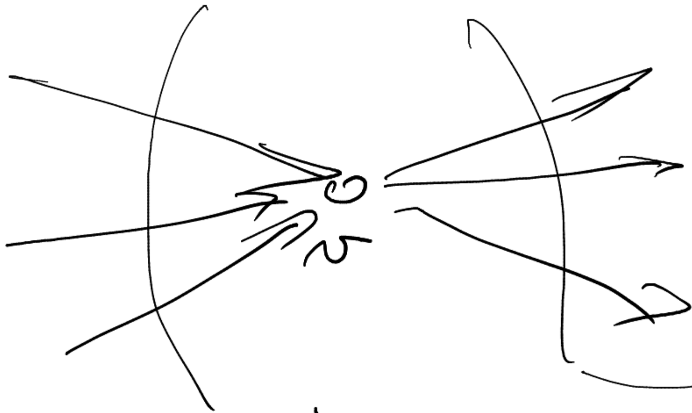
Definition 1. *Let G be a digraph, Γ a group. A mapping $f : E(G) \to \Gamma$ is called a* flow *(or, more explicitly, a Γ-flow), if for every vertex $v \in V(G)$ the Kirchhoff law is valid:*

$$\sum_{e=(v,u)} f(e) = \sum_{e=(u,v)} f(e)\,.$$

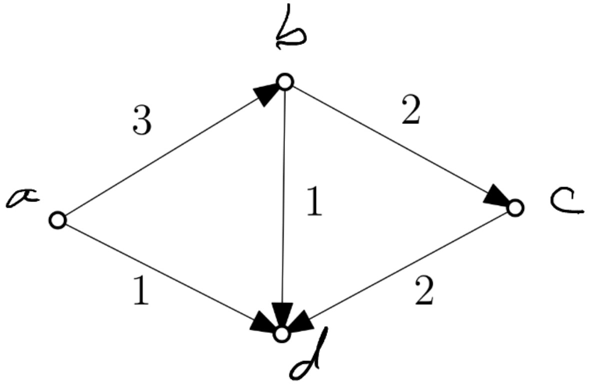
$f^+(v)$ = *the left-hand side of the above equation, the amount of flow that leaves v,*

$f^-(v)$ = *the right-hand side of the above equation, the amount of flow that enters v.*



$f^+(v)$

$$\sum_{e=(u,v)} f(e) = f^-(v)$$



$f^+(b) = 1+2$

$f^-(b) = 3$

$f^+(a) = 1+3$

$f^-(a) = 0$

$\Gamma = \mathbb{Z}_4$

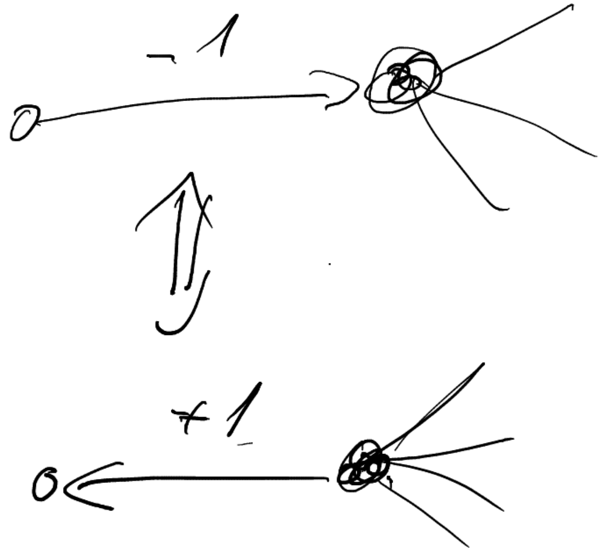
Reversing orientations

We need directed edges for the definition of flows. However, we will in fact study undirected graphs. To understand why, let us define a simple notation. Let G be a digraph, f a mapping $E(G) \to \Gamma$ and $F \subseteq E(G)$ any set of edges. We let G_F denote the digraph obtained from G after reorienting all edges in F. We define a mapping f_F as follows:

$$f_F(e) = \begin{cases} -f(e) & \text{if } e \in F \\ f(e) & \text{otherwise} \end{cases}$$

Observation 7. *Let f be a Γ-flow on a digraph G, let $F \subseteq E(G)$. Then f_F is a Γ-flow on G_F. Moreover, if f is NZ then f_F is also NZ.*

We can consider all pairs (G_F, f_F) to be different representations of "the same flow" and we pick the most convenient one.



