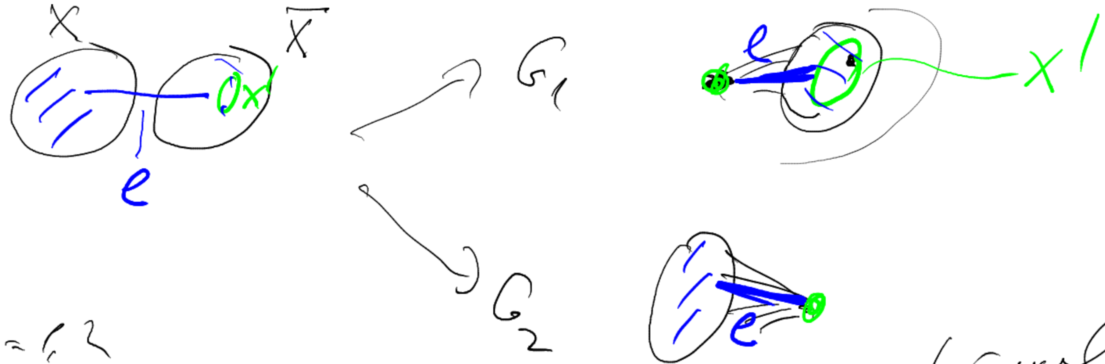
lichá X

tedy je $= \ \forall i$

To put this formally, we let $G_1 = G/X$ – all vertices of X are identified to a single vertex, we keep possible multiedges) – and $G_2 = G/\bar{X}$ (where $\bar{X} = V(G) \setminus X$). Again, let f_i be the restriction of f to the edge-set of G_i ($i = 1, 2$). It is easy to check that $f_i \in P_{G_i}$, which implies (Exercise!) that there are perfect matchings $(M_{i,k})_{k=1}^{N}$ of G_i such that

$$f_i = \frac{1}{N} \sum_{k=1}^{N} c_{M_{i,k}} . \qquad (2)$$

Recall that each $M_{i,k}$ contains exactly one of the edges of $\delta(X)$ (we abuse the notation slightly, we identify the edges of $\delta(X)$ in G, and the corresponding edges of G_1, G_2). Moreover, if e is one of these edges, then the number of perfect matchings $M_{i,k}$ of G_i for which $e \in M_{i,k}$ is $N f_i(e)$ (just look at the e-th coordinate of (2)). However, $N f_1(e) = N f_2(e) =$



$f_i = f/E(G_i)$

$f_i(\delta(v)) = 1$ … v starý vrchol; $v = X$ … z vrcholu X

$f_i(\delta(X')) = f(\delta(X')) \geq 1$ … bo X' je spec. vrchol

Rozdělíme $\{M_{1,k}\}$, $\{M_{2,k}\}$ do skupin podle párování s $\delta(X)$.

$f_i \in P_{G_i} = PMP_{G_i}$ $\quad \exists M_{i,1} \ldots M_{i,N_i}$ $\quad f_i = \sum \alpha_{i,k} \, c_{M_{i,k}}$

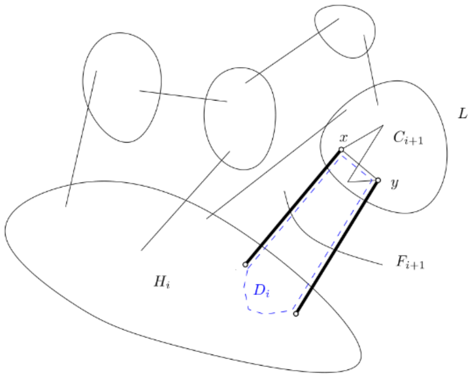
$\exists \alpha_{i,1} \ldots \alpha_{i,M_i}$

$\boxed{- \ \alpha_{i,k} \in \mathbb{Q} \ !}$ $\quad \frac{p_{i,k}}{q_{i,k}} \rightsquigarrow \frac{p_{i,k}}{N}$

