Small families under subdivision

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Abstract

Let H be a graph with maximum degree d, and let $d' \geq 0$. We show that for some $c > 0$ depending on H, and all integers $n \geq 0$, there are at most c^n unlabelled simple d-connected n-vertex graphs with maximum degree at most d' that do not contain H as a subdivision. On the other hand, the number of unlabelled simple $(d-1)$-connected n-vertex graphs with minimum degree d and maximum degree at most $d+1$ that do not contain K_{d+1} as a subdivision is superexponential in n.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. We say H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. A *subdivision* of a graph H is a graph J such that H can be obtained from J by repeatedly contracting an edge incident with a vertex of degree two; and we say G *contains a subdivision of* H if some subgraph of G is a subdivision of H.

It is known [1, 3] that for every graph H, the number of unlabelled n-vertex graphs not containing H as a minor is at most c^n, for some constant c depending on H, as we discuss in the next section. Here we investigate the number of n-vertex graphs that do not contain a subdivision of H.

When is it true that the number of (unlabelled) n-vertex graphs that do not contain H as a subdivision is only exponential in n? When H has maximum degree at most three, this is true, since in that case containing a subdivision of H is equivalent to containing H as a minor. But when H has maximum degree at least four, it is false: in fact there are superexponentially many graphs of maximum degree three that do not contain H. So, this question is not very interesting. Let us ask instead, if H has maximum degree d, when is it true that the number of n-vertex graphs with minimum degree at least d not containing a subdivision of H is at most exponential?

This is still not true in general. For instance, let G be four-connected, with a vertex v such that $G \setminus v$ is a cubic graph. Then G contains no subdivision of the octahedron graph (the 4-regular graph on six vertices); and yet there are superexponentially many such graphs G, since there are superexponentially many cubic graphs that become 4-connected if we add one extra vertex adjacent to all other vertices. For this reason we will impose an upper bound on degree.

This still is not enough: we shall show in section 3 that

1.1 *For all $d \geq 5$, and infinitely many $n > 0$, there are superexponentially many d-regular $(d-1)$-connected n-vertex graphs that do not contain a subdivision of K_{d+1}.*

(This is not true for $d = 4$.) So let us assume d-connectivity as well. Then we have a positive result:

1.2 *Let $d, d' \geq 0$ and let H have maximum degree at most d. Then there exists $c > 0$ such that there are at most c^n unlabelled d-connected n-vertex graphs with maximum degree at most d' that do not contain a subdivision of H.*

This is an immediate consequence of 2.2 below, and the following:

1.3 *Let $d, d' \geq 0$, and let H be a graph with maximum degree at most d. Then there exists $t \geq 0$ such that for every graph G, if G is d-connected and has maximum degree at most d', and G contains K_t as a minor, then G contains a subdivision of H.*

Again, this becomes false if we omit the upper bound d' on degree; the same example (a cubic graph with an extra vertex adjacent to all others) can have arbitrarily large complete graphs as minors, and this generalizes to other values of d.

As far as we know, the result 1.3 is new, despite the extensive research that has already been carried out in this area. Chun-Hung Liu (private communication) tells us that it can be deduced from theorem 6.8 in [4], but we will give a proof not using such heavy machinery. A related result was proved by Irene Muzi in her thesis [6]:

1.4 *Let H_1 be a graph of maximum degree at most four, and let H_2 be a graph such that for some vertex v, $H_2 \setminus v$ has maximum degree at most three. Then there exists t such that every 4-connected graph that has a K_t minor contains a subdivision of one of H_1, H_2.*

As we said, the statement of 1.1 is false when $d = 4$, and is worth a closer look. For infinitely many n there are superexponentially many 3-connected n-vertex graphs that do not contain a subdivision of K_5, in which every vertex has degree 4 or 5; but the number of such graphs that are 4-regular graphs is only exponential. That follows from:

1.5 *Let $d \in \{4, 5\}$ and $d' \geq 0$, and let H be a graph with maximum degree at most d. Then there exists $t \geq 0$ such that for every graph G, if G is $(d-1)$-connected and d-edge-connected and has maximum degree at most d', and G contains K_t as a minor, then G contains a subdivision of H.*

This provides a substitute for 1.3 for 4-regular 3-edge-connected graphs, since for $d = 4$, d-regular $(d-1)$-connected graphs are d-edge-connected.

2 Minor containment

It was proved in [7] that

2.1 *For every graph H, there exists c such that for all n, the number of labelled n-vertex graphs not containing H as a minor is at most $n!c^n$.*

A strengthening was found by Amini, Fomin and Saurabh [1], and the same result is implicit in the proof of theorem 4 of [3]:

2.2 *For every graph H, there exists c such that for all n, the number of unlabelled n-vertex graphs not containing H as a minor is at most c^n.*

This paper is concerned with subdivision containment rather than minor containment, but let us take the opportunity to give another proof of 2.2, simpler than those in print. Two (distinct) vertices u, v of a graph G are *twins* if they have the same neighbours in $V(G) \setminus \{u, v\}$. We use the following easy lemma of [7]:

2.3 *For every graph H there exist c, d with $0 < c \leq 1$ and $d \geq 1$ such that for all $n \geq 2$ and for every n-vertex graph G not containing H as a minor, there is a subset $S \subseteq V(G)$ with $|S| \geq cn$, with the following properties. Every vertex in S has degree at most d; and for every $v \in S$, there exists $u \in S$ with $u \neq v$, such that either u, v are nonadjacent twins, or u, v are adjacent.*

Proof of 2.2. If J is a graph, we say that a graph G is a *k-growth* of J if there are $2k$ distinct vertices $u_1, v_1, \dots, u_k, v_k$ of G, such that either

- for $1 \leq i \leq k$, u_i, v_i are nonadjacent and have the same neighbours in G, and J is obtained from G by deleting $v_1, \dots, v_k$; or
- for $1 \leq i \leq k$, u_i, v_i are adjacent and both have degree at most d, and J is obtained from G by contracting the edges u_iv_i for $1 \leq i \leq k$ (and deleting any resultant parallel edges).

Let us call these k-growths of the *first* and *second kind* respectively.

(1) *If J is an m-vertex graph and $k \geq 0$, the number of distinct unlabelled k-growths of J is at most $3^{2dk}2^m$.*

Certainly J has at most 2^m k-growths of the first kind, since every such k-growth is determined by a knowledge of the set $\{u_1, \ldots, u_k\} \subseteq V(J)$, and there are at most 2^m such sets. We claim that J has at most $2^m 3^{(2d-2)k}$ k-growths of the second kind. To see this, let G be a k-growth of the second kind, let $u_i, v_i \in V(G)$ $(1 \leq i \leq k)$ be as in the definition, and let $w_1, \ldots, w_k$ be the vertices of J formed by contracting the edges $u_1v_1, \ldots, u_kv_k$. Thus each w_i has degree at most $2d-2$. There are at most 2^m choices for the set $\{w_1, \ldots, w_k\}$. For each such choice $\{w_1, \ldots, w_k\}$, to construct G we replace each w_i by the edge u_iv_i, and for each edge w_ix we replace it by either u_ix or v_ix, or both. Let us follow this process one edge at a time. We have three choices for each edge w_ix, and there are only at most $(2d-2)k$ such edges since $w_1, \ldots, w_k$ have degree at most $2d-2$ (and we may assume this property is preserved as we do the uncontractions one by one). Thus, each choice of $w_1, \ldots, w_k$ gives rise to at most $3^{(2d-2)k}$ k-growths of the second kind; and hence in total there are at most $3^{(2d-2)k}2^m$ k-growths of the second kind. Since there are only 2^m k-growths of the first kind, we deduce that altogther there are at most

$$3^{(2d-2)k}2^m + 2^m \leq 3^{2dk}2^m$$

k-growths of J. This proves (1).