Easy properties of flows

The following easy observation connects $\mathbb{Z}_2$-flows with cycles ($\neq$ circuits).

Observation 8 ($\mathbb{Z}_2$-flow)**.** *Let G be a graph and f any $\mathbb{Z}_2$-flow on G. Then the* support *of f (that is, the set of edges with nonzero value of f) is a cycle.*

In particular a graph has a NZ $\mathbb{Z}_2$-flow iff it is a cycle.

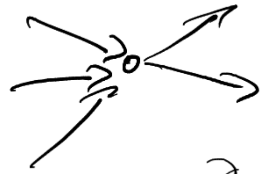
Theorem 9 ($\mathbb{Z}_3$-flow of cubic graphs)**.** *Let G be a cubic (i.e., 3-regular) graph. Then G admits a NZ $\mathbb{Z}_3$-flow iff G is bipartite.*

Proof. If G is bipartite, we direct all edges from one part to the other and assign 1 to each edge, clearly this is the desired flow. On the other hand, … □

$0 = 1+1+1$

$\operatorname{supp} f = \{e : f(e) \neq 0\}$

f je NZ … $\operatorname{supp} f = E(G)$

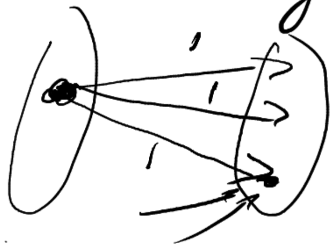


$f^+(v) = f^-(v)$

v $\mathbb{Z}_2$: $\underbrace{f^+(v) + f^-(v)}_{\text{sudý počet}} = 0$!

v supp f má každý vrchol sudý stupeň

$\Rightarrow$ jde rozložit na kružnice



$1+1+1 = 0$

Definition 12. *Let G be a digraph, f a $\mathbb{Z}$-flow on G.*
*f is a k-*flow *if* $|f(e)| < k$ $(\forall e)$.
f is a nowhere-zero k-flow *if* $0 < |f(e)| < k$ $(\forall e)$.

$\pm 1, \pm 2, \dots, \pm (k-1)$

k-NZF := nowhere-zero k-flow
Γ-NZF := nowhere-zero Γ-flow

Note: Many authors use k-flow to mean NZ k-flow.

Theorem 13 (Tutte)**.** *A graph has a k-NZF iff it has $\mathbb{Z}_k$-NZF.*

$\Rightarrow$ lehké
f ... k-NZF ... $f(e) \in$
... f splňuje Kirch. zák. v $\mathbb{Z}$