$-$ bdwo $\alpha_{i,k} = \frac{1}{N_k}$ ✓ zapakuj jedno pudełko wielokrotnie

$-$ $N_1 = N_2$ (bdwo) ✓

$\frac{3}{N} c_{M_1} \sim \frac{1}{N} c_{M_1} + \frac{1}{N} c_{M_1} + \frac{1}{N} c_{M_1}$



$= \delta(X)$

$Nf(e)$ (recall f_i was defined as a restriction of f to $E(G_i)$). Consequently, we may pair up the matchings of G_1 and of G_2 to agree on the edges of $\delta(X)$, indeed we may assume that $M_{1,k}$ and $M_{2,k}$ contain the same edge from the cut Z. We put $M_k = M_{1,k} \cup M_{2,k}$. It is easy to check that f is the average of c_{M_k}, which finishes the proof. □

Theorem 38 (Seymour)**.** *Every bridgeless graph G has a 6-NZF.*

Proof. Equivalently, we will show it has NZ

$\mathbb{Z}_2 \times \mathbb{Z}_3$-flow. First, we can assume that G is 3-edge-connected (with the same proof as in the case of 8-NZF). We will find a spanning cycle C and carefully chosen edges between various components of C. The plan is to use a $\mathbb{Z}_2$-flow f with support $E(C)$ and a $\mathbb{Z}_3$-flow g that is NZ outside of $E(C)$.

We will recursively define subgraphs $(H_i)_{i\geq 0}$ of G, cycles $(C_i)_{i\geq 1}$ and sets of edges $(F_i)_{\geq 1}$. To start, let H_0 be any vertex of G. If H_i is defined, we consider a decomposition of $G' = G - V(H_i)$ into 2-edge-connected components–blocks. (If $V(G) = V(H_i)$, we stop and put $n := i$.) The structure of this decomposition is such that after contracting each of the blocks, we obtain a forest. We take any leaf of this forest at let L be a block of G' corresponding to it.

We observe that $|\delta_{G'}(L)| \leq 1$ (by the choice of L), while $|\delta_G(L)| \geq 3$ (as G is 3-edge-connected). This implies there are at least two edges connecting L with H_i, we let F_{i+1} be the set of some two of them, and x, y be the ends of those edges in L. As L is 2-edge-connected, there are two edge-disjoint $x - y$ paths, and their union is a connected cycle, let it be denoted C_{i+1}. (If $x = y$, we may choose C_{i+1} to be empty.) We put $H_{i+1} = H_i + C_{i+1} + F_{i+1}$ (We do not add spanned edges.)

We let $C = \cup_{i=1}^n C_i$, $F = \cup_{i=1}^n F_i$, $H = H_n$. All edges of G are of three types: $E(C)$, F, and the rest, denoted by R. As claimed above, C is a spanning cycle, so it is easy to take a $\mathbb{Z}_2$-flow with support $E(C)$. We now define a $\mathbb{Z}_3$-flow that is non-zero on $R \cup F$

We observe (by induction on i) that all graphs

H_i are connected, so we take a spanning tree $T \subseteq H$ and let g_n be a $\mathbb{Z}_3$-flow that equals 1 on $E(G) \setminus E(T)$. Next, we define $g_{n-1}, \ldots, g_0$ so that each g_k is a $\mathbb{Z}_3$-flow that is nonzero on R and on F_j with $j > k$. If g_{i+1} is already defined, we consider a cycle D_i containing both edges of F_{i+1}, some $x - y$ path in C_{i+1} and any path in H_i that connects the other ends of the edges of F_{i+1}. Let φ_i be a $\mathbb{Z}_3$-flow that is nonzero on D_i. Consider flows $g_{i+1} + \alpha\varphi_i$ for $\alpha = 0, 1, 2$. At least one of them is nonzero on both edges of F_{i+1}, while we didn't change edges of R neither of $F_{>i+1}$. Consequently, the mapping $g = g_0$ is nonzero on $R \cup F$ and (f, g) is the desired $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF on G. $\square$

Notes: 1) Recall the standard proof of the fact that graph of maximum degree at most k is $(k+1)$-colorable. The second phase of the above proof is an analogue of this for $k = 2$. Indeed, if the graph G/C (each component of C is contracted to a vertex) is planar, then we are using the fact that the dual $(G/C)^*$ is 2-degenerate. As we saw, the argument works even for non-planar graphs. The nontrivial part is, of course, to find the cycle C such that G/C has this degenerate property.

