Let G be a graph and let w be a weight function on E. For $E' \subset E$ let $w(E') = \sum_{e \in E'} w(e)$. We define the generating functions $\mathcal{C}(G,x), \mathcal{E}(G,x), \mathcal{P}(G,x)$ of the edge-cuts, the even sets and the perfect matchings as follows.

$$\mathcal{C}(G,x) = \sum_{E' \text{ edge-cut}} x^{w(E')},$$

$$\mathcal{E}(G,x) = \sum_{E' \text{ even set}} x^{w(E')},$$

$$\mathcal{P}(G,x) = \sum_{E' \text{ perfect matching}} x^{w(E')}.$$

The transformation of Theorem 6.2.2 shows that $\mathcal{E}(G,x) = \mathcal{P}(G',x)$.

6.3 Van der Waerden's theorem

The Ising partition function for a graph G may be expressed in terms of the generating function of the even sets of the same graph G. This is a theorem of Van der Waerden. We will use the following standard notation: $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

Theorem 6.3.1.

$$Z(G,\beta) = 2^{|V|} \left(\prod_{uv \in E} \cosh(\beta w(uv)) \right) \left(\sum_{E' \subset E \text{ even}} \prod_{uv \in E'} \tanh(\beta w(uv)) \right).$$

Proof. We have

$$Z(G,\beta) = \sum_\sigma e^{\beta \sum_{uv} w(uv)\sigma_u \sigma_v} =$$

$$\sum_\sigma \prod_{uv \in E} (\cosh(\beta w(uv)) + \sigma_u \sigma_v \sinh(\beta w(uv))) =$$

$$\prod_{uv \in E} \cosh(\beta w(uv)) \sum_\sigma \prod_{uv \in E} (1 + \sigma_u \sigma_v \tanh(\beta w(uv))) =$$

$$\prod_{uv \in E} \cosh(\beta w(uv)) \sum_\sigma \sum_{E' \subset E} \prod_{uv \in E'} \sigma_u \sigma_v \tanh(\beta w(uv)) =$$

$$\prod_{uv \in E} \cosh(\beta w(uv)) \sum_{E' \subset E} (U(E') \prod_{uv \in E'} \tanh(\beta w(uv))),$$

where

$$U(E') = \sum_\sigma \prod_{uv \in E'} \sigma_u \sigma_v.$$

The proof is complete after noticing that $U(E') = 2^{|V|}$ if E' is even and $U(E') = 0$ otherwise. □

Chapter 9

2D Ising and dimer models

Since the solution of the 2-dimensional (planar) Ising problem was achieved by Onsager, the physicists have been trying to reproduce his solution by more understandable methods. In the fifties and in the beginning of sixties two discrete methods appeared: the *Pfaffian method* of Kasteleyn and independently Fisher, Temperley, and the *path method* of Kac, Ward, Potts, Feynman and Sherman. Both methods start by reducing the Ising partition function $Z(G,\beta)$ to the generating function $\mathcal{E}(G,x)$ of even subsets of edges. This is accomplished by van der Waerden's theorem (Theorem 6.3.1). The Pfaffian method seems to be better known to discrete mathematicians. It further reduces $\mathcal{E}(G,x)$ to the generating function $\mathcal{P}(G',x)$ of the perfect matchings (dimer arrangements) of a graph G' obtained from G by a local operation at each vertex (see Section 6.2). It is important that these operations are locally planar, i.e., G' may be embedded on the same surface as G.

9.1 Pfaffians, dimers, permanents

Let $G = (V,E)$ be a graph and let M, N be two perfect matchings of G. We recall that $M \subset E$ is a matching if $e \cap e' = \emptyset$ for each pair e, e' of edges of M, and a matching is perfect if its elements contain all the vertices of graph G. A cycle is alternating with respect to a perfect matching M if it contains alternately edges of M and out of M; each alternating cycle thus has an even length. We further recall that Δ denotes the symmetric difference, $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. If m and N are two perfect matchings then $M \Delta N$ consists of vertex disjoint alternating cycles.
Let C be a cycle of G of an even length and let D be an orientation of G. C is said to be *clockwise even* in D if it has an even number of edges directed in D in agreement with a chosen direction of traversal. Otherwise C is called *clockwise odd.*