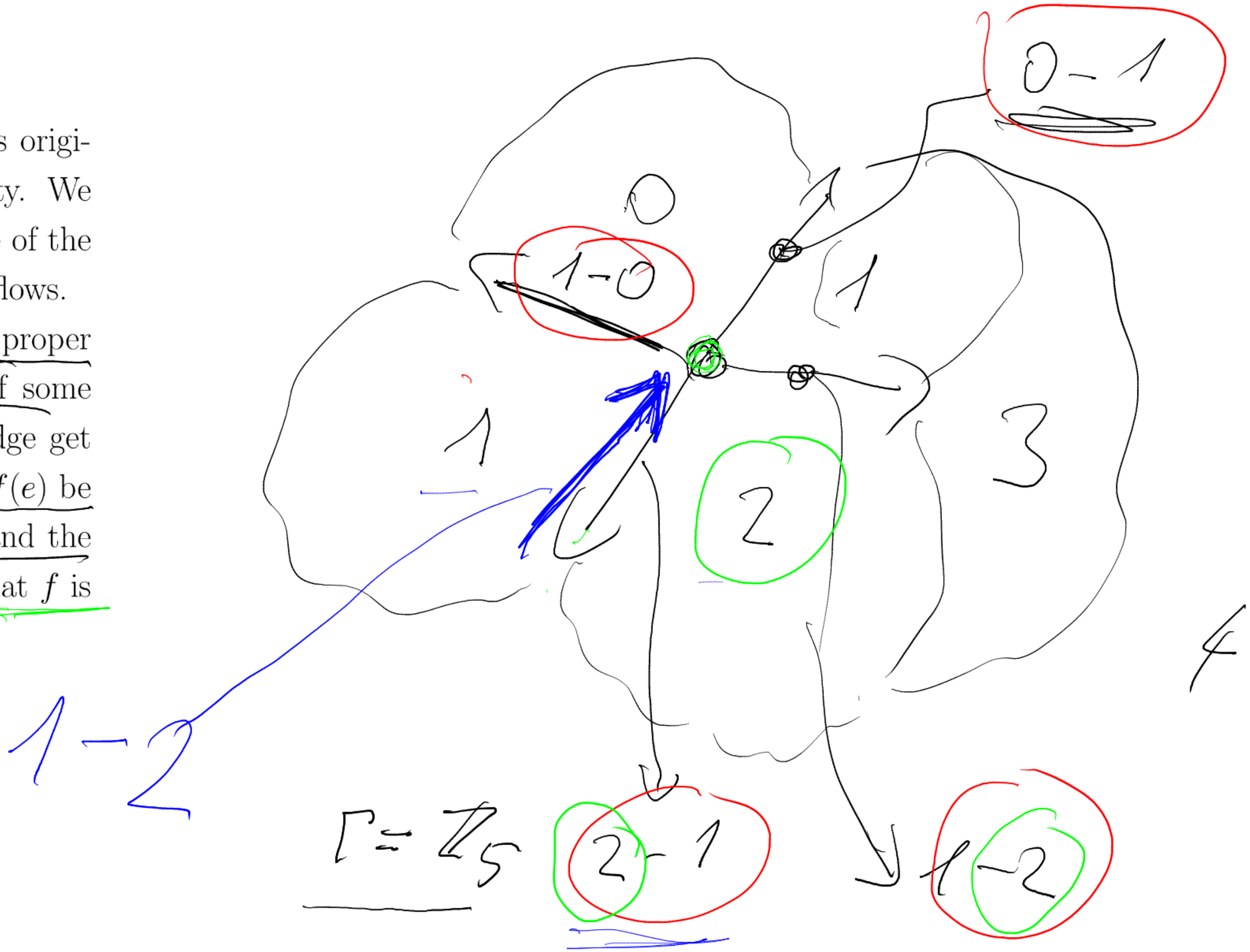
Motivated by this result we will sometimes use *k-flow* to mean Γ-flow for any Γ of size k.

f' ... $f \bmod k$ ... zobr. $E(G) \to \mathbb{Z}_k$
$\Leftarrow$ těžší, později ... f' splňuje K.z. v $\mathbb{Z}_k$
$f(e) \neq 0$

NZ flows in planar graphs

A general way to construct NZ flows originates from colorings and planar duality. We now present just a sample to show one of the early motivations for the study of NZ flows.

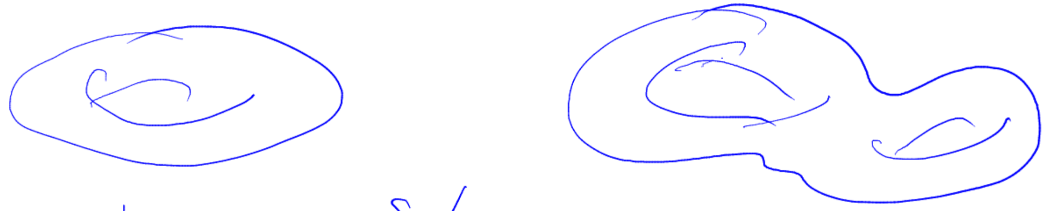
Let G be a planar digraph, consider a proper coloring of faces of G by elements of some group Γ – so that faces sharing an edge get distinct colors. Now for an edge e let $f(e)$ be the difference of the left face's value and the right face's value. It's easy to check that f is a NZ Γ-flow.



- It works for graphs drawn on arbitrary orientable surface.
- For planar graphs all NZ flows arise in this way,
- thus $\varphi(G) = \chi(G^*)$. (Proof later.)
- $\varphi(G) \le 4$ whenever G is planar.
- OTOH $\varphi(\mathrm{Pt}) = 5$ (where Pt is the Petersen graph).
- It is open, whether $\varphi(G) > 5$ is possible.