Given H, let c, d be as in 2.3.

(2) *If G is an n-vertex graph with no H minor, then G is a k-growth of some graph where $k = \lceil cn/(2d+3) \rceil$.*

Let G be a n-vertex graph with no H minor, and let S be as in 2.3. Let S_2 be the set of vertices in S with a neighbour in S, and let $S_1 = S \setminus S_2$. For each $v \in S_1$ there exists $u \in S$ such that u, v are nonadjacent twins; and consequently $u \in S_1$. Let us partition S_1 into pairs of twins, as far as possible; then we obtain at least $|S_1|/3$ pairs. Consequently G is a k_1-growth of the first kind, where $k_1 \geq |S_1|/3$. The subgraph induced on S_2 has minimum degree at least one, and maximum degree at most d, and so has a matching of cardinality at least $|S_2|/(2d)$, as is easily seen. Hence G is a k_2-growth of the second kind, where $k_2 \geq |S_2|/(2d)$. Now

$$cn \leq |S_1| + |S_2| \leq 3k_1 + 2dk_2 \leq (2d+3)\max(k_1, k_2)$$

and so $\max(k_1, k_2) \geq k$. This proves (2).

For $n \geq 0$, let there be $f(n)$ unlabelled n-vertex graphs not containing H as a minor. Define $b = 3^{2d}2^{(2d+3)/c-1}$. Thus $b \geq 1$. We prove by induction on n that $f(n) \leq b^n$. This is true if $n \leq 1$, so we assume that $n \geq 2$. Let k be as in (2). By (1) and (2), $f(n) \leq f(n-k)3^{2dk}2^{n-k}$. From the inductive hypothesis, $f(n-k) \leq b^{n-k}$; and so

$$f(n)b^{-n} \leq b^{-k}3^{2dk}2^{n-k}.$$

It remains to show that $b^{-k}3^{2dk}2^{n-k} \leq 1$, that is, that $3^{2d}2^{n/k-1} \leq b$. But $n/k \leq (2d+3)/c$ from the definition of k, and so $3^{2d}2^{n/k-1} \leq 3^{2d}2^{(2d+3)/c-1} = b$. This proves 2.2. ∎

3 A construction

In this section we prove 1.1. We need

3.1 *For all $b \geq 0$, and for infinitely many $m > 0$, there are more than b^m different unlabelled graphs with m vertices that are d-regular and d-connected.*

In fact Bender and Canfield [2], and independently Wormald [11], proved the following, which for $d \geq 3$ is much stronger than 3.1:

3.2 *For all integers $d \geq 0$ and $m \geq 0$ with dm even, the number of labelled d-regular graphs with m vertices is approximately*

$$\sqrt{2}e^{1-d^2/4}\left(\frac{d^d m^d}{e^d (d!)^2}\right)^{m/2},$$

where "$f(m)$ is approximately $g(m)$" means $f(m)/g(m) \to 1$ as $m \to \infty$.

(In fact these papers estimated the number of labelled d-regular graphs; to count just the d-connected ones, a result of Łuczak [5] shows that for fixed $d \geq 3$, as $m \to \infty$ almost all labelled d-regular graphs with m vertices are d-connected.) To see that 3.2 implies 3.1, observe that for any constant b, the expression in 3.2 is more than $m!b^m$ for m sufficiently large.

The next result is a lemma that will help to show that the graphs we construct do not contain a subdivision of K_{d+1}.

3.3 *Let G be a d-regular graph, and let $Z \subseteq V(G)$ with $|Z| = 2d-2$. Let $X \subseteq Z$ with $|X| \leq d-1$, such that at least three vertices in X have exactly one neighbour in $V(G) \setminus Z$, and each vertex in $Z \setminus X$ has no neighbours in $V(G) \setminus Z$. Let H be a subgraph of G that is a subdivision of K_{d+1}, and let U be the set of vertices that have degree d in H. Then every vertex in $Z \cap U$ belongs to X and has at least two neighbours in $V(G) \setminus Z$.*

Proof. Let $U = \{u_1, \ldots, u_{d+1}\}$. Then evidently:

(1) *For $1 \leq i < j \leq d+1$ there are d paths of H between u_i and u_j, pairwise internally disjoint.*

(2) *Every vertex of G adjacent in G to at least three members of U belongs to U.*

For $1 \leq i \leq d+1$, every edge of G incident with u_i is an edge of H, since G is d-regular and u_i has degree d in H. So every vertex of G with a neighbour in U belongs to $V(H)$. Suppose that $v \in V(G)$ is adjacent in G to at least three members of U. Then $v \in V(H)$, and has degree at least three in H; and since H is a subdivision of K_{d+1}, it follows that $v \in U$. This proves (2).

Suppose first that $U \cap (Z \setminus X) \neq \emptyset$, and let $u_1 \in (Z \cap U) \setminus X$. Since X is a cutset of cardinality at most $d-1$, (1) implies that $U \subseteq Z$. Let X' be the set of all vertices in X with at least two neighbours in $V(G) \setminus Z$. For each $v \in Z \setminus X'$, since v has at most one neighbour in $V(G) \setminus Z$, it has at least $d-1$ neighbours in Z, and since $|Z \setminus U| = d-3$, it follows that either $v \in U$ or v has at least three neighbours in U; and from (2), it follows that $v \in U$. But then U includes $Z \setminus X$ and all the (at least three) vertices of $X \setminus X'$, which is impossible since $|Z \setminus X| \geq d-1$ and $|U| = d+1$.

This proves that $U \cap Z \subseteq X$, and we claim that $U \cap Z \subseteq X'$. Suppose not, and let $u \in (U \cap Z) \setminus X'$, and let y be the unique neighbour of $u \in V(G) \setminus Z$. Since $|U| = d+1$ and $|X| \leq d-1$, there are at least two vertices of U not in Z, and so there is one of them, say u', that is different from y. Then $(X \setminus \{u\}) \cup \{y\}$ is a cutset of cardinality at most $d-1$ separating u and u', contrary to (1). This proves 3.3. ∎

Now we prove 1.1, which we restate:

3.4 *Let $d \geq 5$ be an integer; then for all $c > 0$, and infinitely many values of $n > 0$, there are more than c^n d-regular $(d-1)$-connected graphs G with n vertices that do not contain a subdivision of K_{d+1}.*

Proof. The proof breaks into two cases, depending whether d is odd or even. The odd case is easier, so we begin with that. Take a complete bipartite graph with bipartition (X, Y), where $|X| = |Y| = d-1$. Partition Y into $(d-1)/2$ pairs, and add an edge joining each pair, forming a graph R say. Thus every vertex in Y has degree d, and every vertex in X has degree $d-1$.

