A Doubly Exponentially Crumbled Cake*

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Abstract

We consider the following cake cutting game: Alice chooses a set P of n points in the square (cake) $[0,1]^2$, where $(0,0) \in P$; Bob cuts out n axis-parallel rectangles with disjoint interiors, each of them having a point of P as the lower left corner; Alice keeps the rest.

It has been conjectured that Bob can always secure at least half of the cake. This remains unsettled, and it is not even known whether Bob can get any positive fraction independent of n. We prove that *if* Alice can force Bob's share to tend to zero, *then* she must use very many points; namely, to prevent Bob from gaining more than $1/r$ of the cake, she needs at least $2^{2^{\Omega(r)}}$ points.

Keywords: Combinatorial Geometry, Cake Cutting, Packing Rectangles

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1 Introduction

Alice has baked a square cake with raisins for Bob, but really she would like to keep most of it for herself. In this, she relies on a peculiar habit of Bob: he eats only rectangular pieces of the cake, with sides parallel to the sides of the cake, that contain exactly one raisin each, and that raisin has to be exactly in the lower left corner (see Fig. 1). Alice gets whatever remains after Bob has cut out all such pieces. In order to give Bob at least some chance, Alice has to put a raisin in the lower left corner of the whole cake.

Mathematically, the cake is the square $[0,1]^2$, the raisins form an n-point set $P \subset [0,1]^2$, where $(0,0) \in P$ is required, and Bob's share consists of n axis-parallel rectangles with disjoint interiors, each of them having a point of P as the lower left corner.

By placing points densely along the main diagonal, Alice can limit Bob's share to $\frac{1}{2} + \varepsilon$, with $\varepsilon > 0$ arbitrarily small. A natural question then is, can Bob always obtain at least half of the cake?

This question (in a cake-free formulation) appears in Winkler [5] ("Packing Rectangles", p. 133), where he claims it to be at least 10 years old and of origin unknown to him. The first written reference seems to be an IBM puzzle webpage [1].

We tried to answer the question and could not, probably similar to many other people before us. We believe that there are no simple examples leaving more than $\frac{1}{2}$ to Alice, but on the other hand, it seems difficult to prove even that Bob can always secure 0.0001% of the cake. We were thus led to seriously considering the possibility that Alice might be able to limit Bob's share to less

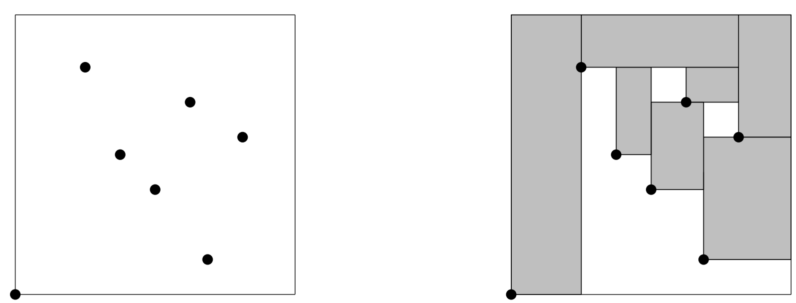


Fig. 1. Example: Alice's points (left) and Bob's rectangles (right)

than $1/r$, for every $r > 0$, but that the number of points n she would need would grow enormously as a function of r.

Here we prove a doubly exponential lower bound on this function. First we introduce the following notation. For a finite $P \subset [0,1]^2$, let $\mathsf{Bob}(P)$ be the largest area Bob can win for P, and let $\mathsf{Bob}(n)$ be the infimum of $\mathsf{Bob}(P)$ over all n-point P as above.[1] Also, for a real number $r > 1$ let $n(r) := \min\{n : \mathsf{Bob}(n) \leq 1/r\} \in \{1, 2, \ldots\} \cup \{\infty\}$.

Theorem 1.1 *There exists a constant r_0 such that for all $r \geq r_0$, $n(r) \geq 2^{2^{r/2}}$.*

The only previous results on this problem we could find is the Master's thesis of Müller-Itten [3]. She conjectured that Alice's optimal strategy is placing the n points on the main diagonal with equal spacing (for which Bob's share is $\frac{1}{2}\left(1 + \frac{1}{n}\right)$). She proved this conjecture for $n \leq 4$, and also in the "grid" case with $P = \{(0,0)\} \cup \{(\frac{i}{n}, \frac{\pi(i)}{n}) : i \in \{1, \ldots, n-1\}\}$, where π is a permutation of $\{1, \ldots, n-1\}$. She also showed that $\mathsf{Bob}(n) \geq \frac{1}{n}$.

The problem considered here can be put into a wider context. Various problems of fair division of resources, often phrased as cake-cutting problems, go back at least to Steinhaus, Banach and Knaster; see, e.g., [4]. Even closer to our particular setting is Winkler's *pizza problem*, recently solved by Cibulka et al. [2].