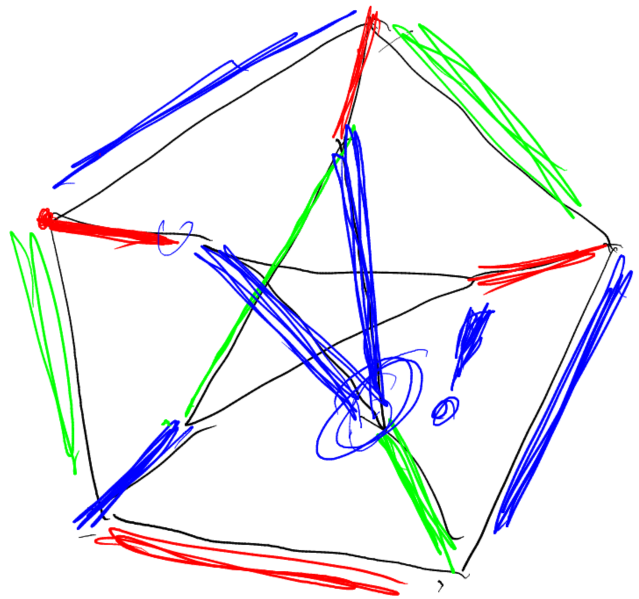
Platí: $\forall G$ bez mostů:

$$\varphi(G) \le 6.$$
$$\le 8$$



$\varphi(G) = \min\{k : \text{ex. } k\text{-NZF}\}$

$\mathbb{Z}_k$-NZF



NZ $\mathbb{Z}_3$-tok $\Leftrightarrow$ bip. ✗

NZ $\mathbb{Z}_4$-tok $\Leftrightarrow$ NZ $\mathbb{Z}_2^2$-tok $\Leftrightarrow$ hranové 3-barvení

More basic properties

Theorem 15 (Jaeger). *The following are equivalent for any graph G*

1. G has a $\mathbb{Z}_2^2$-NZF

2. $E(G)$ is a union of two cycles

Proof. Let f be a NZ $\mathbb{Z}_2^2$-flow on G, observe it only uses values $(0,1)$, $(1,0)$, $(1,1)$. …

In the other direction: let $E(G) = E_1 \cup E_2$ and each E_i is a cycle. We take a $\mathbb{Z}_2$-flow f_i that is 1 precisely on E_i. Putting $f = (f_1, f_2)$ we get the desired flow.

An alternative proof: consider (integer) 2-flows g_i on E_i. Then $g = 2g_1 + g_2$ is a NZ 4-flow. □

Theorem 16 (Tutte). *Let $k \geq 2$ be an integer. A graph has a k-NZF iff it has $\mathbb{Z}_k$-NZF.*

$2 \cdot (\pm 1) + 1 \cdot (\pm 1) = \{\pm 1, \pm 3\}$

$\boxed{1 \Rightarrow 2}$: $f = (f_1, f_2)$

$E_1 = \text{supp}\, f_1 = \{e : f_1(e) \neq 0\}$

$E_2 = \text{supp}\, f_2$

$\forall e : f(e) \neq (0,0) \Rightarrow f_1(e) \neq 0$ nebo $f_2(e) \neq 0$

$\Rightarrow e \in E_1 \cup E_2$

f_1 … $\mathbb{Z}_2$-tok

f_2 ……

$\Rightarrow (f_1, f_2) \neq (0,0)$ pro každou hranu

$\boxed{2 \Rightarrow 1}$ $E(G) = E_1 \cup E_2$

E_i je cyklus $\Rightarrow f_i$ je $\mathbb{Z}_2$-tok

Proof. The forward implication is obvious. For the other one, let g be a $\mathbb{Z}_k$-NZF in a graph G. For any mapping $f : E(G) \to \mathbb{Z}$ we let $f(v)$ be the net flow out of a vertex v, that is $\boxed{f(v) = \sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e).}$ Recall that f is a flow iff $f(v) = 0$ for every vertex v. We won't achieve this directly, however, but by certain optimization.

Let $f : E(G) \to \mathbb{Z}$ be such that

1. $f(e) \equiv g(e) \pmod k$ for each edge e,
2. $|f(e)| < k$ for each edge e, and
3. subject to the above, $\sum_{v\in V(G)} |f(v)|$ is as small as possible.

(If the sum in part 3. is zero, then f is a flow and we are done.)