Let $n = (2d-2)m$, where $m \geq 2$ is an integer. Let D be a $(d-1)$-regular, $(d-1)$-connected graph with m vertices $v_1, \ldots, v_m$. Take the disjoint union of m copies of R, say $R_1, \ldots, R_m$, and for $1 \leq i \leq n$ let $X_i \subseteq V(R_i)$ be the set of vertices of R_i with degree $d-1$ in R_i, and $Y_i = V(R_i) \setminus X_i$. For each edge v_iv_j of D, add an edge between X_i and X_j, so that these new edges form a matching (this is possible, since each v_i has degree $d-1$ in D, and each set X_i has cardinality $d-1$). Let the graph we form by this process be G_D. It is easy to see that G_D is $(d-1)$-connected and d-regular, and has n vertices. We claim that G_D contains no subdivision of K_{d+1}. Suppose that it does, and $H \subseteq G_D$ is a subdivision of K_{d+1}. Let U be the vertices of H with degree d. By 3.3 applied to $Z = V(R_i)$, it follows that $U \cap V(R_i) = \emptyset$, for $1 \leq i \leq m$, which is impossible. This proves that G_D contains no subdivision of K_{d+1} (in the case when d is odd).

We observe that if D, D' are nonisomorphic graphs, both $(d-1)$-regular and $(d-1)$-connected, then G_D and $G_{D'}$ are nonisomorphic; because with notation as before, every subgraph of G_D isomorphic to R is one of the graphs $R_1, \ldots, R_m$, and so D can be obtained from G_D by contracting all edges of every R-subgraph of G_D.

Now let $c > 0$, and let $b = c^{2d-2}$. By 3.1, for infinitely many $m > 0$ there are more than b^m graphs D on m vertices that are $(d-1)$-regular and $(d-1)$-connected; and all the corresponding graphs G_D are distinct. Hence, since each such G_D has $(2d-2)m = n$ vertices, for infinitely many $n > 0$ there are more than $b^m = c^n$ d-regular $(d-1)$-connected graphs with n vertices, not containing a subdivision of K_{d+1}. This completes the proof when d is odd.

Now we turn to the case when d is even. Define a graph R as follows. Take six pairwise disjoint sets A, B, C, C', B', A' of cardinalities $d/2, d-1, d/2-1, d/2-1, d-1, d/2$ respectively, and choose $s, u \in B, t \in C, t' \in C'$ and $s', u' \in B'$. Make every vertex in $A \cup C$ adjacent to every vertex in B, except the pair st; and similarly make $A' \cup C'$ complete to B' except for the pair $s't'$. Add a perfect matching between C, C', in which t, t' are not adjacent (this is possible since $d/2 - 1 \geq 2$); and add one more edge tt'. Pair up the vertices in $B \setminus \{u\}$, and join each pair with an edge; and add one more edge su. (Thus s is incident with two of these edges.) Add edges within B' similarly. This defines R. We see that

- every vertex in $A \cup A'$ has degree $d-1$, and all other vertices have degree d;

- every edge of R either has both ends in $A \cup B \cup C$, or both ends in $A' \cup B' \cup C'$, or both ends in $C \cup C'$.

Let $n = (4d-4)m$, where $m \geq 2$ is an integer. Let D be a d-regular, d-connected graph with m vertices $v_1, \dots, v_m$. Take the disjoint union of m copies of R, say $R_1, \dots, R_m$, and for $1 \leq i \leq m$ for $1 \leq i \leq m$, let

$$A_i, B_i, C_i, C'_i, B'_i, A'_i, s_i, t_i, u_i$$

correspond to

$$A, B, C, C', B', A', s, t, u$$

respectively. For each edge $v_i v_j$ of D, add an edge between X_i and X_j, so that these new edges form a matching (this is possible, since each v_i has degree d in D, and each set X_i has cardinality d.) Let the graph we form by this process be G_D. It is easy to see that G_D is d-regular, and has n vertices. We claim that G_D contains no subdivision of K_{d+1}. Suppose that it does, and $H \subseteq G_D$ is a subdivision of K_{d+1}. Let U be the vertices of H with degree d. Choose some i with $1 \leq i \leq m$. Since every vertex in A_i has only one neighbour in $V(G) \setminus Z$ and $|A_i| = d/2 \geq 3$, 3.3 applied to $Z = A_i \cup B_i \cup C_i$ tells us that $U \cap (A_i \cup B_i) = \emptyset$, and similarly $U \cap (A'_i \cup B'_i) = \emptyset$. Suppose that there exists $u \in U \cap C_i$. Since $A_i \cup C'_i$ is a cutset of G of cardinality $d-1$, and every two vertices in U are joined by d internally disjoint paths, it follows that $U \subseteq A_i \cup B_i \cup C_i \cup C'_i$, and hence $U \subseteq C_i \cup C'_i$. But this is impossible since $|C_i \cup C'_i| = d-2$. Thus $U \cap C_i = \emptyset$, and similarly $U \cap C'_i = \emptyset$; and so $U \cap V(R_i) = \emptyset$. Since this holds for all i, this is impossible. Consequently G_D contains no subdivision of K_{d+1}.

For $1 \leq i \leq m$, let $X_i = A_i \cup A'_i$, and $Y_i = V(R_i) \setminus X_i$. Next we show that the graph G_D is $(d-1)$-connected. Suppose not; then there is a set $W \subseteq V(G_D)$ with $|W| \leq d-2$ such that deleting W from G_D makes a graph with at least two components. Choose a partition P, Q of $V(G_D) \setminus W$ with P, Q nonempty, such that no vertex in P has a neighbour in Q.