Definition 9.1.1. Let G be a graph with a weight function w on the edges. Let D be an orientation of G. Let M be a perfect matching of G. For each perfect matching P of G let $\text{sign}(D, M\Delta P) = (-1)^z$ where z is the number of clockwise even alternating cycles of $M\Delta P$. Moreover let

$$\mathcal{P}(D, M) = \sum_{P \text{ perfect matching}} \text{sgn}(D, M\Delta P)x^{w(P)}.$$

Let $G = (V, E)$ be a graph with $2n$ vertices and D an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by V, where $a_{uv} = x^{w(u,v)}$ in case (u, v) is an arc of D, $a_{u,v} = -x^{w(u,v)}$ in case (v, u) is an arc of D, and $a_{u,v} = 0$ otherwise.

Definition 9.1.2. The *Pfaffian* of $A(D)$ is defined as

$$\text{Pf}(A(D)) = \sum_P s^*(P)a_{i_1j_1} \cdots a_{i_nj_n},$$

where $P = \{\{i_1j_1\}, \cdots, \{i_nj_n\}\}$ is a partition of the set $\{1, \cdots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \cdots, n$, and $s^*(P)$ equals the sign of the permutation $i_1j_1 \cdots i_nj_n$ of $12\cdots(2n)$. Hence, each nonzero term of the expansion of the Pfaffian equals $x^{w(P)}$ or $-x^{w(P)}$ where P is a perfect matching of G. If $s(D, P)$ denotes the sign of the term $x^{w(P)}$ in the expansion, we may write

$$\text{Pf}(A(D)) = \sum_P s(D, P)x^{w(P)}.$$

The following theorem was proved by Kasteleyn.

Theorem 9.1.3. *Let G be a graph and D an orientation of G. Let P, M be two perfect matchings of G. Then*

$$s(D, P) = s(D, M)\text{sign}(D, M\Delta P).$$

Corollary 9.1.4.

$$\text{Pf}(A(D)) = s(D, M)\mathcal{P}(D, M).$$

The relevance of the Pfaffians in our context lies in the fact that the Pfaffian is a determinant-type function. The determinants are invariant under elementary row/column operations and these can be used in the Gaussian elimination to calculate a determinant. The Pfaffian may be computed efficiently by a variant of Gaussian elimination. Let A be an antisymmetric $2n \times 2n$ matrix. A *cross* of the matrix A is the union of a row and a column of the same index: the k-th cross is the following set of elements:

$$A_k = \{a_{ik}; 1 \le i \le 2n\} \cup \{a_{kj}; 1 \le j \le 2n\}.$$

Multiplying a cross A_k by a scalar α means multiplying each element of A_k by α.
Swapping crosses A_k and A_l means exchanging both the respective rows and columns. Another way of regarding the swap operation is that it . . .

the values of k and l in both of the index positions. The resulting matri[x is] antisymmetric again;
Adding cross A_k to cross A_l means adding first the k-th row to the l-th on[e,] then adding the respective columns. The matrix remains antisymmetric.

These operations may be used to transform matrix A by at most $O(n^2)$ operations into a form where the Pfaffian can be determined trivially. Mo[reover,] for graphs with some restrictive properties, e.g. for graphs with bounded [...] there are more efficient ways to perform the elimination. Apart of the Ga[ussian] elimination, we also have the following classical theorem of Cayley.

Theorem 9.1.5.

$$(\text{Pf}(A(D)))^2 = det(A(D)).$$

Kasteleyn introduced the following seminal notion:

Definition 9.1.6. A graph G is called Pfaffian if it has a *Pfaffian orien[tation]*, i.e., an orientation such that all alternating cycles with respect to an ar[bitrary] fixed perfect matching M of G are clockwise odd.

If G has a Pfaffian orientation D, then by Theorem 9.1.3 the signs s [...] are all equal and $\mathcal{P}(G, x)$ is equal to $\text{Pf}(A(D))$ up to a sign. Kasteleyn [...] that each planar graph has a Pfaffian orientation.