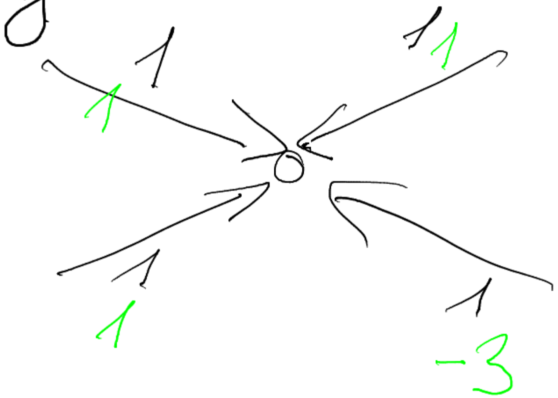
By possibly reorienting the edges of G we may assume that $f(e) > 0$ for each edge e.

f k-NZF $\Rightarrow$ f mod k ... $\mathbb{Z}_k$-NZF

$f(v) = f^+(v) - f^-(v)$ $\quad$ ($= 0$ $\forall v$ pokud f je tok)

g ... $\mathbb{Z}_4$-tok

$1+1+1+1 = 0$



f $\quad$ $1+1+1-3 = 0$

v $\quad$ $\mathbb{Z}_4$

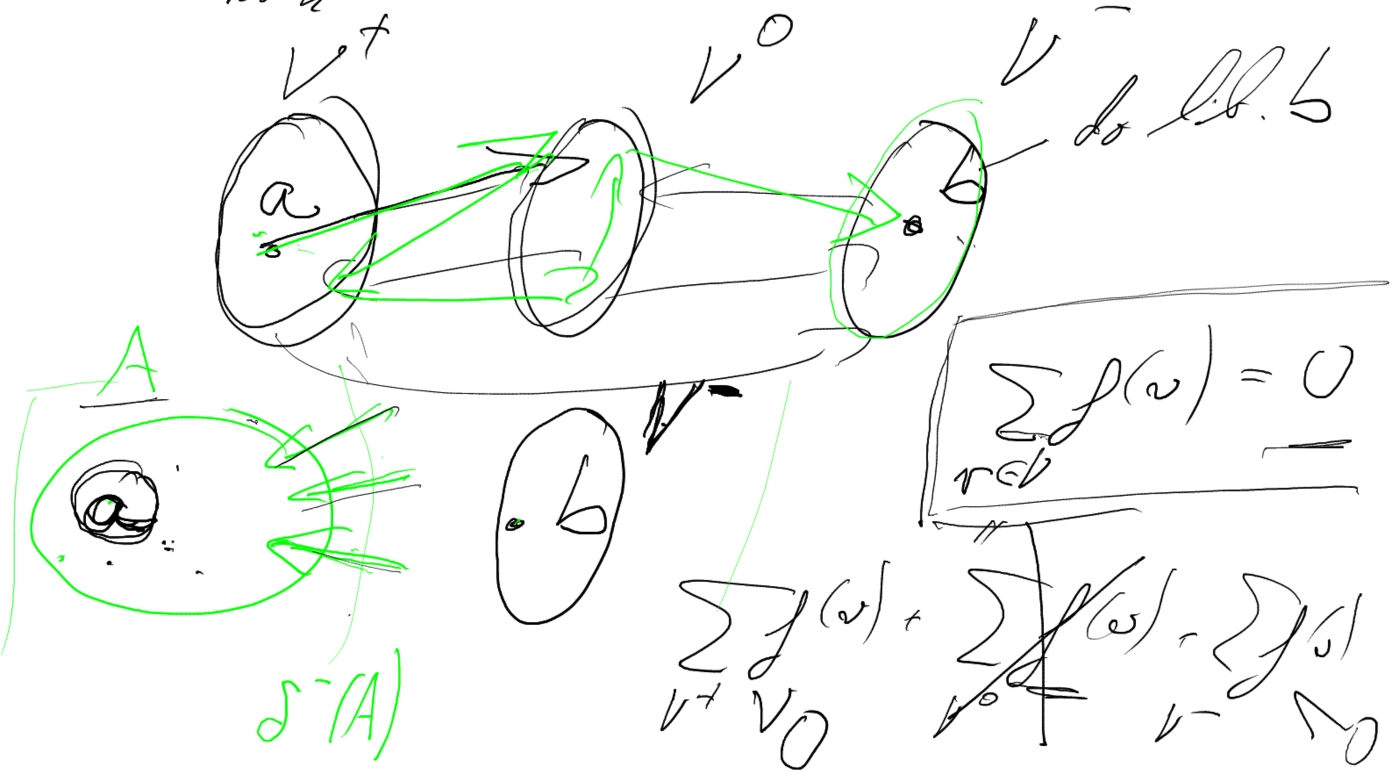
$f(v) \in \{0, \pm k, \pm 2k, \dots\}$

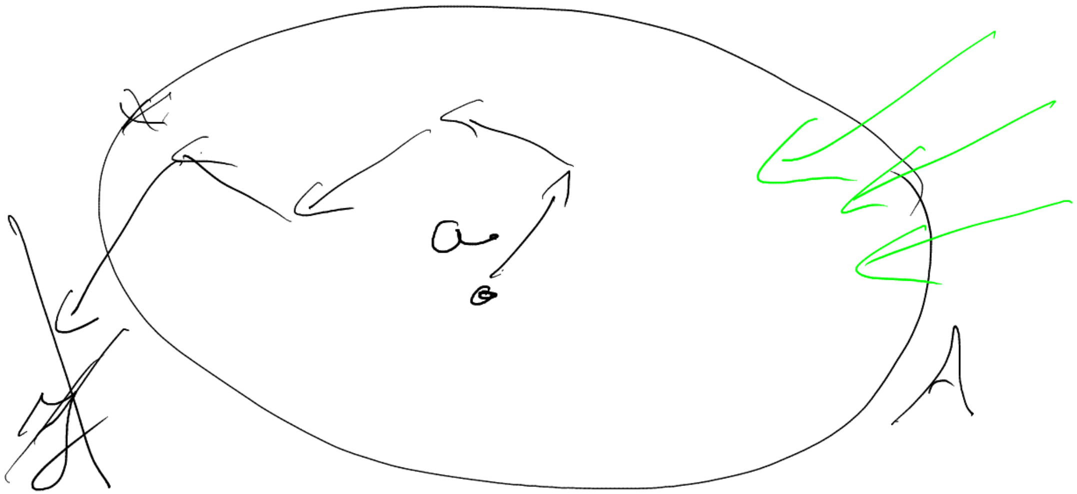
- $f(v) \equiv 0 \pmod{k} \qquad \forall v$
- $V^+ := \{v : f(v) > 0\}$
- $V^0 := \{v : f(v) = 0\}$
- $V^- := \{v : f(v) < 0\}$
- If $V^0 = V$ we are done.
- Otherwise, observe that both V^+ and V^- are nonempty and pick $a \in V^+$, $\cancel{b \in V^-}$.
- Either there is a directed $a-b$ path or there is a set A containing a but not b such that no directed edge leaves A.
- The second possibility immediately yields a contradiction:

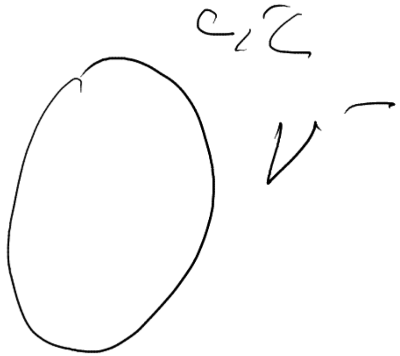
$$\underbrace{\sum_{v \in A} f(v)}_{\geq 0} = -\sum_{e \in \delta^-(A)} f(e) < 0$$

