$f : 2^X \longrightarrow \mathbb{R}$ <u>submodular</u> if $(\forall Y, Z \subseteq X)\ \big(f(Y) + f(Z) \geq f(Y \cap Z) + f(Y \cup Z)\big)$

Combinatorial auctions X : set of items

- $f(S)$: <u>value</u> of a subset S of items
- $x \in X \Rightarrow \Delta f_x(T) = f(T \cup \{x\}) - f(T)$, $T \subseteq X$ [marginal values]
- f submodular iff $\forall x \in X$, Δf_x non-increasing
  "Additional items have less and less value, as the set we possess grows."
- <u>The submodular welfare problem</u> : allocate items to multiple players, each of which values subsets of items according to a submodular utility fctn
- <u>The general assignment problem</u> : assign items to bins of limited capacity, where both "values" and "sizes" of items can depend on the bin where the item is placed.

Příklady

1. $\{A_j\}_{j\in X}$ finite collection of sets, define

$$f(S) = \left|\bigcup_{j\in S} A_j\right|$$

coverage-type fction f is monotone, submodular.

<u>Max k-cover problem</u>: find $\max f(S)$; $|S| = k$

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2. **cut-type**

- $G = (V,E)$ graph; $\Delta(S) = |\{e \in E ; |e \cap S| = 1\}|$ $(S \subseteq V)$
- $D = (V,A)$ digraph, $\Delta(S) = \#$ arcs pointing from S to $\overline{S}$
- rank fction of a matroid

Matroid Polytope $P(M) = \{x \in \mathbb{R}^E_+ ;\ \forall S \subseteq X,\ \sum_{j\in S} x_j \leq r(S)\} =$

$\mathrm{Conv}(\chi_I ; I \text{ independent})$.

Representation of submodular fctions

(a) coverage-type, cut-type ... $\Rightarrow$ represent the comb. object (graph ...)

(b) <u>Value oracle</u>: What is the value of $f(S)$?

<u>Demand oracle</u> (more powerful) Given assignment of prices to items $p: X \to \mathbb{R}$, what is $\max_{S \subseteq X} \left[f(S) - \sum_{j \in S} p_j \right]$?

- <u>Nonadaptive algorithms</u>: issue a polynomial number of queries and the answers are processed by a polynomial time computation
- <u>Adaptive algorithms</u>: issue queries in the process of polynomial time computation

Problems

1. Given a submodular fction $f : 2^X \to \mathbb{R}$, value-oracle access, $\min(f(S);\ S \subseteq X)$. Polynomial

2. Given a $\boxed{\text{non-negative}}$ submodular fction $f : 2^X \to \mathbb{R}$, value-oracle, $\max(f(S);\ S \subseteq X)$. NP-hard (e.g. Max-Cut)

   Approximations

- e.g. 0,878-approximation for Max-Cut by Goemans, Williamson [semidefinite programming]

3. Given a monotone submodular fction $f : 2^X \to \mathbb{R}^+$ and matroid $M = (X, \mathcal{I})$, $\max(f(S);\ S \in \mathcal{I})$.
   oracle model for f and also membership oracle for M.

Example f is coverage-type, M <u>uniform</u>: S independent iff $|S| \le k$.

Greedy Algorithm

$S := \emptyset$

<u>While</u> S not a maximal independent set

{ Compute $f_S(j) = f(S \cup \{j\}) - f(S)$ for all $j \notin S$
such that $S \cup \{j\}$ is independent.

Pick j maximising $f_S(j)$ and include it in S. }

<u>Output</u> S.

- If f linear, i.e., $f(S) = \sum_{j \in S} w_j$, then ↑ is GA for max indep. set of a matroid.

- GA provides $(1 - 1/e)$ approximation for <u>Max k-cover</u>; best possible unless $P = NP$

⊙ TRUE for general monotone submodular f, general matroid

Submodular Welfare Problem

Given n players with utility fctions $w_i : 2^X \to \mathbb{R}_+$ assumed to be monotone submodular, find partition $X = S_1 \dot\cup \dots \dot\cup S_n$ in order to maximise $\sum_{i=1}^{n} w_i(S_i)$.

- demand oracle not unnatural: given an assignment of prices, a player can decide which set of items is most valuable (for her).
- in combinatorial auctions the utility fctions are unknown [incentive-compatible mechanisms]

- The submodular welfare problem is special case of submodular maximisation under a matroid constraint.

⊙ The submodular Welfare problem can be $(1-1/e)$-approximated in the value oracle model.

Submodular welfare $\subsetneq$ submodular max under matroid constraint

P set of players, Q set of items, $i \in P \Rightarrow w_i : 2^Q \longrightarrow \mathbb{R}_+$ utility fctions

<u>Define</u>: $X = P \times Q$, $f : 2^X \to \mathbb{R}_+$ defined as follows:

- $S \subseteq X \Rightarrow S = \bigcup_{i \in P} (\{i\} \times S_i)$ (uniquely)
- define $f(S) = \sum_{i \in P} w_i(S_i)$. ⊙ [f submodular]

"Many copies of each item"

- Define matroid $M = (X, \varphi)$; $\varphi = \{ S \subseteq X ; (\forall j) \, |S \cap (P \times \{j\})| \leq 1 \}$.

⊙ M partition matroid.

$\Rightarrow$ Submodular welfare problem equivalent to $\max(f(S) ; S \in \varphi)$.

The general assignment problem | GAP

- n bins, m items
- item j, bin i < value v_{ij}, size s_{ij}

Want: assignment of items to bins s.t. ① total size of items in each bin ≤ 1 and ② total value of all packed items max.

- Reduction to submodular maximization under a matroid constraint:

$\mathcal{F}_i :=$ collection of sets of items feasible for bin i

$X = \{ (i, S) ; 1 \leq i \leq n, S \in \mathcal{F}_i \}$

$f : 2^X \to \mathbb{R}^+ \quad f(S) = \sum_j \max_i \{ v_{ij} ; \exists (i,S) \in X, j \in S \}$ [submodular]

M_j: (i,S) has weight v_{ij} if $j \in S$ and 0 otherwise. A set indep. iff card. ≤ 1.

max f with matroid constraint: $M = (X, \mathcal{I})$

$A \in \mathcal{I}$ iff A contains at most one pair (i, S) for each i.

- Such A corresponds to an assignment of set S to bin i for each $(i,S) \in A$.
- equivalent to GAP: the bins can be assigned overlapping sets, we only count value of the most valuable assignment for each item.
- $f(A) = \sum_j \underbrace{\max_i (v_{ij} : \exists (i,S) \in A, j \in S)}$. $= g_j(A)$ weighted rank fction of a $\underbrace{\text{matroid}}_{M_j}$ on X.