Theorem 9.1.7. *Every topological planar graph has a Pfaffian orienta[tion in] which all inner faces are clockwise odd.*

Proof. Let G be a topological planar graph, and let M be a perfect ma[tching] in it. Without loss of generality we assume that G is 2-connected. Th[en by] Proposition 2.10.10, each face is bounded by a cycle. Starting with an ar[bitrary] inner face, we can gradually construct an orientation D such that in D [each] inner face is clockwise odd. Next we observe, e.g. by induction on the n[umber] of faces, that this orientation D satisfies: A cycle is clockwise odd if and [only if] it encircles an even number of vertices. However, each alternating cycle [...] perfect matching in the complement must encircle an even number of ve[rtices] and hence it is clockwise odd.

As a consequence we obtain the following theorem of Kasteleyn.

Theorem 9.1.8. *Each planar graph has an orientation D so that*

$$\mathcal{P}(G, x) = \text{Pf}(A(D)).$$

Kasteleyn stated that for a graph of genus g, $\mathcal{P}(G, x)$ is a linear co[mbina]tion of 4^g Pfaffians. This was proved by Galluccio, Loebl and indepen[dently] by Tesler. There were earlier partial results towards the proof by Reg[ge and] Zecchina. Tesler extended the result to the non-orientable surfaces. Ga[...]

Theorem 9.1.9. *If G is a graph of genus g then it has 4^g orientations $D_1, \cdots, D_{4^g}$ so that*

$$\mathcal{P}(G,x) = 2^{-g}\sum_{i=1}^{4^g} \mathrm{sign}(D_i)Pf(A(D_i),x),$$

for well-defined $\mathrm{sign}(D_i) \in \{1,-1\}$.

The proof can be found in [GL1]. Such a linear combination repair of a non-zero genus complication is a basic technique used both by mathematicians and physicists. The earliest work I have seen it in is by Kac and Ward; we will get to it in the next section. The next section also contains a theorem analogous to Theorem 9.1.9; there we will include the proof. Theorem 9.1.9 has attractive algorithmic consequences.

Corollary 9.1.10. *The Ising partition function $Z(G,\beta)$ can be determined efficiently for the topological graphs on an arbitrary surface of bounded genus. Also, the whole density function of the weighted edge-cuts, or weighted perfect matchings, may be computed efficiently for such graphs. Another well-known problem which is efficiently solvable for these graphs by the method of Theorem 9.1.9 is the exact matching problem: Given a positive integer k, a graph G and let the edges of G be colored by blue and red. It should be decided if there is a perfect matching with exactly k red edges.*
The efficiency is in the following sence: if we have integer weights, then the complexity is polynomial in the sum of the absolute values of the edge-weights.

We remark that a stronger notion of efficiency, where the complexity needs to be polynomial in the size of the graph plus the maximum of the logarithms of the edge-weights, is more customary. The existence of a polynomial algorithm in this sence is still open.
Curiously, there is no other polynomial method known to solve the max-cut problem alone even for the graphs on the torus. The method of Theorem 9.1.9 led to a useful implementation by Vondrák ([GLV1], [GLV2]).

Question 9.1.11. Is there an efficient combinatorial algorithm for the toroidal max-cut problem?

A lot of attention was given to the problem of characterizing graphs which admit a Pfaffian orientation. The problem of recognizing the Pfaffian bipartite graphs goes implicitly back to 1913, when Pólya asked for a characterization of convertible matrices (this is the 'Pólya scheme'). A matrix A is *convertible* if one can change some signs of its entries to obtain a matrix B such that $\mathrm{Per}(A) = \det(B)$. A polynomial-time algorithm to recognize the Pfaffian bipartite graphs (this problem is equivalent to the Pólya problem described above) has been obtained by McCuaig, Robertson, Seymour and Thomas. For the recognition of the Pfaffian graphs embeddable on an arbitrary 2-dimensional surface, there is a polynomial algorithm by Galluccio and Loebl (using Theorem 9.1.9). Theorem

Corollary 9.1.12. *For each matrix A there are matrices B_i, $i = 1, \cdots,$ obtained from A by changing signs of some entries, so that $\mathrm{Per}(A)$ is an al nating sum of the $\det(B_i)$'s. The parameter g is the genus of the bipartite gr determined by the non-zero entries of A.*

Several researchers (Hammersley, Heilmann, Lieb, Godsil, Gutman) ticed that $\mathrm{Per}(A)$, A a general complex matrix, is equal to the expectatior $(\det(B))^2$, where B is obtained from A by taking the square root of the mini argument of each non-zero entry and then multiplying each non-zero entry an element of $\{1,-1\}$ chosen independently uniformly at random. This lead a Monte-Carlo algorithm for estimating the permanent (see Karmarkar, K Lipton, Lovász and Luby [KKLLL] for the rate of convergence analysis).