$$f(v) = \sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e)$$

$$\underset{\text{mod } k}{\equiv} \sum g(e) - \sum g(e) = 0$$

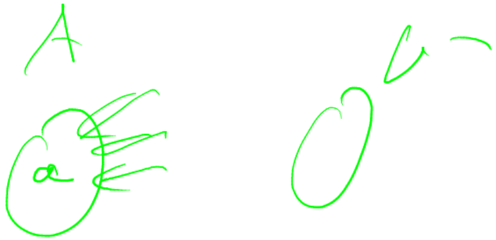






$A = \{x : \text{ex. orient. } a\text{–}x \text{ cesta}\}$

z A nevedou hrany

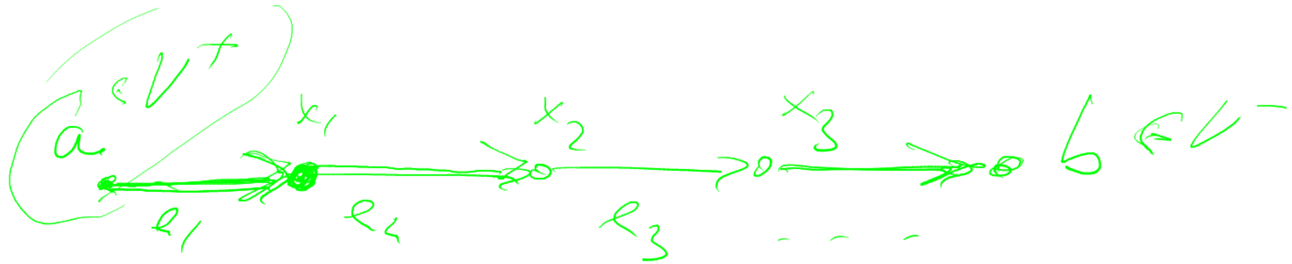


BUĎ $A \cap V^- = \emptyset$

NEBO $A \cap V^- \ni \{b\}$ ... ex. or. a–b cesta

- So there is a directed $a - b$ path P with $a \in V^+$, $b \in V^-$.
- We define a mapping f' by letting $f'(e) = f(e) - k$ for $e \in E(P)$, and $f'(e) = f(e)$ otherwise.

$\square$



$f(e) \in \{1, 2, \dots, k-1\}$

$\Downarrow$

$f'(e) \in$ —||— $\in \{-1, -2, \dots, -k+1\}$

$f(e_i) = f(e_i) - k$

$f'(e) = f(e)$ jinak

$f'(x_1) = f(x_1)$ $\quad f'(x_1) = f'^{+}(x_1) - f'^{-}(x_1)$

$f'(a) = f(a) - k$ $\quad f(a) > f'(a) \geq 0$

$f'(b) = f(b) + k$ $\quad f(b) < f'(b) \leq 0$

$\sum |f'(v)| = \sum |f(v)| - 2k$

- The *existence* of a k-NZF and $\mathbb{Z}_k$-NZF are equivalent,
- but the *numbers* of them not (in general)
- However, the number of k-NZF's of a given graph is also a polynomial in k.
- (Proof using Ehrhart method).

P ... mnohostěn v $\mathbb{R}^k$

$f(k) = |k.P \cap \mathbb{Z}^k|$

... polynom

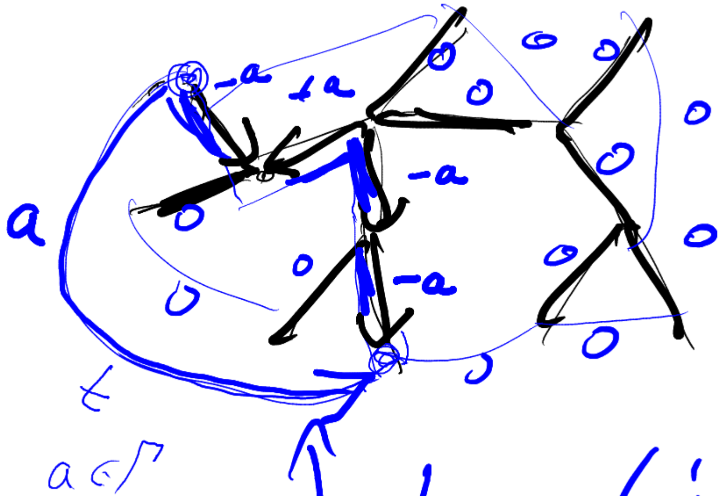
Flows and spanning trees – sum

Let T be a spanning tree of G. Now for every edge $t \in E(G) \setminus E(T)$ and every $a \in \Gamma$ we let $\varphi_{t,a}$ be the (unique) flow in G such that

- $\varphi_{t,a}(t) =$ a
- $\varphi_{t,a}(e) = 0$ for $e \neq t$ and $e \in E(G) \setminus E(T)$

$\therefore$ *elementary flow with respect to T.*

- $\mathcal{F}_\Gamma(G) :=$ the vector space of all flows $\subset \Gamma^{E(G)}$
- (we need Γ to be a field).
- For any fixed spanning tree T the elementary flows $\{\varphi_{t,1} : t \in E(G) \setminus E(T)\}$ form a basis of $\mathcal{F}_\Gamma(G)$.
- Any mapping $\varphi : E(G) \setminus E(T) \to \Gamma$ can be uniquely extended to a Γ-flow on G.
- No control over the edges of T, thus we can't use this easily to construct a NZ flow.