2 Preliminaries

We call a point a a *minimum* of a set $X \subseteq [0,1]^2$ if there is no $b \in X \setminus \{a\}$ for which both $x(b) \leq x(a)$ and $y(b) \leq y(a)$. Let $p_1, p_2, \ldots, p_k$ be an enumeration of the minima of $P \setminus \{(0,0)\}$ in the order of decreasing y-coordinate (and increasing x-coordinate). Let $\mathsf{stairs}(P)$ be the union of all the axis-parallel rectangles with lower left corners at $(0,0)$ whose interior avoids P; see Fig. 2(a).

Furthermore, let s be the area of $\mathsf{stairs}(P)$, and let α be the largest area of an axis-parallel rectangle contained in $\mathsf{stairs}(P)$. Let us also define $\rho := \frac{s}{\alpha}$. For a point $p \in P$ and an axis-parallel rectangle $B \subseteq [0,1]^2$ with lower left corner at p, we denote by a be the maximum area of the cake Bob can gain in B using only rectangles with lower left corner in points of $B \cap P$. By re-scaling, we have $a = \beta \cdot \mathsf{Bob}(P_B)$, where β is the area of B and P_B denotes the set $P \cap B$ transformed by the affine transform that maps B onto $[0,1]^2$.

We will use the monotonicity of $\mathsf{Bob}(\cdot)$, i.e., $\mathsf{Bob}(n+1) \leq \mathsf{Bob}(n)$ for all $n \geq 1$. Indeed, Alice can always place an extra point on the right side of the square, say, which does not influence Bob's share.

3 The decomposition

We decompose the complement of $\mathsf{stairs}(P)$ into horizontal rectangles $B_1, \ldots, B_k$ as indicated in Fig. 2(a), so that p_i is the lower left corner of B_i. Let β_i be the area of B_i; we have $s + \sum_{i=1}^{k} \beta_i = 1$.

By the above and by an obvious superadditivity, we have

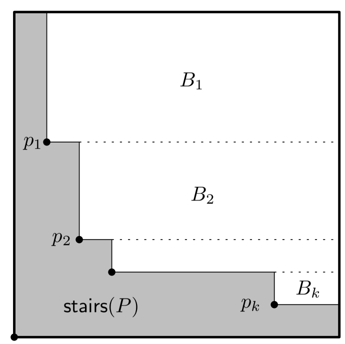
$$\mathsf{Bob}(P) \geq \alpha + \sum_{i=1}^{k} \beta_i \, \mathsf{Bob}(P_i), \tag{1}$$

where $P_i := P_{B_i}$. (This is a somewhat simple-minded estimate, since it doesn't take into account any interaction among the B_i).

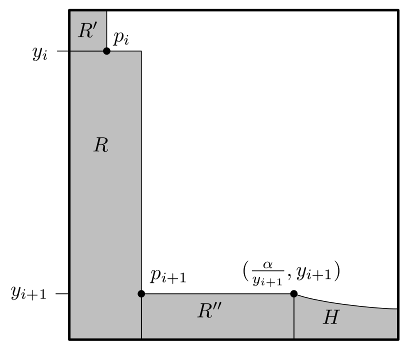
The following lemma captures the main properties of this decomposition.

Lemma 3.1 *Let us assume that $\rho = \frac{s}{\alpha} \geq r_0$, where r_0 is a suitable (sufficiently large) constant. Then*

- *$s \leq \frac{1}{4} \cdot 2^{-\rho}$ (the staircase has a small area), and*



(a) $\mathsf{stairs}(P)$ and the sub-problems.



(b) Illustration to the proof of Lemma 3.1.

Fig. 2.

[1] It is easily checked that, given P, there are finitely many possible placement of Bob's *inclusion-maximal* rectangles, and therefore, $\mathsf{Bob}(P)$ is attained by some choice of rectangles. On the other hand, it is not so clear whether $\mathsf{Bob}(n)$ is attained; we leave this question aside.

- $\sum_{j:j\neq i}\beta_i \geq 2^\rho s$ *for every* $i = 1, 2, \ldots, k$ *(none of the subproblems occupies almost all of the area).*

Proof. First we note that since no rectangle with lower left corner $(0, 0)$ and upper right corner in $\mathsf{stairs}(P)$ has area bigger than α, the region $\mathsf{stairs}(P)$ lies below the hyperbola $y = \frac{\alpha}{x}$. Thus $s \leq \alpha + \int_\alpha^1 \frac{\alpha}{x}\,dx = \alpha + \alpha\ln\frac{1}{\alpha}$. This yields $\alpha \leq e^{-\rho+1}$, and so $s = \rho\alpha \leq \rho e^{-\rho+1} \leq \frac{1}{4}\cdot 2^{-\rho}$ (for ρ sufficiently large).

It remains to show that $\sum_{j:j\neq i}\beta_i \geq 2^\rho s$; since $\sum_{j=1}^k \beta_j = 1 - s$, it suffices to show $\beta_i \leq 1 - 2\cdot 2^\rho s$ for all i.

Let y_i be the y-coordinate of p_i for $i \geq 1$, and let $y_0 = 1$; we have $\beta_{i+1} \leq y_i - y_{i+1}$ for $i \geq 0$.