2) It's tempting to try and use similar ideas to get a 5-flow conjecture. For this, one may say the above proof in an alternative way: we find a 2-flow f and a 3-flow g that are not both equal to zero at the same edge. Then $2g + f$ is a NZ 6-flow. Now one may try to find a 2.5-flow g instead that is a real-valued flow such that $1 \le g(e) \le 1.5$ for each edge e for which $f(e) = 0$. This would indeed produce a 5-flow. Exercise: discuss why does the above approach fail.

Exercises: 4. Describe PMP_G when G is a disjoint union of even circuits.

5. Let P be a polytope $\{x \in \mathbb{R}^d : Ax \leq b\}$. Let V denote the vertices of P. Let x be a point of P.

(a) x is a convex combination of at most $d+1$ elements of V.

(b) If A and b have rational entries then x is a convex combination of some elements of V *with rational coefficients.*

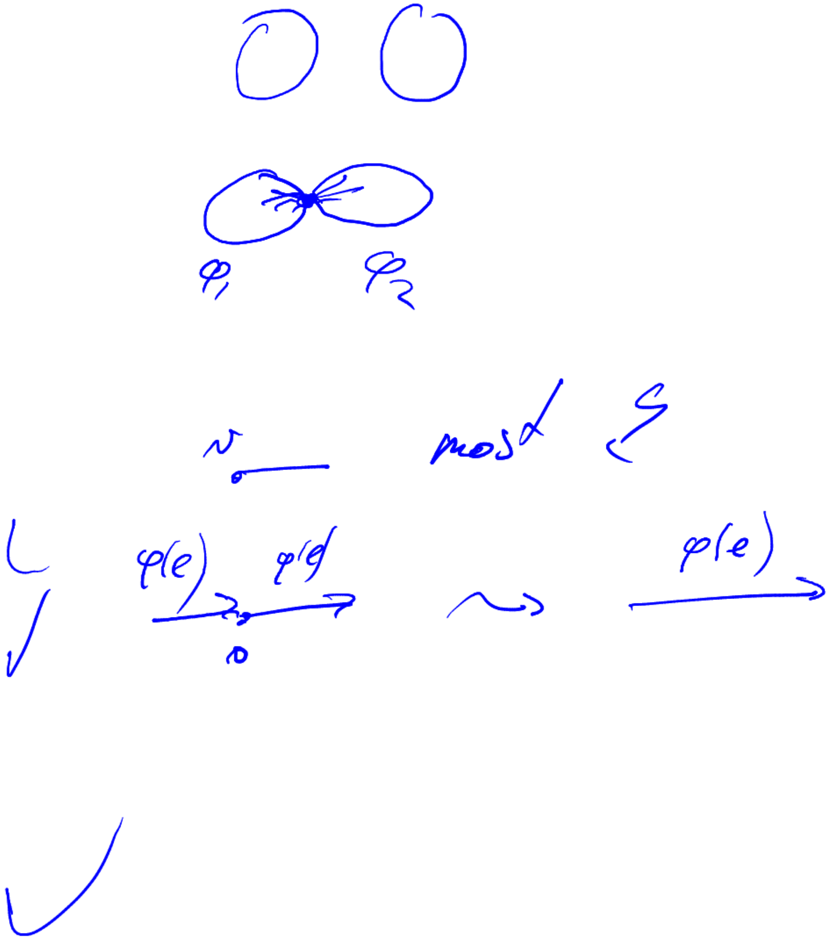
(c) There is a list $v_1, \ldots, v_n$ of vertices from V (possibly with repetition), such that $x = (v_1 + \cdots + v_n)/n$.

6. * Try to modify the proof of 6-NZF theorem to work for 5-NZF (as indicated in the notes below the proof). Describe what makes this approach fail. (If you succeed in proving the existence of 5-NZF, let humanity know! – see the list of open problems . . .)