(1) *For $1 \leq i \leq m$, not both P, Q have nonempty intersection with $A_i \cup B_i \cup C_i$.*

Suppose they do, for $i = 1$ say. Since $|A_1 \cup C_1| = |B_1| = d-1$ and $|W| \leq d-2$, and $P \cup Q$ contains all vertices not in W, it follows that $P \cup Q$ has nonempty intersection with both $A_1 \cup C_1$ and B_1. If $P \cup Q$ contains a vertex of $A_1 \cup C_1$ different from s_1, and contains a vertex of B_1 different from t_1, then they are adjacent, and since every vertex of $A_1 \cup B_1 \cup C_1$ is adjacent to one of these two vertices, it follows that the subgraph induced on $(P \cup Q) \cap (A_1 \cup B_1 \cup C_1)$ is connected, a contradiction. Thus either W includes $A_1 \cup C_1 \setminus \{t_1\}$ or W includes $B_1 \setminus \{s_1\}$. Since these sets both have cardinality $d-2$, and $|W| \leq d-2$, it follows that W is one of $A_1 \cup C_1 \setminus \{t_1\}, B_1 \setminus \{s_1\}$. If $W = A_1 \cup C_1 \setminus \{t_1\}$, we may assume that $t_1 \in P$; and since $B_1 \cap W = \emptyset$, and all vertices in B_1 except b_1 are adjacent to t_1, it follows that $B_1 \setminus \{b_1\} \subseteq P$; and in particular $u_1 \in P$, and since u_1, s_1 are adjacent it follows that $s_1 \in P$, contradicting that $Q \cap (A_1 \cup B_1 \cup C_1) \neq \emptyset$. Thus $W = B_1 \setminus \{s_1\}$. We may assume that $s_1 \in P$, and so $A_1 \cup C_1 \setminus \{t_1\} \subseteq P$; and since $Q \cap (A_1 \cup B_1 \cup C_1) \neq \emptyset$, it follows that $t_1 \in Q$. Since $G[B_1 \cup C'_1 \cup \{t_1\}]$ is connected and none of its vertices are in W, it follows that $B_1 \cup C'_1 \subseteq Q$; but there is a vertex in $C_1 \setminus \{t_1\}$, and it belongs to P and has a neighbour in C'_1, a contradiction. This proves (1).

(2) *There do not exist $i, j \in \{1, \dots, m\}$ such that $V(R_i) \subseteq P \cup W$ and $V(R_j) \subseteq Q \cup W$.*

Suppose such i, j exist; then $i \neq j$, since $|V(R_i)| > |W|$. Now the graph D is d-connected, and so there are d paths of G_D, pairwise vertex-disjoint, between $V(R_i)$ and $V(R_j)$ (note that these paths are vertex-disjoint and not just internally disjoint, since the edges joining $V(R_i)$ to $V(G_D) \setminus V(R_i)$ form a matching, and the same for R_j). But then one of these paths is disjoint from W, and so its vertex set is a subset of P or a subset of Q, in either case a contradiction since $V(R_i) \subseteq P \cup W$ and $V(R_j) \subseteq Q \cup W$. This proves (2).

(3) *For $1 \leq i \leq m$, if P, Q both have nonempty intersection with $V(R_i)$ then one of A_1, A'_i is a subset of $P \cup W$ and the other a subset of $Q \cup W$, and $|W \cap (C_i \cup C'_i)| \geq d/2 - 1$.*

By (1) we may assume that $A_i \cup B_i \cup C_i \subseteq P \cup W$ and $A'_i \cup B'_i \cup C'_i \subseteq Q \cup W$, and the first claim follows; and since there is a matching between C_i, C'_i of cardinality $d/2 - 1$, it follows that $|W \cap (C_i \cup C'_i)| \geq d/2 - 1$. This proves (3).

From (2) we may assume that $P \cap V(R_i) \neq \emptyset$ for $1 \leq i \leq m$; and so from (3), there are at most two values of $i \in \{1, \ldots, m\}$ such that $Q \cap V(R_i) \neq \emptyset$. Since $Q \neq \emptyset$, we may assume that $Q \cap V(R_1) \neq \emptyset$, and from (3), $A_1 \subseteq Q \cup W$ and $|W \cap (C_1 \cup C'_1)| \geq d/2 - 1$. If $Q \cap V(R_2) \neq \emptyset$, then similarly $|W \cap (C_2 \cup C'_2)| \geq d/2 - 1$, and so $W \subseteq C_1 \cup C'_1 \cup C_2 \cup C'_2$; but some vertex in A_1 has a neighbour in $V(R_i)$ where $i \neq 1, 2$, and this provides an edge between Q, P, a contradiction. So $V(R_i) \subseteq P \cup W$ for $2 \leq i \leq m$. There is a matching of cardinality $d/2$ between A_1 and $V(G_D) \setminus V(R_1)$, and one of these edges has no end in W, and therefore joins Q, P, a contradiction. This completes the proof that G_D is $(d-1)$-connected.

We observe that if D, D' are nonisomorphic graphs, both d-regular and d-connected, then G_D and $G_{D'}$ are nonisomorphic; because with notation as before, every subgraph of G_D isomorphic to R is one of the graphs $R_1, \ldots, R_m$, and so D can be obtained from G_D by contracting all edges of every R-subgraph of G_D.

Now let $c > 0$, and let $b = c^{4d-4}$. By 3.1, for infinitely many $m > 0$ there are more than b^m graphs D on m vertices that are d-regular and d-connected; and so all the corresponding graphs G_D are distinct. Hence, since each such G_D has $(4d-4)m = n$ vertices, for infinitely many $n > 0$ there are more than $b^m = c^n$ d-regular $(d-1)$-connected graphs with n vertices, not containing a subdivision of K_{d+1}. This completes the proof when d is even, and so proves 3.4. ∎

This result 3.4 is about subdivisions of K_{d+1}, but we believe we have a proof that the analogous statement is true for subdivisions of H, where H is any d-regular graph. We omit the proof, which is similar but more complicated.

4 Tangles

A *separation* of a graph G is a pair (A, B) of subgraphs with union G and with $E(A \cap B) = \emptyset$; and its *order* is $|V(A \cap B)|$. Let $\theta \geq 1$ be an integer. A *tangle* in a graph G of *order* θ is a set $\mathcal{T}$ of separations of G, all of order less than θ, such that:

- for every separation (A, B) of order $< \theta$, one of $(A, B), (B, A)$ belongs to $\mathcal{T}$
- if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$, and

- if $(A,B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

These are called the "tangle axioms". Tangles were central to the "Graph Minors" series of papers by Robertson and the third author, and much of the theory behind them was developed in [8].

If $(A_1,B_1),\dots,(A_k,B_k)$ are separations of G, then so is

$$(A_1 \cup \cdots \cup A_k, B_1 \cap \cdots \cap B_k),$$

and we call this the *union* of $(A_1,B_1),\dots,(A_k,B_k)$.

The following is a consequence of theorem 2.9 of [8].

4.1 *Let $\mathcal{T}$ be a tangle of order θ in a graph G, and let $(A_1,B_1) \in \mathcal{T}$. Let (A_2,B_2) be a separation of G of order less than θ. If either $V(B_1) \subseteq V(B_2)$, or $V(A_2) \subseteq V(A_1)$, then $(A_2,B_2) \in \mathcal{T}$.*

The next result is theorem (8.5) of [8].

4.2 *Let $\mathcal{T}$ be a tangle in G of order θ, and let $W \subseteq V(G)$ with $|W| < \theta$. Let $\mathcal{T}'$ be the set of all separations (A',B') of $G \setminus W$ of order less than $\theta - |W|$, such that there exists $(A,B) \in \mathcal{T}$ with $W \subseteq V(A \cap B)$ and $A' = A \setminus W$ and $B' = B \setminus W$. Then $\mathcal{T}'$ is a tangle in $G \setminus W$ of order $\theta - |W|$.*

We denote the tangle $\mathcal{T}'$ in 4.2 by $\mathcal{T} \setminus W$.

