What about 5-flows?

(1) A bridgeless cubic graph admits a nowhere-zero 3-flow iff it is bipartite.
(2) A bridgeless cubic graph admits a nowhere-zero 4-flow iff it has a 2-factor M such that all cycles of M have even lengths.

Let $G = (V, E)$ be a cubic bridgeless graph. A subset of edges $F \subset E$ such that the degree of any vertex v in the subgraph induced by F is 1 or 2 is called a *(1, 2)-factor* of G. Given a (1, 2)-factor F of a graph G, we say that an edge is F-*balanced* if it belongs to F or it does not and its ends have the same degree in the subgraph induced by F in G. Call a cycle of G F-*even* if it has an even number of F-balanced edges. A (1, 2)-factor F is *even* if each cycle in G is F-even.

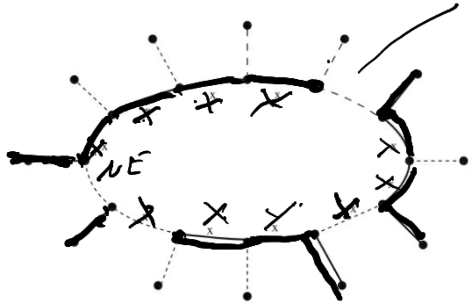


Fig. 1 Example of an F-even cycle. *Solid edges* are in F. *Dashed edges* are in $E \setminus F$. F-balanced edges are indicated with an x

F-even

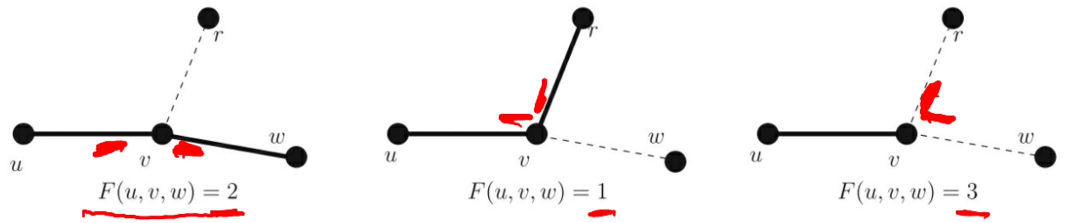


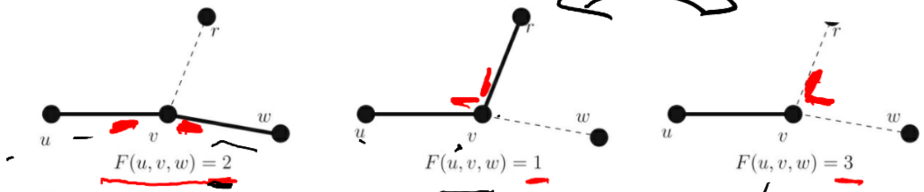
Fig. 2 Definition of $F(u, v, w)$. *Solid edges* are in F. *Dashed edges* are in $E \setminus F$. In the *first diagram* uv and vw are F-related; in the *second diagram* uv and vr are F-related; in the *third diagram* vr and vw are F-related. Notice that we have shown only one of the two possible cases for each value of $F(u, v, w)$

Theorem 1 *A bridgeless cubic graph G admits a nowhere-zero 5-flow if and only if there exists an even* (1, 2)*-factor F of G.*

Let F be a $(1, 2)$-factor of a cubic graph G. Let v be a vertex of G and let u, w, r be its three neighbors. We say that the edges uv and vw are *F-related* if $uv, vw \in F$ or $uv, vw \notin F$. We define the *F-parity* of the tuple (u, v, w), denoted by $F(u, v, w)$, by (Fig. 2):

$F(w, v, u) = 3$

$$F(u, v, w) = \begin{cases} 2 & \text{if } uv \text{ and } vw \text{ are } F\text{-related} \\ 1 & \text{if } uv \text{ and } vr \text{ are } F\text{-related} \\ 3 & \text{if } vr \text{ and } vw \text{ are } F\text{-related.} \end{cases}$$