Theorem 9.1.13. *Let A be a matrix and let B be the random matrix obta from A by taking the square root of minimal argument of each non-zero try and then multiplying each non-zero entry by an element of $\{1,-1\}$ cho independently uniformly at random. Then $\mathbb{E}((\det(B))^2) = \mathrm{Per}(A)$.*

Proof. Since $\det(B) = \sum_\pi \mathrm{sign}(\pi)\prod_i B_{i\pi(i)}$, we have

$$(\det(B))^2 = \sum_{(\pi_1,\pi_2)} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2))\prod_i B_{i\pi_1(i)}B_{i\pi_2(i)} =$$

$$\sum_\pi \mathrm{sign}(\pi)^2 \prod_i B^2_{i\pi(i)} +$$

$$\sum_{(\pi_1,\pi_2);\pi_1\neq\pi_2} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2))\prod_i B_{i\pi_1(i)}B_{i\pi_2(i)} =$$

$$\mathrm{Per}(A) + \sum_{(\pi_1,\pi_2);\pi_1\neq\pi_2} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2))\prod_i B_{i\pi_1(i)}B_{i\pi_2(i)}.$$

It remains to show that the expectation of the last sum is zero. Let A be $n \times n$ matrix and let $\pi_1 \neq \pi_2$ be two permutations of n. We can associate w them a graph $G(\pi_1,\pi_2)$. Its vertex-set is the set of all pairs (i,j) for $j = \pi$ or $j = \pi_2(i)$. Two vertices $(i,j),(i',j')$ are connected by an edge if and c if $i = i'$ or $j = j'$. We recall that $c(G)$ denotes the number of the connec components of G.
Clearly, each $G(\pi_1,\pi_2)$ has at least one edge, and the non-empty component each $G(\pi_1,\pi_2)$ are cycles of an even length. Let $\mathcal{G}$ be the set of all such gra $G(\pi_1,\pi_2)$ for some $\pi_1 \neq \pi_2$. If $G \in \mathcal{G}$ then we let $eq(G) = \{(\pi_1,\pi_2) : G$ $G(\pi_1,\pi_2)\}$. We observe that $|eq(G)| = 2^{c(G)}$. Finally let us denote by (ij) an arbitrary vertex of G which belongs to a cycle. Now, we can write

$$\sum_{(\pi_1,\pi_2);\pi_1\neq\pi_2} \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2))\prod_i B_{i\pi_1(i)}B_{i\pi_2(i)} =$$

$$\sum \sum \mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2))\prod B_{i\pi_1(i)}B_{i\pi_2(i)} =$$

$$\sum_{G\in\mathcal{G}} B_{(ij)(G)}y(G),$$

where $y(G)$ is a random variable independent of $B_{(ij)(G)}$. Since the expectation of $B_{(ij)(G)}$ is equal to zero, the proof is finished.

□

However, for the matrices with 0, 1 entries, there is something better. Jerrum, Sinclair and Vigoda constructed a *fully polynomial randomized approximation scheme* (FPRAS, in short) for approximating permanents of matrices with *nonnegative entries*. Briefly, a FPRAS for the permanent is an algorithm which, when given as input an $n \times n$ nonnegative matrix A together with an accuracy parameter $\epsilon \in (0,1]$, outputs a number Z (a random variable of the coins tossed by the algorithm) such that

$$\text{Prob}[(1-\epsilon)Z \leq \text{Per}(A) \leq (1+\epsilon)Z] \geq \frac{3}{4}$$

and runs in time polynomial in $n, \sum |log(A_{ij})|$ and ϵ^{-1}. The probability 3/4 can be increased to $1-\delta$ for any desired $\delta \in (0,1]$ by outputting the median of $O(\log \delta^{-1})$ independent trials.

