Písemka E

1) $$\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = \exp\left(\frac{1}{x^2}\ln\frac{\sin x}{x}\right)$$

+ exp je spoj.

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{z}, \quad \text{kde}$$

$$z = \lim_{x\to 0}\frac{\ln\frac{\sin x}{x}}{x^2}$$

dle l'H $\frac{0}{0}$ $\left(\frac{\sin x}{x}\to 1 \Downarrow \ln\frac{\sin x}{x}\to 0\right)$

$$= \lim \frac{\frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \underbrace{\lim\frac{x}{\sin x}}_{=1}\cdot\lim\frac{x\cos x - \sin x}{2x^3} \quad \Big/ \text{l'H } \frac{0}{0}$$

$$= \lim\frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{6x^2} = -\frac{1}{6}$$

$$\boxed{\Rightarrow \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}}$$

2) $\sum \frac{2^n+3^n}{3^n+5^n}$

dle limitního podílového kritéria:

$$\frac{2^{n+1}+3^{n+1}}{3^{n+1}+5^{n+1}} \cdot \frac{3^n+5^n}{2^n+3^n}$$

$$= \frac{\cancel{3^n}\left(2\cdot\left(\frac{2}{3}\right)^n+3\right)}{\cancel{5^n}\left(3\cdot\left(\frac{3}{5}\right)^n+5\right)} \cdot \frac{\cancel{5^n}\left(\left(\frac{3}{5}\right)^n+1\right)}{\cancel{3^n}\left(\left(\frac{2}{3}\right)^n+1\right)}$$

$$\xrightarrow{n\to\infty} \frac{3}{5} < 1$$

$\boxed{\Rightarrow \text{řada konverguje}}$

3) $$f(x) = e^{|x|} \sin x$$

$x > 0$: $$f'(x) = e^{x} \sin x + e^{x} \cos x$$

$x < 0$: $$f'(x) = -e^{-x} \sin x + e^{-x} \cos x$$

$$f'_{+}(0) = \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} e^{x} \sin x + e^{x} \cos x$$

$$= 1$$

$$f'_{-}(0) = \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} -e^{-x} \sin x + e^{-x} \cos x$$

$$= 1$$

$$\boxed{\Longrightarrow f'(0) = 1}$$

4) $g(x) = \sqrt[3]{x^3 + 3x}$

$D_g = R$, spoj. na R

průsečík s osami $[0;0]$

($\Rightarrow$ není per.)

g je lichá $\left(g(-x) = \sqrt[3]{-(x^3+3x)} = -\sqrt[3]{x^3+3x} = -g(x)\right)$

asymptoty: $\lim\limits_{x \to +\infty} \frac{g(x)}{x} = \lim\limits_{x \to \infty} \frac{x\sqrt[3]{1+\frac{3}{x^2}}}{x} = 1$

$\lim\limits_{x \to -\infty} \frac{g(x)}{x} = \lim\limits_{x \to -\infty} \frac{x\sqrt[3]{1+\frac{3}{x^2}}}{x} = 1$

$\lim g(x) - x = \lim \frac{\cancel{x^3} + 3x - \cancel{x^3}}{\sqrt[3]{x^3+3x}^2 + \sqrt[3]{x^3+3x} \cdot x + x^2}$

$= 0$

obdobně v $-\infty$

$$g'(x) = \left(\sqrt[3]{x^3 + 3x}\right)'$$

$$= \frac{1}{3}\underbrace{\left(x^3+3x\right)^{-\frac{2}{3}}}_{\substack{>0 \\ \text{pro } x>0}} \cdot \underbrace{\left(3x^2+3\right)}_{>0} \quad \Rightarrow \text{ na } (0, \infty) \text{ rostoucí}$$

$$\left(g_+'(0) = \infty\right)$$

$$g(x) = -g(-x) \Rightarrow g'(x) = g'(-x)$$

$\Rightarrow g' > 0$ a g rostoucí pro $x \in (-\infty, 0)$

& $g_-'(0) = \infty$

$$g'' = -\frac{2}{\cancel{3}}\left(x^3+3x\right)^{-\frac{5}{3}} \cdot \cancel{3}\left(x^2+1\right)^2$$

$$+ \left(x^3+3x\right)^{-\frac{2}{3}}\left(2x\right)$$

$$= 2\left(x^3+3x\right)^{-\frac{5}{3}}\left[-\left(x^2+1\right)^2 + x\left(x^3+3x\right)\right]$$