We extend the definition of F-parity to cycles. Clearly, given a cycle C the number of vertices whose incident edges in C are not F-related is even. Hence if the vertices of C are numbered by $u_0, u_1, \ldots, u_{n-1}, u_n = u_0$, then the following quantity is even:

$$T(C) := \sum_{i=0}^{n-2} F(u_i, u_{i+1}, u_{i+2}) + F(u_{n-1}, u_0, u_1).$$

$F(C) := T(C) \bmod 4$

$F(\overleftarrow{C}) = -F(\overrightarrow{C})$

Lemma 1 *Let G be a bridgeless cubic graph. For each $(1, 2)$-factor F and each cycle C of G we have the following:*

$$\boxed{F(C) = 2|C_F| \mod 4,}$$

where C_F is the set of F-balanced edges of C.

F-even

$\Updownarrow$

$F(C) = 0$

Lemma 1 *Let G be a bridgeless cubic graph. For each $(1,2)$-factor F and each cycle C of G we have the following:*

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where C_F is the set of F-balanced edges of C.

Proof Let $C = (u_0, \dots, u_{n-1}, u_0)$. The relation is clear when $E(C) \subseteq F$ because the F-parity of each tuple of the cycle is 2, then $F(C) = 2|E(C)|$. Let Q be a non trivial (at least one edge) connected component of $C \setminus F$. Let us assume that $Q = (u_1, \dots, u_i)$, $i \geqslant 2$. Let $\alpha(Q)$ be defined by $\alpha(Q) := \sum_{j=1}^{i} F(u_{j-1}, u_j, u_{j+1})$. We prove that

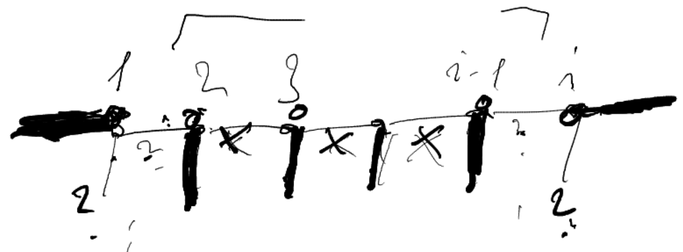
$$\alpha(Q) = 2|E(Q) \cap C_F| + 2 \mod 4.$$

$\boxed{i = 1}$

$F(0,1,2) = 3$	$F(0,1,2) = 1$	$F(0,1,2) = 3$	$F(0,1,2) = 1$
$F(1,2,3) = 1$	$F(1,2,3) = 1$	$F(1,2,3) = 3$	$F(1,2,3) = 3$
$\alpha = 0$	$\alpha = 2$	$\alpha = 2$	$\alpha = 0$

Let us consider first the case $i \geqslant 3$. Notice that $F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 0 \mod 4$ when u_1u_2 and $u_{i-1}u_i$ are F-balanced and $F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 2 \mod 4$ otherwise. Hence,

$$\begin{aligned}\alpha(Q) &= \sum_{j=1}^{i} F(u_{j-1}, u_j, u_{j+1})\\ &= F(u_0, u_1, u_2) + 2(i-3) + 2 + F(u_{i-1}, u_i, u_{i+1})\\ &= 2|E(O) \cap C_F| + 2 \mod 4.\end{aligned}$$



x => F(0, 1, 2) = 3
0 => 1

x => F(i-1, i, i+1) = 1
0 => 3

1	1	3
2	2	1
3	3	4
4	4	2
0	0	0

$x+y+z = 0$ $\quad x+y+z \neq x \mod 5 \in \{1, 3\}$

$1+y+z = 0 \mod 5$

$y+z \in \{4, 9, 14, \ldots\}$

$F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 2$ # bal.

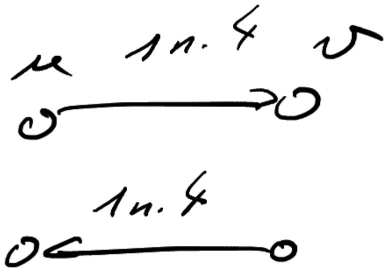
[illegible] mod:

$\{1, 2\}$ $\{i-1, i\}$

✓

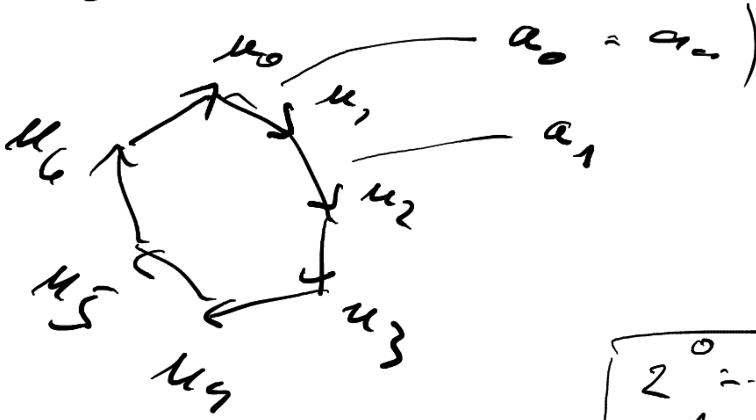
Proposition 1 *Let $G = (V, E)$ be an undirected cubic graph. If G admits a nowhere-zero $\mathbb{Z}_5$-flow, then there exists an even $(1, 2)$-factor F of G.*

Proof Let us assume that G has a nowhere-zero $\mathbb{Z}_5$-flow associated with an orientation $H = (V, A)$ and a function φ. Let F be defined as follows.



$$F = F_\varphi := \{\underline{uv \in E} : \varphi(u, v) \in \underline{\{1, 4\}} \text{ or } \varphi(v, u) \in \{1, 4\}\}.$$

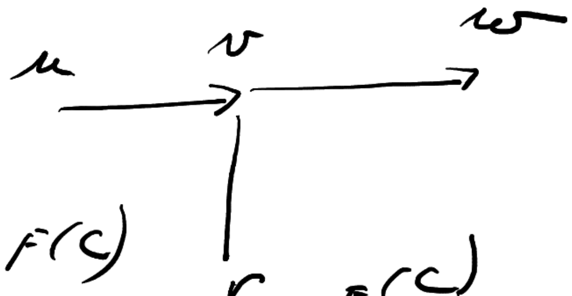
In $\mathbb{Z}_5$ the equation $x + y + z = 0$ has exactly 5 distinct solutions given by $\{\{a, a, 3a\} : a \in \mathbb{Z}_5\}$. Then, for each vertex v at least one arc incident with v has flow in the set $\{1, 4\}$ and at least one arc incident with v has flow in the set $\{2, 3\}$. Therefore, F is a $(1, 2)$-factor of G.



Let $C = (u_0, \ldots, u_{n-1}, u_0)$ be a cycle of length n in G. To ease the notation, let us define $a_i = (u_i, u_{i+1})$ for $i = 0, \ldots, n-1$. w.l.og we can assume that $a_i \in A$, for $i = 0, \ldots, n-1$. To see this, let $a \in A$ and let $A' = A \setminus \{a\} \cup \{-a\}$ and $\varphi' : A' \to \mathbb{Z}_5$, where $\varphi'(b) = \varphi(b)$, if $b \neq -a$ and $\varphi'(-a) = -\varphi(a)$. Then, $F_{\varphi'} = F$. Hence, by modifying the orientation of subgraphs of H and the associated flow, the set F remains the same.

$2^0 = 1$
$2^1 \quad 2$
$2^2 \quad 4$
$2^3 \quad 8$
$2^4 \quad 16 = 1$

~~From the choice of F and~~ the definition of $F(u, v, w)$, the reader can check that $\boxed{\varphi(v, w) = \varphi(u, v)2^{F(u,v,w)}}$ (see Fig. 3), for each path (u, v, w) in H. Hence, for each $i = 0, \ldots, n-2$ it holds:



$$\underline{\varphi(a_{i+1})} = \underline{\varphi(a_i)}2^{F(u_i,u_{i+1},u_{i+1})}. \tag{1}$$

$$\varphi(a_0) = \varphi(a_n) \cdot \varphi(a_{n-1}) \cdot 2^{F(u_{n-1}, u_0, u_1)}$$

$$= \ldots\ldots = \varphi(a_0) \cdot \prod 2^{F(\ldots)} = \varphi(a_0) \cdot 2^{F(C)} \Rightarrow 2^{F(C)} = 1 \Rightarrow \boxed{F(C) = 0 \bmod 4}$$