4.3 *Let $\mathcal{T}$ be a tangle in G of order θ, and let $(A_1,B_1),(A_2,B_2) \in \mathcal{T}$, with order d_1,d_2 respectively. Let $(A,B) = (A_1 \cup A_2, B_1 \cap B_2)$ and $(A',B') = (A_1 \cap A_2, B_1 \cup B_2)$, with orders d,d' respectively. Then*

- *$d + d' = d_1 + d_2$;*
- *if $d < \theta$ then $(A,B) \in \mathcal{T}$;*
- *if $d' < \theta$ then $(A',B') \in \mathcal{T}$.*

Proof. The first statement is clear. The second statement follows from 4.1, and the third from theorem 2.2 of [8]. This proves 4.3. ▮

If $\mathcal{T}$ is a tangle of order θ in G, we say the *rank* of $X \subseteq V(G)$ relative to $\mathcal{T}$ is the minimum order of a separation $(A,B) \in \mathcal{T}$ with $X \subseteq V(A)$, if there is such a separation, and θ otherwise. A set is *free* (relative to the tangle) if its rank equals its cardinality. The rank of a subgraph is the rank of its vertex set. If $v \in V(G)$, a *v-nexus* in G is a set $\mathcal{P}$ of paths of G, all with one end v. If $\mathcal{P}$ is a v-nexus, we write $V(\mathcal{P})$ for $\cup_{P \in \mathcal{P}} P$.

The next result is the main theorem of this section.

4.4 *Let $d',k \geq 0$ be integers. Let G be a graph with maximum degree at most d', with a tangle $\mathcal{T}$ of order at least $(k+1)^2$. Let $v \in V(G)$, and let $\mathcal{P}$ be a v-nexus, such that each member of $\mathcal{P}$ has rank at most k. Then $V(\mathcal{P})$ has rank at most $(kd')^k$.*

Proof. If $d' = 0$ then G has no edges, so $k = 0$ and $\mathcal{P}$ has at most one path, and its rank is zero. If $d' = 1$ then at most one member of $\mathcal{P}$ has maximal vertex set, and since $k \leq (kd')^k$ the claim holds. Thus we may assume that $d' \geq 2$.

Define $r_0 = 0$, and $r_k = (kd')^k - 1$ if $k > 0$. For inductive purposes we prove a slightly stronger statement, that $V(\mathcal{P})$ has rank at most r_k. We proceed by induction on k, and so we may assume the result holds for all $k' < k$ with $k' \geq 0$. Let $G, \mathcal{T}, v$ and $\mathcal{P}$ be as in the theorem. For each $P \in \mathcal{P}$, since the rank of P is at most $k < (k+1)^2$, there exists $(A_P, B_P) \in \mathcal{T}$ with $V(P) \subseteq V(A_P)$. Choose a separation $(A, B) \in \mathcal{T}$ of order at most k, such that the number of $P \in \mathcal{P}$ with $V(P) \subseteq V(A)$ is maximum. We may assume that this number is at least one, and so $v \in V(A)$.

Suppose that $k = 0$, and that there exists $P \in \mathcal{P}$ with $V(P) \not\subseteq V(A)$. Since $(A_P, B_P), (A, B) \in \mathcal{T}$, 4.3 implies that $(A_P \cup A, B_P \cap B) \in \mathcal{T}$ and has order zero, contrary to the choice of (A, B). Thus if $k = 0$ than $V(P) \subseteq V(A)$ for each $P \in \mathcal{P}$, and so $V(\mathcal{P})$ has rank zero. Hence we may assume that $k > 0$.

(1) *If $P \in \mathcal{P}$ and $V(P) \not\subseteq V(A)$ then $V(A_P \cap B_P \cap A) \neq \emptyset$.*

Suppose that $V(A_P \cap B_P \cap A) = \emptyset$. From the choice of (A, B), since $V(P) \subseteq V(A_P)$ and $V(P) \not\subseteq V(A)$, it follows that there exists $P' \in \mathcal{P}$ with $V(P') \subseteq V(A)$ and $V(P') \not\subseteq V(A_P)$. Since $V(P') \subseteq V(A)$, and $V(A_P \cap B_P \cap A) = \emptyset$, it follows that $V(A_P \cap B_P \cap P') = \emptyset$. But $v \in V(P) \subseteq V(A_P)$, and since $v \in V(P')$ and P' is connected, it follows that $P' \subseteq A_P$, a contradiction. This proves (1).

Let $X = V(A \cap B)$. Let W be the set of vertices in $V(G) \setminus X$ with a neighbour in X. Thus $|W| \leq kd'$. Let $\mathcal{Q}$ be the set of all paths Q of $B \setminus X$ such that Q is a component of $P \setminus X$ for some $P \in \mathcal{P}$. Thus each member Q of $\mathcal{Q}$ is a path of $G \setminus X$ and has an end in W. We claim that

(2) *Each $Q \in \mathcal{Q}$ has rank at most $k - 1$ relative to $\mathcal{T} \setminus X$.*

First, we observe that $(A \setminus X, B \setminus X) \in \mathcal{T} \setminus X$, of order zero. Choose $P \in \mathcal{P}$ such that Q is a component of $P \setminus X$. Consequently $V(P) \not\subseteq V(A)$. Since (A_P, B_P) is a separation of order at most k, it follows that there is a separation (A_P^+, B_P^+) with $X \subseteq V(A_P^+ \cap B_P^+)$ and $A_P^+ \setminus X = A_P \setminus X$ and $B_P^+ \setminus X = B_P \setminus X$, of order at most $k + |X|$; and since $(A_P, B_P) \in \mathcal{T}$, 4.1 implies that $(A_P^+, B_P^+) \in \mathcal{T}$, because $k+|X| \leq 2k < (k+1)^2$. Consequently $(A_P \setminus X, B_P \setminus X) \in \mathcal{T} \setminus X$. Define $A' = (A_P \setminus X) \cup (A \setminus X)$ and $B' = (B_P \setminus X) \cap (B \setminus X)$. Then from 4.3, $(A', B') \in \mathcal{T} \setminus X$. But

$$A' \cap B' = ((A_P \cup A) \cap (B_P \cap B)) \setminus X \subseteq A_P \cap B_P \cap (B \setminus X),$$

and $|V(A_P \cap B_P \cap (B \setminus X))| < |V(A_P \cap B_P)| \leq k$, by (1). Thus (A', B') has order at most $k - 1$. Since $Q \subseteq A_P$ and $V(Q) \cap X = \emptyset$ it follows that $Q \subseteq A'$; and so Q has rank at most $k - 1$ relative to $\mathcal{T} \setminus X$. This proves (2).

