(1) $I = \int \frac{1}{x(\ln^2 x - 3\ln x + 2)(\ln^2 x + \ln x + 1)}\,dx$

$t = \ln x \qquad t: (0,\infty) \to (-\infty, \infty)$

$dt = \frac{1}{x}\,dx$

$= \int \frac{1}{(t-1)(t-2)\,(t^2+t+1)}\,dt$

$= \int \frac{-\frac{1}{3}}{t-1} + \frac{1/7}{t-2} + \frac{At+B}{t^2+t+1}$

$1 = (t^2+t+1)\left(-\frac{1}{3}(t-2) + \frac{1}{7}(t-1)\right) + (At+B)(t^2-3t+2)$

$t^3: \quad A = \frac{1}{3} - \frac{1}{7} = \boxed{\frac{4}{21}}$

$1: \quad 1 = 1\cdot\left(\frac{2}{3} - \frac{1}{7}\right) + 2B \quad \Rightarrow 2B = 1 + \frac{1}{7} - \frac{2}{3} = \frac{21+3-14}{21} = \frac{10}{21}$

$\boxed{B = \frac{5}{21}}$

kontrola:

$$(t^2+t+1)\left(-\frac{4}{21}t+\frac{11}{21}\right)+\left(\frac{4}{21}t+\frac{5}{21}\right)(t^2-3t+2)$$

t^3 ✓

t^2 $\frac{11}{21}-\frac{4}{21}+\frac{5}{21}-\frac{12}{21}$ ✓

t $11-4+8-15$ ✓

1 ✓

$$I=\int\left(\frac{-\frac{1}{3}}{t-1}+\frac{1/7}{t-2}+\frac{1}{21}\,\frac{4t+5}{t^2+t+1}\right)dx$$

zvlášť spočteme:

$$I_1=\int\frac{4t+5}{t^2+t+1}\,dt=2\underbrace{\int\frac{2t+1}{t^2+t+1}}_{2\ln t^2+t+1}+3\underbrace{\int\frac{1}{t^2+t+1}}_{I_2}$$

platí $\forall t$, protože $t^2+t+1>0 \quad \forall t\in\mathbb{R}$

$$I_2 = \int \frac{1}{t^2+t+1} = \int \frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{\sqrt{3}}{2}\,\frac{4}{3} \int \frac{1}{\left[\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right]^2 + 1} \cdot \frac{2}{\sqrt{3}}\, dt$$

(subst. $s = \frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)$)

$$= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t+1}{\sqrt{3}} + C$$

A jsme hotovi :

$$I = \int \left(\frac{-\frac{1}{3}}{t-1} + \frac{1/7}{t-2} + \frac{1}{21}\,\frac{4t+5}{t^2+t+1} \right) dx$$

$$= -\frac{1}{3} \lg|t-1| + \frac{1}{7} \lg|t-2| + \underbrace{\frac{1}{21} I_1} + C$$

$$\frac{1}{21}\left(2 \ln (t^2+t+1) + 3 \underbrace{I_2}\right)$$

$$\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t+1}{\sqrt{3}}$$

pro
$t \in (-\infty, 1)$
$(1, 2)$
a $(2, \infty)$

$$= -\frac{1}{3}\log|\log x - 1| + \frac{1}{7}\log|\log x + 2|$$

$$+ \frac{2}{21}\log\left((\log x)^2 - \log x + 1\right) + \frac{1}{7}\cdot\frac{2}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2\log x - 1}{\sqrt{3}} + C$$

pro $x \in (0,e)$

(e,e^2)

a (e^2,∞)