First, if $y_i \leq \frac{1}{2}$, then $\beta_{i+1} \leq \frac{1}{2} \leq 1 - 2\cdot 2^\rho s$ by the above, and we are done. So we assume $y_i > \frac{1}{2}$.

The area of $\mathsf{stairs}(P)$ can be bounded from above as indicated in Fig. 2(b). Namely, the rectangle R has area at most α (since it is contained in $\mathsf{stairs}(P)$), and the rectangle R' above it also has area no more than α (using $y_i > \frac{1}{2}$). The top right corner of R'' lies on the hyperbola $y = \frac{\alpha}{x}$ used above, and thus R'' has area at most α as well. Finally, the region H on the right of R'' and below the hyperbola has area $\int_{\alpha/y_{i+1}}^1 \frac{\alpha}{x}\,dx = \alpha\ln(y_{i+1}/\alpha)$.

Since $\mathsf{stairs}(P) \subseteq R \cup R' \cup R'' \cup H$, we have $s \leq \alpha(3 + \ln(y_{i+1}/\alpha))$. Using $\rho = \frac{s}{\alpha}$ we obtain $y_{i+1} \geq \alpha e^{\rho-3} = se^{\rho-3}/\rho \geq 2\cdot 2^\rho s$ (again using the assumption that ρ is large).

Finally, we have $\beta_{i+1} \leq 1 - y_{i+1} \leq 1 - 2\cdot 2^\rho s$, and the lemma is proved.□

4 Proof of Theorem 1.1

Proof. Let $r \geq r_0$. We may assume that r is of the form $r = 1/\mathsf{Bob}(n)$, where $n = n(r)$. In particular, $\mathsf{Bob}(m) > \frac{1}{r}$ for all $m < n$.

We will derive the following recurrence for such an r:

$$n(r) \geq 2n(r - 2^{-(r+1)/2}). \tag{2}$$

Applying it iteratively $t := 2^{r/2}$ times, we find that $n(r) \geq 2^t n(r-1) \geq 2^t$ as claimed in the theorem.

We thus start with the derivation of (2). Let us look at the inequality (1) for an n-point set P that attains $\mathsf{Bob}(n)$.[2] Since $n_i := |P_i| < n$ for all i, we have $\mathsf{Bob}(P_i) > \frac{1}{r}$ for all i.

[2] Or rather, since we haven't proved that $\mathsf{Bob}(n)$ is attained, we should choose n-point P with $\mathsf{Bob}(P) < \mathsf{Bob}(n')$ for all $n' < n$.

Let α and s be as above. First we derive $\rho = \frac{s}{\alpha} \geq r$. Indeed, if we had $\alpha > \frac{s}{r}$, then the right-hand of (1) can be estimated as follows:

$$\alpha + \sum_{i=1}^k \beta_i\,\mathsf{Bob}(P_i) > \frac{1}{r}\left(s + \sum_{i=1}^k \beta_i\right) = \frac{1}{r},$$

which contradicts the inequality (1). So $\rho \geq r \geq r_0$ indeed.

Let us set $\gamma_i := \mathsf{Bob}(P_i) - \frac{1}{r}$; this is Bob's "gain" over the ratio $\frac{1}{r}$ in the ith subproblem. From (1) we have

$$\frac{1}{r} \geq \sum_{i=1}^k \beta_i\left(\frac{1}{r} + \gamma_i\right) \geq \frac{1}{r}\left(\sum_{i=1}^k \beta_i\right) + \sum_{i=1}^k \beta_i\gamma_i = \frac{1-s}{r} + \sum_{i=1}^k \beta_i\gamma_i,$$

and so

$$\sum_{i=1}^k \beta_i\gamma_i \leq \frac{s}{r}. \tag{3}$$

According to Lemma 3.1, we can partition the index set $\{1, 2, \ldots, k\}$ into two subsets I_1, I_2 so that $\sum_{i\in I_j}\beta_i \geq 2^\rho s \geq 2^r s$ for $j = 1, 2$.

Let i_1 be such that $\gamma_{i_1} = \min_{i\in I_1}\gamma_i$, and similarly for i_2. Then (3) gives, for $j = 1, 2$,

$$\frac{s}{r} \geq \sum_{i\in I_j}\beta_i\gamma_i \geq \gamma_{i_j}\sum_{i\in I_j}\beta_i \geq \gamma_{i_j}2^r s,$$

and so $\gamma_{i_j} \leq \gamma^* := 2^{-r}/r$.

Let us define $r^* < r$ by $\frac{1}{r^*} = \frac{1}{r} + \gamma^*$. Then we know that at least two of the sets P_i contain at least $n(r^*)$ points each, and hence $n(r) \geq 2n(r^*)$. We calculate $r^* = \frac{r}{1+r\gamma^*} \geq r(1 - r\gamma^*) = r - r2^{-r} \geq r - 2^{-(r+1)/2}$ (again using $r \geq r_0$).

So we have derived the desired recurrence (2), and Theorem 1.1 is proved.□

Acknowledgments

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