Now $\mathcal{Q}$ can be partitioned into at most kd' subsets, each a w-nexus for some $w \in W$. Since $\mathcal{T} \setminus X$ has order at least $(k+1)^2 - |X| \geq k^2$, the inductive hypothesis implies that each such w-nexus has rank at most r_{k-1} relative to $\mathcal{T} \setminus X$; and so $V(\mathcal{Q})$ has rank at most $kd' r_{k-1}$ relative to $\mathcal{T} \setminus X$, and hence $V(\mathcal{Q}) \cup A$ has rank at most $kd' r_{k-1} + k$ relative to $\mathcal{T}$. Consequently $V(\mathcal{P})$ has rank at most $kd' r_{k-1} + k$ relative to $\mathcal{T}$. Since $kd' r_{k-1} + k \leq r_k$ (because $k > 0$ and $d' \geq 2$), this proves 4.4. ∎

5 From minors to subdivisions

In this section we use 4.4 to prove 1.3 and 1.5. We need the following, a case of theorem 7.2 of [10]:

5.1 *Let $\mathcal{T}$ be a tangle in a graph G, and let $W \subseteq V(G)$ be free relative to $\mathcal{T}$, with $|W| \le w$. Let $h \ge 1$ be an integer, and let $\mathcal{T}$ have order at least $(w+h)^{h+1}+h$. Then there exists $W' \subseteq V(G)$ with $W \subseteq W'$ and $|W'| \le (w+h)^{h+1}$ such that for every $(C,D) \in \mathcal{T}$ of order $< |W|+h$ with $W \subseteq V(C)$, there exists $(A',B') \in \mathcal{T}$ with $W' \subseteq V(A' \cap B')$, such that $|V(A' \cap B') \setminus W'| < h$ and $C \subseteq A'$.*

We also need:

5.2 *If $\mathcal{T}$ is a tangle in a graph G, and $W \subseteq V(G)$ is free relative to $\mathcal{T}$, there exists $(A_1,B_1) \in \mathcal{T}$ of order $|W|$, with $W \subseteq V(A_1)$, such that $A \subseteq A_1$ and $B_1 \subseteq B$ for every $(A,B) \in \mathcal{T}$ of order $|W|$ with $W \subseteq V(A)$.*

Proof. Let $\mathcal{S}$ be the set of all members of $\mathcal{T}$ of order $|W|$ with $W \subseteq V(A)$. Now $\mathcal{S} \neq \emptyset$, because $(A,B) \in \mathcal{S}$ where $V(A) = W, E(A) = \emptyset$, and $B = G$. If $(A_1,B_1),(A_2,B_2) \in \mathcal{S}$, then their intersection has order at least $|W|$ (because otherwise it would belong to $\mathcal{T}$, by 4.3, contradicting that W is free); and so by 4.3, their union, (A,B) say, has order at most $|W|$. Hence $(A,B) \in \mathcal{T}$ by 4.3, and so it has order exactly $|W|$, since W is free; and so $(A,B) \in \mathcal{S}$. This proves that the union of every two members of $\mathcal{S}$ is also a member of $\mathcal{S}$; and so the union of all members of $\mathcal{S}$ is a member of $\mathcal{S}$. This proves 5.2. ∎

We deduce:

5.3 *Let $d,d' \ge 0$ be integers, and let G be a graph with maximum degree at most d'. Suppose that either G is d-connected, or $d \in \{4,5\}$ and G is $(d-1)$-connected and d-edge-connected. Let $s \ge 0$ be an integer, and let $\mathcal{T}$ be a tangle in G of order*

$$\theta \ge (s-1)(d+1) + (dd')^d(d'(s-1+(sd)^{d+1}) + d(s-1)) + (sd)^{d+1}.$$

Then there exist distinct $z_1,\dots,z_s \in V(G)$, and pairwise disjoint subsets $W_1,\dots,W_s$ of $V(G) \setminus \{z_1,\dots,z_s\}$, such that

- *for $1 \le i \le s$, $|W_i| = d$, and z_i is adjacent to each vertex in W_i; and*
- *$W_1 \cup \dots \cup W_s$ is free relative to $\mathcal{T} \setminus \{z_1,\dots,z_s\}$ in $G \setminus \{z_1,\dots,z_s\}$.*

Proof. We proceed by induction on s, and so we may assume that there exists $Z = \{z_1,\dots,z_{s-1}\} \subseteq V(G)$, and pairwise disjoint subsets $W_1,\dots,W_{s-1}$ of $V(G) \setminus Z$, such that

- for $1 \le i \le s-1$, $|W_i| = d$, and z_i is adjacent to each vertex in W_i;
- $W_1 \cup \dots \cup W_{s-1}$ is free relative to $\mathcal{T} \setminus Z$ in $G \setminus Z$.

Let $G' = G \setminus Z$ and $\mathcal{T}' = \mathcal{T} \setminus Z$; so $\mathcal{T}'$ is a tangle in G' of order $\theta - s + 1$. Let $W = W_1 \cup \dots \cup W_{s-1}$; then $|W| = (s-1)d$ and W is free relative to $\mathcal{T}'$. From 5.2, there exists $(A_1,B_1) \in \mathcal{T}'$, of order $|W|$ and with $W \subseteq V(A_1)$, such that $A \subseteq A_1$ and $B_1 \subseteq B$ for every $(A,B) \in \mathcal{T}'$ of order $|W|$ with $W \subseteq V(A)$.

(1) *W is free relative to $\mathcal{T}' \setminus \{v\}$, for each $v \in V(B_1) \setminus V(A_1)$.*

Suppose not; then there is a separation (A, B) of $\mathcal{T}' \setminus \{v\}$ of order $< |W|$, with $W \subseteq V(A)$. Hence there is a separation (A', B') of $\mathcal{T}'$ of order $\leq |W|$ with $W \subseteq V(A')$ and $v \in V(A' \cap B')$. But (A', B') has order exactly $|W|$, since W is free relative to $\mathcal{T}'$; and so $A' \subseteq A_1$, from the property of (A_1, B_1). This contradicts that $v \in V(B_1) \setminus V(A_1)$, and so proves (1).

Let $w' = (sd)^{d+1}$. By 5.1 applied to $G', \mathcal{T}'$ (taking $w = (s-1)d$ and $h = d$) there exists $W' \subseteq V(G')$ with $W \subseteq W'$ and $|W'| \leq w'$, such that for every $(C, D) \in \mathcal{T}'$ of order $< sd$ with $W \subseteq V(C)$, there exists $(A', B') \in \mathcal{T}'$ with $W' \subseteq V(A' \cap B')$, such that $|V(A' \cap B') \setminus W'| < d$ and $C \subseteq A'$.

Let $G'' = G' \setminus W'$, and $\mathcal{T}'' = \mathcal{T}' \setminus W'$. Hence $\mathcal{T}''$ is a tangle in G'' of order at least $\theta - s + 1 - w'$. Let N be the set of vertices of G that are not in $W' \cup Z$ but have a neighbour in $W' \cup Z$. Hence $N \subseteq V(G'')$. Since every vertex of G has degree at most d', it follows that $|N| \leq d'|W' \cup Z| \leq d'(s - 1 + w')$. Let $N' = N \cup V(A_1 \cap B_1 \cap G'')$. Thus $|N'| \leq d'(s - 1 + w') + d(s - 1)$.