$$2\left(x^3+3x\right)^{-\frac{5}{3}}\left[-\left(x^4+2x^2+1\right) + x^4+3x^2\right]$$

$$\boxed{g''(x) = 2\left(x^3+3x\right)^{-\frac{5}{3}}\left[x^2-1\right]}$$

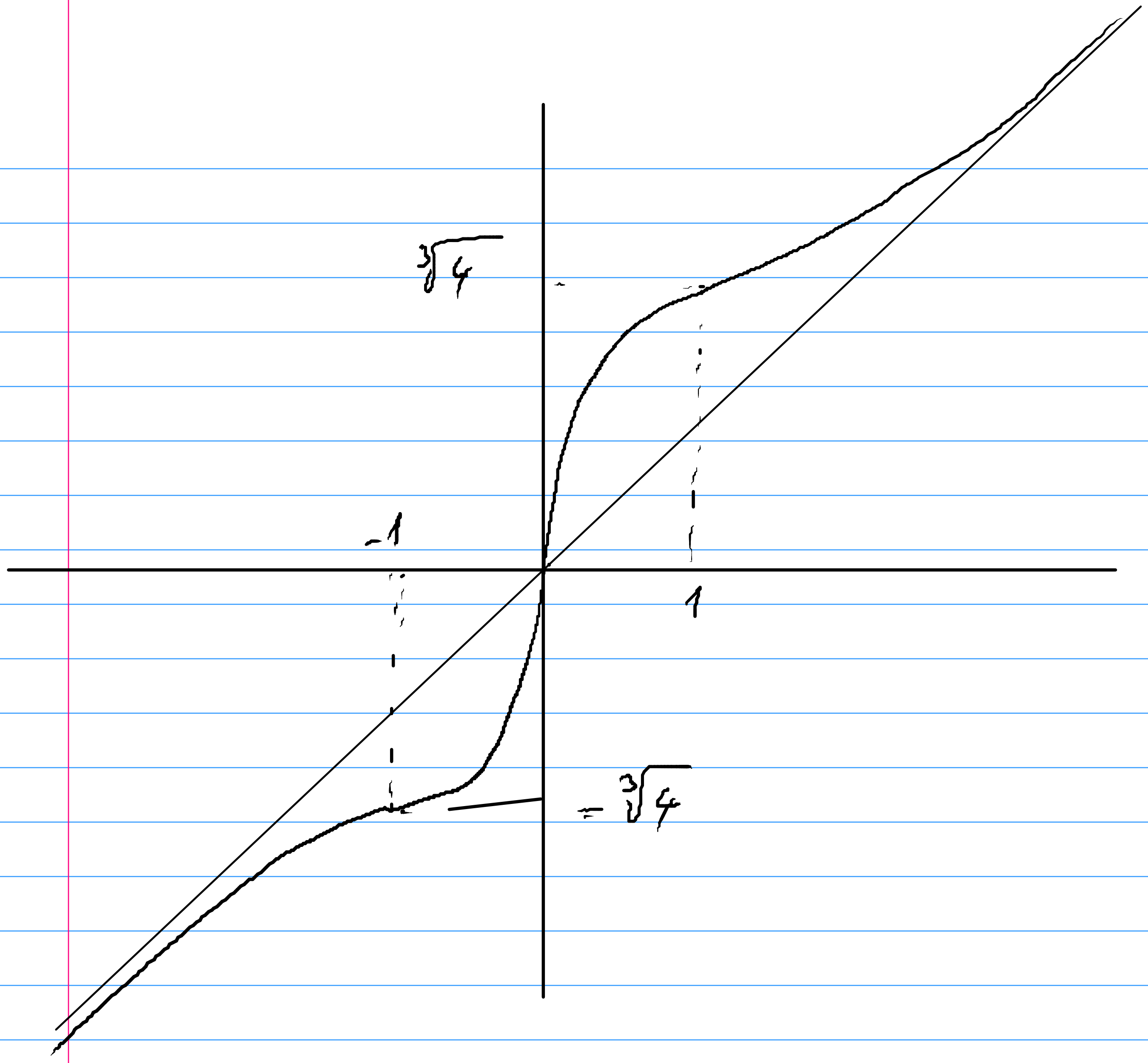
& $g''(x) = -g''(-x)$ / $g''(x) > 0$ $x > 1$

< 0 $x \in (0, 1)$

> 0 $x \in (-1, 0)$

< 0 $x < -1$

$= 0$ $x = \pm 1$



$\Rightarrow M_g = \mathbb{R}$