9.2 Products over aperiodic closed walks

The following solution to the 2-dimensional Ising model has been developed by Kac, Ward and Feynman. This theory is closely related to that of Section 7.1. Let $G=(V,E)$ be a planar topological graph. It is convenient to associate a variable x_e instead of a weight to each edge e. If $e \in E$ then a_e will denote the orientation of e and a_e^{-1} will be the reversed orientation. We let $x_a = x_e$ for each orientation a of e. A circular sequence $p = v_1, a_1, v_2, a_2, ..., a_n, (v_{n+1} = v_1)$ is called a *prime reduced cycle*, if the following conditions are satisfied: $a_i \in \{a_e, a_e^{-1} : e \in E\}$, $a_i \neq a_{i+1}^{-1}$ and $(a_1, ..., a_n) \neq Z^m$ for some sequence Z and $m > 1$. We let $X(p) = \prod_{i=1}^n x_{a_i}$ and if each degree of G is at most 4 then we let $W(p) = (-1)^{\text{rot}(p)}X(p)$ where rot(p) was defined in Chapter 5.
If $E' \subset E$ then we also let $X(E') = \prod_{e\in E'} x_e$. There is a natural equivalence on the prime reduced cycles: p is equivalent to reversed p. Each equivalence class has two elements and will be denoted by $[p]$. We let $W([p]) = W(p)$ and note that this definition is correct since equivalent walks have the same sign. The following theorem was proposed by Feynman and proved by Sherman. It provides, for a planar graph G, an expression for the generating function $\mathcal{E}(G,x)$ of the even sets of edges (see Section 2.1), in terms of the Ihara-Selberg function of G (see Definition 7.1.1).

Theorem 9.2.1. *Let G be a planar topological graph with each degree even and at most 4. Then*

$$\mathcal{E}(G,x) = \prod(1 - W([p]))$$

where we denote by $\prod(1-W([p])$ the formal product of $(1-W([p])$ over all eq alence classes of prime reduced cycles of G (the formal product was consid in Chapter 7).

Note that the product is infinite even for a very simple graph consistin one vertex and two loops.
When each transition between a pair of directed edges is decorated by its r tion contribution (see Section 5.3), Theorem 9.2.1 implies that $\mathcal{E}^2(G,x)$ beco an Ihara-Selberg function (see Section 7.1). Hence we get the following co lary, whose statement (and incorrect proof) by Kac and Ward was in fact starting point of the whole path approach.

Theorem 9.2.2. *Let G be a topological planar graph with all degrees even an most 4. Then $\mathcal{E}^2(G,x)$ equals the determinant of the transition matrix betu directed edges; each transition is decorated by its rotation contribution.*

Theorem 9.2.1 is formulated for those topological planar graphs where e degree is even and at most 4. It is not difficult to reduce $\mathcal{E}(G,x)$, G a g eral topological planar graph, to this case: First we make each degree ever doubling each edge. If we set the variables of the new edges to zero then e term containing a contribution of at least one new edge disappears. Next make each non-zero degree equal to 2 or 4 as follows. We replace each ve v with incident edges $e_1, ..., e_{2k}$, $k > 2$, listed in the circular order given the embedding of G in the plane, by a path P of $2k-2$ vertices. We set variables of the edges of P equal to 1. Next we double each edge of the uni perfect matching of P and set the variables of the new edges to zero. Finally join the edges $e_1, ..., e_{2k}$ to the vertices of the auxiliary path so that the or is preserved along the path and each degree is four: there is a unique way to that.
In order to prove Theorem 9.2.1, Sherman formulated and proved the follow generalization which we now state. Let v be a vertex of degree 4 of G and l be an aperiodic closed walk of G. We say that p satisfies the *crossover condi* at v if the way p passes through v is consistent with the crossover pairing of four edges incident with v.
Let U be a subset of vertices of degree 4. An even subset $E' \subset E$ is ca *acceptable* for U if, for each $u \in U$ and for both pairs of edges incident w u and paired by the crossover pairing at u, if E' contains one edge of the p then it also contains the other one.

Theorem 9.2.3. *Let $G=(V,E)$ be a topological planar graph where each de is even and at most 4. Let U be a subset of vertices of G of degree 4. $\prod'_{G,U}(1-W([p]))$ denote the product over all equivalence classes of the aperi closed walks of G which satisfy the crossover condition at each $u \in U$. Then*

$$\prod{}'_{G,U}(1 - W([p])) = \sum (-1)^{c(E')}X(E'),$$

where the sum is over all acceptable even subsets $E' \subset E$ and $c(E')$ is equa the number of vertices of U such that E' contains all four edges incident w