By 4.4, for each $n \in N'$ there is a separation $(A_n, B_n) \in \mathcal{T}''$ of order at most $(dd')^d$, such that $P \subseteq A_n$ for each path P of G'' of rank at most d (relative to $\mathcal{T}''$) with one end n. Let (A_0, B_0) be the union of these separations. Thus (A_0, B_0) has order at most $(dd')^d(d'(s - 1 + w') + d(s - 1))$, and so belongs to $\mathcal{T}''$ by 4.3.

(2) *There exists $v \in V(G'')$ such that $v \notin V(A_0 \cup A_1)$.*

Since $(A_0, B_0) \in \mathcal{T}''$, there is a separation (A_2, B_2) of $\mathcal{T}'$ with $W' \subseteq V(A_2 \cap B_2)$, such that $A_2 \setminus W' = A_0$ and $B_2 \setminus W' = B_0$. Its order is at most $(dd')^d(d'(s - 1 + w') + d(s - 1)) + w'$; and so the union (A, B) of $(A_1, B_1), (A_2, B_2)$ has order at most

$$(dd')^d(d'(s - 1 + w') + d(s - 1)) + w' + (s - 1)d$$

and so belongs to $\mathcal{T}'$. From the third tangle axiom, applied to $\mathcal{T}'$ and (A, B), there is a vertex $v \in V(B) \setminus V(A)$. In particular, $v \notin W'$ since $W' \subseteq V(A_2)$, and so $v \in V(G'')$. This proves (2).

(3) *There exists $u \in V(G'') \setminus V(A_1)$ such that there is no $(A, B) \in \mathcal{T}''$ of order $< d$ with $u \in V(A) \setminus V(B)$.*

Choose v as in (2); we may therefore assume that there is a separation $(A, B) \in \mathcal{T}''$ of order $< d$ with $v \in V(A) \setminus V(B)$. Choose (A, B) with B minimal. Let C be the component of A that contains v, and suppose first that $V(C) \cap N' \neq \emptyset$. Hence there is a path of A between v and N', say P. Thus P has rank at most $d - 1$ relative to $\mathcal{T}''$, since $P \subseteq A$; and so $P \subseteq A_0$, contradicting that $v \notin V(A_0)$. This proves that $V(C) \cap N' = \emptyset$, and in particular, G is not d-connected.

Consequently $d \in \{4, 5\}$, and G is $(d - 1)$-connected and d-edge-connected. Thus (A, B) has order $d - 1$. Since G is d-edge-connected, there are at least d edges of G between $V(C) \setminus V(B)$ and its complement in $V(G)$. Since all of these edges belong to G'' (since $V(C) \cap N = \emptyset$), they are all between $V(C) \setminus V(B)$ and $V(A \cap B)$; and so some vertex $u \in V(A \cap B \cap C)$ has at least two neighbours $u_1, u_2 \in V(C) \setminus V(B)$. In particular, $u \in V(C)$; let P be a path of C between u, v. If

$u \in V(A_1)$, then since $v \notin V(A_1)$, some vertex of P belongs to $V(A_1 \cap B_1)$, and since P has rank at most $d-1$ relative to $\mathcal{T}''$, it follows that $P \subseteq A_0$ from the definition of (A_0, B_0), contradicting that $v \notin V(A_0)$. Thus $u \notin V(A_1)$.

Suppose that there is a separation (A', B') of G'' of order $< d$ with $u \in V(A') \setminus V(B')$. We may assume that A' is connected (for instance, by choosing (A', B') with A' minimal). If $N' \cap V(A') \neq \emptyset$, then there is a path of A' between N' and u, and since there is a path of C between u, v included in A, it follows that there is a path P between N', v included in $A \cup A'$. But P has rank at most the order of $(A \cup A', B \cap B')$, and hence at most $2d-5 \leq d$, and so $P \subseteq A_0$ from the definition of (A_0, B_0), contradicting that $v \notin V(A_0)$. This proves that $N' \cap V(A') = \emptyset$. If $(A \cup A', B \cap B')$ has order at most $d-1$, then it belongs to $\mathcal{T}''$ by 4.3, contradicting the minimality of B since $A \cup A' \neq A$. Thus $(A \cup A', B \cap B')$ has order at least d, and since $(A, B), (A', B')$ have order at most $d-1$, it follows that $(A \cap A', B \cup B')$ has order at most $d-2$. Since G is $(d-1)$-connected and $N' \cap V(A') = \emptyset$, it follows that $V(A \cap A') \subseteq V(B \cup B')$, and in particular $u_1, u_2 \in V(B')$. Thus $|V(A' \cap B') \setminus V(B)| \geq 2$. Since $|V(A' \cap B')| \leq d-1$, it follows that $|V(A' \cap B') \cap V(B)| \leq d-3$. Since $(A \cap A', B \cap B')$ has order greater than the order of (A, B), it follows that

$$2 \geq d-3 \geq |V(A' \cap B') \setminus V(A)| > |V(A \cap B) \setminus V(B')| \geq 1,$$

and so equality holds throughout; and in particular, $d = 5$, $V(A \cap B) \setminus V(B') = \{u\}$, and $|V(A' \cap B') \setminus V(A)| = 2$, and so $V(A \cap B \cap A' \cap B') = \emptyset$. Thus the separation $(A' \cap B, A \cup B')$ has order at most three, and since G is 4-connected and $N' \cap V(A') = \emptyset$, it follows that $V(A' \cap B) \subseteq V(B' \cup A)$, and in particular, all neighbours of u belong to $V(A' \cap B')$; which is impossible since G is d-edge-connected. Thus there is no such separation (A', B'). This proves (3).

Define $z_s = u$, where u is chosen as in (3).

(4) *There is no $(C, D) \in \mathcal{T}'$ of order $< sd$ such that $W \subseteq V(C)$ and $z_s \in V(C) \setminus V(D)$.*

For suppose that there is such a separation (C, D). From the choice of W', there exists $(A', B') \in \mathcal{T}'$ with $W' \subseteq V(A' \cap B')$ such that $|V(A' \cap B') \setminus W'| < d$ and $C \subseteq A'$. Hence $z_s \in V(A')$. Choose (A', B') with B' minimal, and suppose that $z_s \in V(B')$. From the minimality of B', it follows that $(A', B' \setminus \{z_s\}) \notin \mathcal{T}'$, and so by 4.1, z_s is adjacent in G' to some vertex $b \in V(B') \setminus V(A')$. But $z_s \in V(C) \setminus V(D)$, and so $b \in V(C) \subseteq V(A')$, a contradiction. Thus $z_s \notin V(B')$, and so $z_s \in V(A') \setminus V(B')$. But $(A' \setminus W', B' \setminus W') \in \mathcal{T}''$, contrary to (3). This proves (4).