In the last section we saw Seymour's proof of the existence of NZ 6-flow. Tutte's 5-flow conjecture is still elusive, let us however look at some simple observations. In the following, G is a minimal counterexample to the conjecture. Explicitly: G is a bridgeless graph that admits no NZ 5-flow and among such graphs G has the smallest $|V(G)| + |E(G)|$.

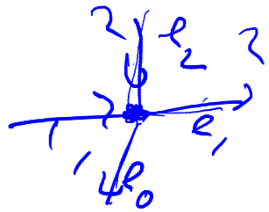
(1) G is 2-connected Suppose not; then $G = G_1 \cup G_2$ where graphs G_1 and G_2 share just one vertex, and both are bridgeless. By minimality of G, both G_1 and G_2 admit a NZ 5-flow, thus G has it, too.

(2) G is cubic Suppose not, let v be a vertex such that $\deg v \neq 3$. If $\deg v = 1$, then G has a bridge, contradiction. If $\deg v = 2$, then we can suppress this vertex (contract one of its incident edges). The graph we obtain

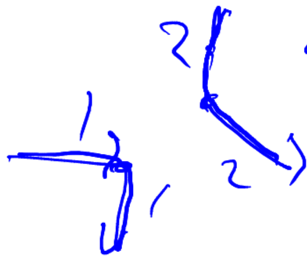


is smaller, so has a NZ 5-flow, which is easily extended back to G. Finally, let $\deg v \geq 4$, let (as in the Fleischner's lemma), e_0, e_1, e_2 be three of the incident edges. As G is 2-connected, the lemma implies that one of graphs $G_i = G_{[v:e_0e_i]}$ $(i = 1, 2)$ is bridgeless. After suppressing the newly created vertex of degree 2, we get a graph G_i' that has the same number of vertices as G but one edge less – thus it admits a NZ 5-flow f_i. It is easy to extend it back to G_i and then to G, which yields contradiction.

(3) G is edge 3-connected Suppose not, let $A \subseteq V(G)$ be such that $|\delta(A)| = 2$, say $\delta(A) = \{e, e'\}$. Now $G' = G/e$ is smaller then G, thus it admits a NZ 5-flow f. We extend it to G by letting $f(e) = \pm f(e')$ (the sign is chosen according to the orientation of e, e'). As we saw already in several occasions,

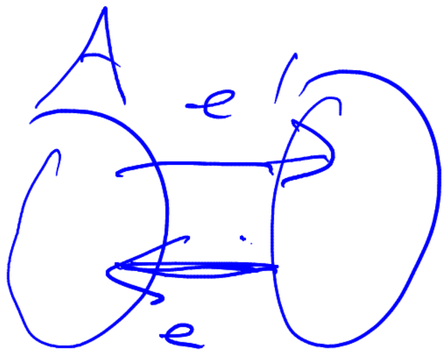


$\exists i$ $G_i = G_{[v:e_0e_i]}$ je bez mostů



menší než G

$\Rightarrow \exists$ 5-tok



$f(e) = f(e')$

$G' = G/e$

má 5-NZT f

this extension yields a flow.

(4) G is cyclically edge 4-connected
Note that a graph G is called *cyclically edge k-connected*, if $|\delta(A)| \geq k$ whenever A is a set of vertices such that both A and $\bar{A}$ contain a circuit. (Exercise: determine the cyclic edge connectivity of the Petersen graph.)

Suppose G fails the above definition with $k = 3$ that is there is A such that $|\delta(A)| = 3$. Put $G_1 = G/A$, $G_2 = G/\bar{A}$ – both G_1 and G_2 are smaller than G, thus admit a NZ 5-flow. Now it is possible to show [Sekine and Zhang] that

$$F_G(x) = \frac{F_{G_i}(x) \cdot F_{G_2}(x)}{F_{K_2^3}(x)}.$$

(Here $K_2^3 = C_3^*$ is the graph with two vertices and three parallel edges.)

Using this with $x = 5$ (CHECK) gives us that G has a NZ 5-flow, a contradiction.

(5) G **is cyclically edge 6-connected** [Kochol 2004]

(6) G **is has no circuit of length less than 9** [Kochol 2006]