elementární kružnice

v $\Gamma^{E(G)}$ je $\mathcal{F}_\Gamma(G)$ podprostor určený $|V(G)|$ rovnostmi (Kirch. zák.)

$$\sum_{t \in E(G) \setminus E(T)} \underbrace{\varphi(t)}_{\in \Gamma} \cdot \underbrace{\varphi_{t,1}}_{\text{tok}} \text{ — tok}$$

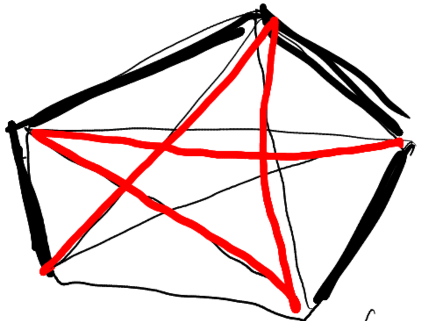
Flows and spanning trees – product

Theorem 17. *Any 4-edge connected graph admits a $\mathbb{Z}_2^2$-NZF.*

Proof. If G is 4-edge connected, then there are two disjoint spanning trees, T_1 and T_2 (proof later).

Let f_i be the $\mathbb{Z}_2$-flow on G that equals 1 on all edges not in T_i. (Such flow exists — see above.)

Now put $f = (f_1, f_2)$. This is indeed a $\mathbb{Z}_2^2$-flow, and if $f(e) = 0 = (0, 0)$ for some edge e then e lies in both T_1 and T_2, a contradiction. □



f_i ... $\mathbb{Z}_2$-tok równy 1 mimo $E(T_i)$

	$E(T_1)$	$E(T_2)$	zbytek
	$e_1\ e_2 \ldots$	$e_k \ldots$	
f_1	? ? 0/1 ?	1 1 ... 1	1 ... 1
f_2	1 1 1 1	? ... ?	1 ... 1

$f = (f_1, f_2) : E(G) \to \mathbb{Z}_2^2 \setminus \{(0,0)\}$

tok

□

Theorem 18 (Jaeger)**.** *Any bridgeless graph admits a* $\mathbb{Z}_2^3$*-NZF.*

Proof. Suppose first that G is 3-edge connected, we will use spanning trees similarly as in the construction of a NZ 4-flow.

We let G' be the (multi)graph obtained from G by adding to each edge a new one, parallel to it.

G' is 6-edge connected ...

So the theorem holds for all 3-edge-connected graphs. To prove it for all bridgeless graphs, suppose there is a counterexample and choose one with minimal number of edges, let it be denoted G. ...

$\square$

PŘÍŠTĚ

Small flows – for bridgeless graphs

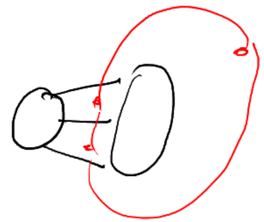
- 1-flow: impossible
- 2-flow: exists precisely in cycles
- 3-flow: for cubic graphs exists precisely in bipartite graphs
- 3-flow: should exist in every 4-edge-connected graph by a **conjecture of Tutte**, 1966. It exists in every 6-edge-connected graph. It suffices to prove it for 5-edge-connected graph.
- 4-flow for a cubic graph is the same as 3-edge-colorability. By a **conjecture of Tutte**, every bridgeless graph that does not have Petersen graph as a minor admits a 4-flow. Proved for cubic graphs by Robinson, Seymour and Thomas (unpublished) by reducing to four-color theorem.
- 4-edge-connected graph has a 4-flow .... dokázali jsme

G rovinný

G hr. 4-souv.

⇓

NE:



G^* nemá C_3 ..... Grötzschova věta

$\chi(G^*) \leq 3$

$\varphi(G) \leq 3$

$G \not\succ K_5, K_{3,3}$

G rovinný $\Rightarrow G \not\succ Pt$

⇓

$\chi(G^*) \leq 4$

$\varphi(G) \leq 4$?

- **Conj. [Tutte 1954]** 5-flow exists in every bridgeless graph
- 6-flow exists in every graph [Seymour 1981]
- 8-flow exists in every graph [Jaeger]
- In particular $\varphi(G) \leq 6$ for each bridgeless graph G.