Pozn. 1) Nezapomínat na intervaly

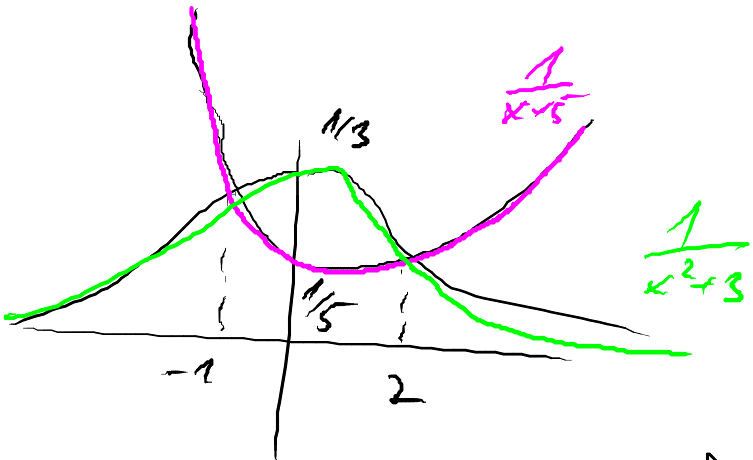
2) Výpočet šlo zjednodušit pokud si pamatujeme (umíme napsat) vzorec pro $\int \frac{1}{x^2+px+q}$.

(2) $\frac{1}{x+5} - \frac{1}{x^2+3}$

$\Downarrow$

$x^2+3 = x+5 \iff x^2 - x - 2 = 0$

$x_{1,2} = \begin{cases} 2 \\ -1 \end{cases}$



$$\int_{-1}^{2} \frac{1}{x^2+3} - \frac{1}{x+5} = \left[\frac{\sqrt{3}}{3} \operatorname{arctg}\left(\frac{x}{\sqrt{3}}\right) - \ln(x+5)\right]_{-1}^{2}$$

pro $x \in (-5, \infty)$, což stačí

$$= \frac{\sqrt{3}}{3}\left(\operatorname{arctg}\frac{2}{\sqrt{3}} - \operatorname{arctg}\frac{-1}{\sqrt{3}}\right) - \ln 7 + \ln 4$$

(3) $f(x,y,z) = xy^2z^3$, $M = \{x+y+3z = a,\ x,y,z > 0\}$

označme jistě $\bar{M} = \{(x,y,z) : x+2y+3z = a;\ x,y,z \geq 0\}$

$g(x,y,z) = x+2y+3z-a$

Lagr. věta $\Rightarrow$ v lok. extr. f na M platí

$\nabla f = \lambda \nabla g$ $\qquad (\nabla g = (1,2,3) \neq \vec{0}\ !!)$

$y^2z^3 = \lambda \cdot 1$ $\qquad \frac{1}{x} = c$

$x \cdot y \cdot z^3 = \lambda \cdot 2$ $\qquad \frac{2}{y} = 2c$

$xy \cdot 3z^2 = \lambda \cdot 3$ $\qquad \frac{3}{z} = 3c$

$c := \frac{1}{xy^2z^3}$ $\qquad \Rightarrow x = y = z \Rightarrow x = y = z = \frac{a}{6}$

$$f\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right) = \left(\frac{a}{6}\right)^6$$

- f je spoj. (zjevně), $\overline{M}$ kompaktní (viz níže), tudíž f na $\overline{M}$ nabývá max. a min.

- $\forall p \in M \quad f(p) > 0$

  $\forall p \in \overline{M} \setminus M \quad f(p) = 0$

- tudíž: na $\overline{M}$: max. v $\left(\frac{a}{6}, \frac{a}{c}, \frac{a}{c}\right)$ $f = \left(\frac{a}{c}\right)^2$

  min. v mnoha bodech, $f = 0$

  (ve všech, co mají v nějaké souř. 0)

$\Rightarrow$ na M: max. v $\left(\frac{a}{6}, \frac{a}{c}, \frac{a}{c}\right)$ $f = \left(\frac{a}{c}\right)^2$

min. není (infimum je 0)

- $\overline{M}$ kompaktní:

1) omezená: $0 \leq x, y, z \leq a$

2) uzavřená: $\overline{M} = \{(x,y,z) : x + 2y + 3z = a\}$

$\cap \{ - " - : x \geq 0\}$

$\cap \{ - " - : y \geq 0\}$

$\cap \{ - " - : z \geq 0\}$

Tedy $\overline{M}$ je průnik těch množin, z nichž každá je uzavřená (neboť je vzor uzavř. množ. $\{a\}$, resp. $[0,\infty)$ ve spoj. zobrazení).