Let P be the set of all neighbours of z_s in $V(G) \setminus (Z \cup W)$.

(5) *There is no $(A, B) \in \mathcal{T}' \setminus \{z_s\}$ of order $< sd$ such that $W \cup P \subseteq V(A)$.*

For suppose there exists such a separation (A, B) of $G' \setminus \{z_s\}$. Hence there is a separation $(C, D) \in \mathcal{T}'$ with $z_s \in V(C \cap D)$, such that $C \setminus \{z_s\} = A$ and $D \setminus \{z_s\} = B$. Choose (C, D) with C maximal. By 4.1, every edge of G' incident with z_s and with its other end in $V(C)$ belongs to C. But every neighbour of z_s in G' belongs to $P \cup W$ and hence to $V(C)$; and so no edge of D is incident with z_s. Consequently $(C, D \setminus \{z_s\})$ is also a separation of G', and by 4.1, it belongs to $\mathcal{T}'$. Its order is that of (A, B) and hence less than sd, contrary to (4). This proves (5).

By theorem 12.2 of [8], the subsets of $V(G' \setminus \{z_s\})$ that are free relative to $\mathcal{T}' \setminus \{z_s\}$ are the independent sets of a matroid, M say, of rank $\theta - s$ (since $\mathcal{T}' \setminus \{z_s\}$ has order $\theta - s$); and by the same theorem, the set $P \cup W$ has rank at least sd in M because of (5). Since W is free relative to $\mathcal{T}' \setminus \{z_s\}$ by (1), and hence independent in M, it can be extended to an independent subset of $W \cup P$ of cardinality sd; and so there exists $W_s \subseteq P$ of cardinality d such that $W \cup W_s$ is free relative to $\mathcal{T}' \setminus \{z_s\}$. This completes the inductive argument, and so proves 5.3. ∎

We will need theorem 5.4 of [9], which states:

5.4 *Let G be a graph and let $Z \subseteq V(G)$. Let $k \geq \lfloor \frac{3}{2}|Z| \rfloor$ and let $C_1, \ldots, C_k$ be connected subgraphs of G, mutually vertex-disjoint, such that for $1 \leq i < j \leq k$ there is an edge between C_i and C_j. Suppose that there is no separation (A, B) of G of order $< |Z|$ with $Z \subseteq V(A)$ and $A \cap C_i$ null for some $i \in \{1, \ldots, k\}$. Then for every partition $(Z_i : 1 \leq i \leq n)$ of Z into nonempty subsets, there are n connected subgraphs $T_1, \ldots, T_n$ of G, mutually vertex-disjoint and with $V(T_i) \cap Z = Z_i$ for $1 \leq i \leq n$.*

Finally, we deduce 1.3 and 1.5, which we combine in:

5.5 *Let $d, d' \geq 0$, and let H be a graph with maximum degree at most d. Then there exists $t \geq 0$ such that for every graph G, if*

- *G has maximum degree at most d',*
- *G contains K_t as a minor, and*
- *either G is d-connected, or $d \in \{4, 5\}$ and G is $(d-1)$-connected and d-edge-connected,*

then G contains a subdivision of H.

Proof. Let $s = |V(H)|$, and we assume $V(H) = \{h_1, \ldots, h_s\}$. Let

$$\theta \geq (s-1)(d+1) + (dd')^d(d'(s-1+(sd)^{d+1}) + d(s-1)) + (sd)^{d+1},$$

let $k = \lceil 3\theta/2 \rceil$, and let $t = k + s$. Now let G be as in the theorem. Choose t disjoint connected subgraphs $C_1, \ldots, C_t$ such that for $1 \leq i < j \leq t$ there is an edge of G between $V(C_i)$ and $V(C_j)$.

For each separation (A, B) of G of order $< \theta$, exactly one of $A \setminus V(B), B \setminus V(A)$ includes one of $C_1, \ldots, C_t$; let $\mathcal{T}$ be the set of all such (A, B) where $B \setminus V(A)$ includes one of $C_1, \ldots, C_t$. Then $\mathcal{T}$ is a tangle in G of order θ, for instance by the argument of theorem 4.4 of [8]. By 5.3 applied to $\mathcal{T}$, there exist $z_1, \ldots, z_s \in V(G)$, and pairwise disjoint subsets $W_1, \ldots, W_s$ of $V(G) \setminus \{z_1, \ldots, z_s\}$, such that

- for $1 \leq i \leq s$, $|W_i| = d$, and z_i is adjacent to each vertex in W_i; and
- $W_1 \cup \cdots \cup W_s$ is free relative to $\mathcal{T} \setminus \{z_1, \ldots, z_s\}$ in $G \setminus \{z_1, \ldots, z_s\}$.

For $1 \leq i \leq s$ and each edge e of H incident with h_i, choose $y_{i,e} \in W_i$, in such a way that all the vertices $y_{i,e}$ are distinct (this is possible since $|W_i| = d$ and h_i has degree at most d in H).

Let $Z = W_1 \cup \cdots \cup W_s$, and for each edge $e = h_i h_j$ of H, let $Z_e = \{y_{i,e}, y_{j,e}\}$. Then $(Z_e : e \in E(H))$ is a partition of Z. We may assume that none of $C_1, \ldots, C_k$ contain any of $z_1, \ldots, z_s$, since $C_1, \ldots, C_t$

are pairwise disjoint, and so $C_1, \dots, C_s$ are all subgraphs of G', where $G' = G \setminus \{z_1, \dots, z_s\}$. Suppose that there is a separation (A', B') of G' of order $< |Z|$ with $Z \subseteq V(A')$ and $A' \cap C_i$ null for some $i \in \{1, \dots, k\}$. It follows that there is a separation (A, B) of G of order less than $|Z| + s$, with $\{v_1, \dots, v_s\} \subseteq V(A \cap B)$, such that $A \setminus \{z_1, \dots, z_s\} = A'$ and $B \setminus \{z_1, \dots, z_s\} = B'$. Consequently $A \cap C_i$ is null, since $z_1, \dots, z_s \notin V(C_i)$; and so $(A, B) \in \mathcal{T}$, since $|Z| + s = s(d+1) < \theta$. Hence $(A', B') \in \mathcal{T} \setminus \{z_1, \dots, z_s\}$, contradicting that Z is free relative to $\mathcal{T} \setminus \{z_1, \dots, z_s\}$ in $G \setminus \{z_1, \dots, z_s\}$. Thus there is no such (A', B').

From 5.4 applied to $G', \mathcal{T} \setminus \{z_1, \dots, z_s\}$, Z and the partition $(Z_e : e \in E(H))$, it follows that for each edge e of H there is a connected subgraph P_e of G' containing the two vertices of Z_e, such that the subgraphs P_e $(e \in E(H))$ are pairwise vertex-disjoint. By choosing each P_e minimal we may assume each P_e is a path joining the two members of Z_e. But then adding the vertices $z_1, \dots, z_s$ and the edges between each z_i and the corresponding W_i, to the union of these paths, gives a subdivision of H. This proves 5.5. ∎

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