$F = 2$ μ ν ω

1 4
4 1

2 | 3
3 | 2

$4 = 1 \cdot 2^2$

$1 = 4 \cdot 2^2$

$2 = 1 \cdot 2^1$

$3 = 4 \cdot 2^1$

$F = 1$

1 4
4 1

$F = 3$

4
1

3
2

$2 = 4 \cdot 2^3$

$3 = 1 \cdot 2^3$

2
3

4
1

$1 = 2 \cdot 2^3$

$4 = 3 \cdot 2^3$

<u>Dk</u> ∃ sudý (1,2)-faktor F ⟹ ∃ $\mathbb{Z}_3$-NZF φ

<u>Plán</u> $F = \{ uv : \varphi(uv) \in \{1,2\}$
$(u{:}\ \varphi(v,u) \in \{1,2\} \}$

pro jednu hranu e_0 $\varphi(e_0)$ zvolíme lib. ze 2 možností

pro ostatní jednoznačně

$F(0,1,2)$
$F(1,2,3)$ zbude

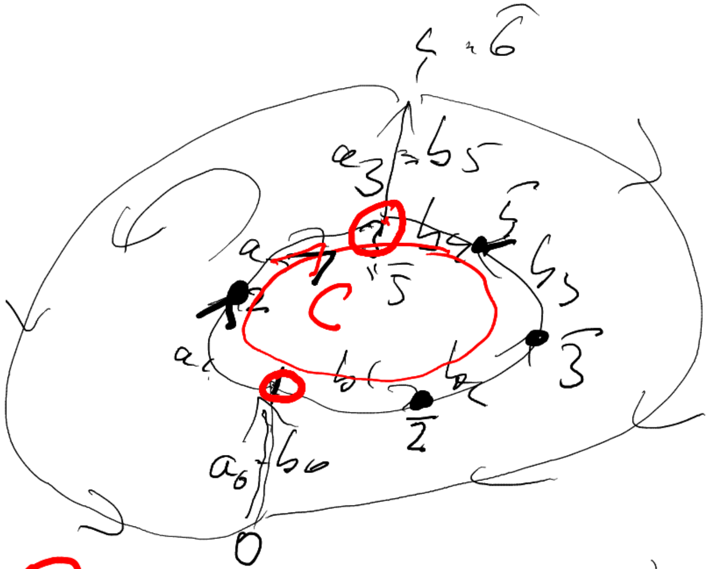
def. $\varphi(a_1) = \varphi(e_0) \cdot 2$

$\varphi(a_2) = \varphi(a_1) \cdot 2$

vír: "po cyklu nenajde spor"

2-souv. ⟹ vícenásobné def.

? nedostaneme spor ?



duci delfa $\varphi(a_3) \overset{?}{=} \varphi(b_5)$

$F(0,1,2)+F(1,2,3)+F(2,3,5)$

$\varphi(a_0)\cdot 2$ $\quad$ $\overset{?}{\parallel}$ mod 5

$\parallel$ $\quad$ $F(0,\bar1,\bar2)+F(\bar1,\bar2,\bar3)+\dots+F(\bar4,\bar5,\bar6)$

$\overset{?}{=} \varphi(b_0)\cdot 2$

$F(C)=0$

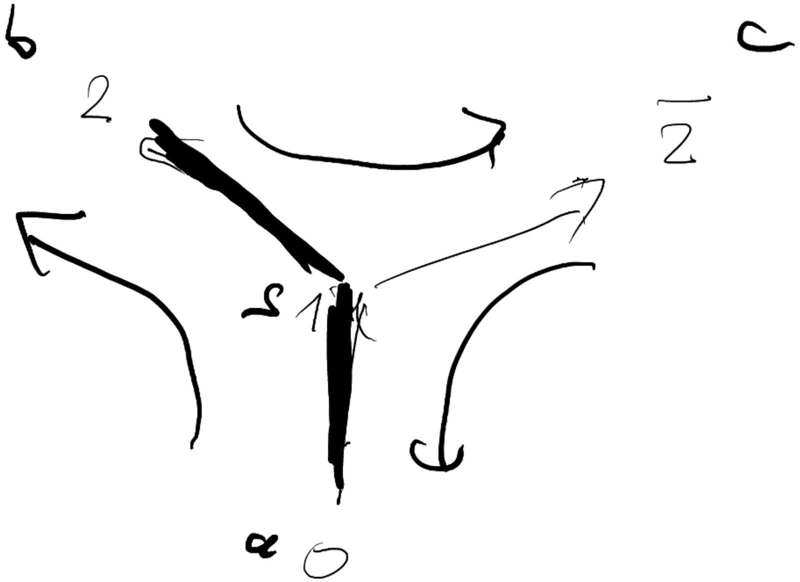
$F(0,1,2)+F(1,2,3)+F(2,3,5) = F(0,\bar1,\bar2)+\dots+F(\bar4,\bar5,\bar6)$ (mod 4)

$F(1,2,3)-F(\bar3,\bar4,\bar5)-F(\ )-F(\ )$

$+F(\bar5,\bar4,\bar3)$

$F(0,1,2)-F(0,\bar1,\bar2) \overset{?}{=} F(\bar2,1,2)+2$

$F(0,1,2)+F(\bar2,1,0)+F(2,1,\bar2) \overset{?}{=} 2$

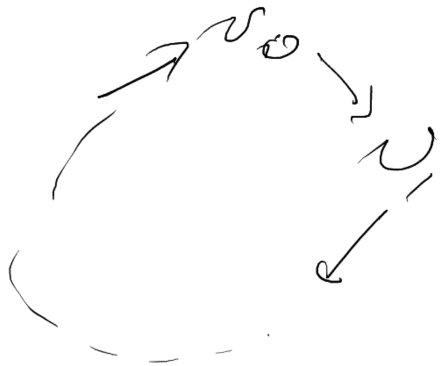


$$F(a, s, b) + F(b, s, c) + F(c, s, a) = 0$$

$$2 + 1 + 3 \qquad 2$$

tenze $t : E(G) \to \Gamma$ ... ab. grupa

$\forall$ cykly $\sum t(v_i v_{i+1}) = 0$



potenciál $p : V(G) \to \Gamma$

$\delta p : uv \mapsto p(v) - p(u)$

Cv. $\delta(p)$ je tenze

Věta $\forall t \; \exists p : \quad t = \delta p$

Dk   dané t

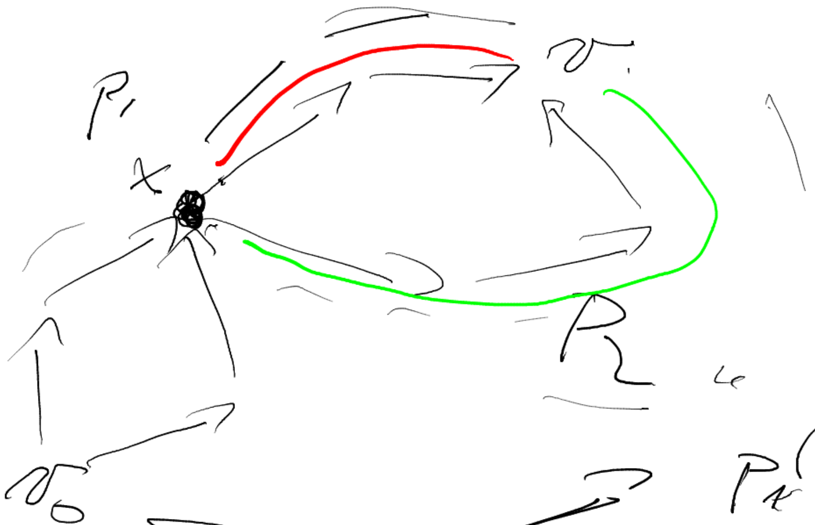
po lib. $v_0 \in V(G)$ $p(v_0) := 0$

hladově definuji p: $p(u)$ je def., $p(v)$ není,

$$p(v) = p(u) \pm t(uv)$$

G souvislý $\Rightarrow$ p def. $\forall v$

? nastane spor ?



$\exists x$ fi.

$x P_1 v$ — $x P_2 v$

$\Rightarrow$ $p_1(v) = p_2(v)$

je cyklus ... po novém je $\sum t = 0$