Matroid (X, φ), $\varphi \subseteq 2^X$ 1

(1) $\emptyset \in \varphi$ (2) $A \in \varphi$, $A' \subseteq A \Rightarrow A' \in \varphi$ (hereditary)

(3) exchange axiom $U, V \in \varphi$, $|U| > |V| \Rightarrow$

$$(\exists x \in U \setminus V)(V \cup \{x\} \in \varphi)$$

(3') $A \subseteq X \Rightarrow$ all maximal (w.r.t. $\subseteq$) independent subsets of A have the same size.

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(3) $\Leftrightarrow$ (3') ; Examples: I A matrix over field $\mathbb{F}$

X set of columns of A, $\varphi = \{Y \subseteq X ; Y$ linearly indep. over $\mathbb{F}\}$ $\boxed{\text{Terminology}}$ elmts of $\varphi \equiv$ independent sets

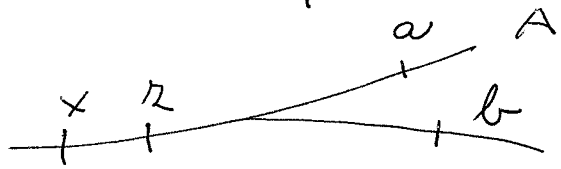
Example III Matroid *simple* if $r(A) = |A|$ whenever $|A| < 3$.

$A \subseteq X$ *closed* if $z \in \cancel{X} \setminus A \Rightarrow r(A \cup \{z\}) > r(A)$.

Each simple matroid of rank 3 ($r(X) = 3$) determined by

$\mathcal{L}(M) = \{A \subseteq X ; |A| > 2, r(A) = 2, A \text{ closed}\}$.

Lemma $A, B \in \mathcal{L}(M) \Rightarrow |A \cap B| \leq 1$



(R3) contr.

Definition $\mathcal{C} \subset 2^X$ *configuration* if (1) $A \in \mathcal{C} \Rightarrow |A| \geq 3$

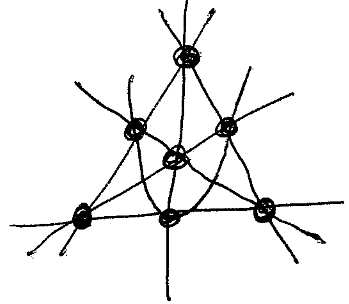
(2) $A, B \in \mathcal{C} \Rightarrow |A \cap B| \leq 1$.

Theorem Each configuration is the set $\mathcal{L}(M)$ of a simple matroid of rank 3 on X.

Define r:
- $r(A) = |A|$ for $|A| \leq 2$
- $A \subseteq$ elmt of $\mathcal{C} \Rightarrow r(A) = 2$
- otherwise $r(A) = 3$

satisfies R1, R2, R3

Fano matroid



F_7

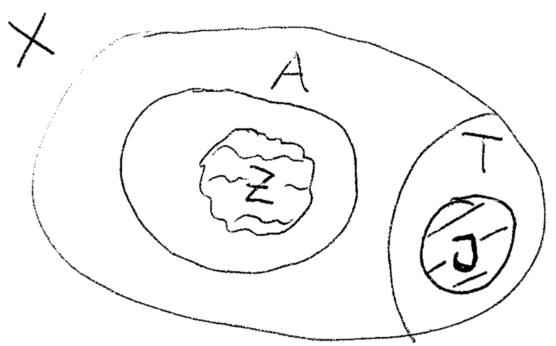
Contraction $T \subseteq X$, $J \subseteq T$ maximal independent.

$M \setminus T$ (contraction of T) $= (X \setminus T, \varphi')$

$$A \in \varphi' \overset{def}{\Leftrightarrow} A \cup J \in \varphi$$

Theorem $M \setminus T$ is matroid; $r'(A) = r(A \cup T) - r(T)$, $A \subseteq X - T$

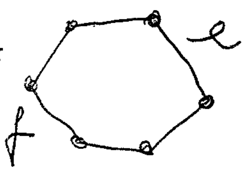
Proof.



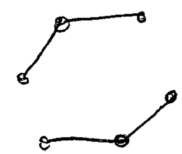
Z: maximal ($\subseteq$) independent in $M \setminus T$ subset of A

●: $Z \cup J$ is maximal independent in φ since any enlargement leads to contradiction. ☒

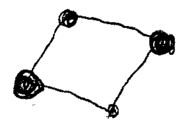
Example Graphic matroids $G =$



$M_G - \{e, f\} = M_{G'}$



$M_G \setminus \{e, f\} = M_{G''}$

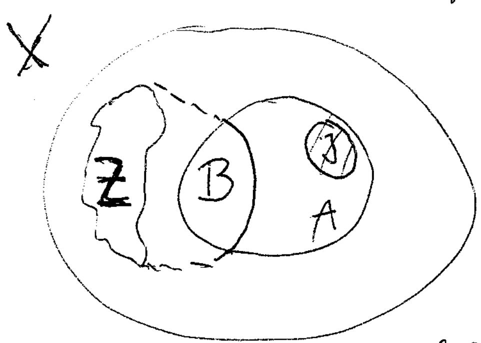


Duality $M = (X, \mathcal{Y})$. Dual $M^* = (X, \mathcal{Y}^*)$

$Y \in \mathcal{Y}^* \overset{\text{def}}{\iff} r(X-Y) = r(X)$

[B^* basis of M^* iff $X - B^*$ basis of M]

Theorem M^* matroid, $r^*(A) = |A| - r(X) + r(X-A)$.



$J \subseteq A$ maximal independent in $\mathcal{Y}^*$

$Z \subset X - A$ max indep. in $\mathcal{Y}$

$Z \subseteq B$ max independent (in $\mathcal{Y}$) of $X - J$.

- $|B| = r(X)$ since ~~not~~ $r(X-J) = r(X)$
- $J = A \setminus B$ since otherwise J is not max indep.-(in $\mathcal{Y}^*$) subset of A.

$\Rightarrow \quad |J| = |A| - |B \cap A| = |A| - (|B| - |Z|)$ $\boxtimes$

Terminology __circuit__ : minimal ($\subseteq$) dependent subset

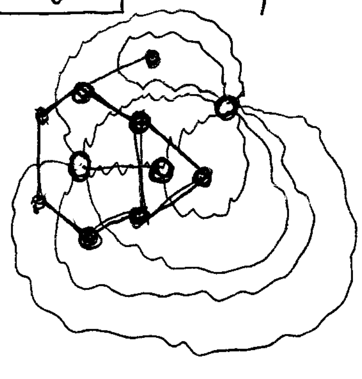
__closed set__ : $A \subseteq X$ closed if $(\forall y \in X \setminus A)(r(A \cup \{y\}) > r(A))$

co-(basis, circuit) $\equiv$ basis, circuit of the dual

Theorem G graph. Cocircuits of M_G are minimal edge-cuts.

__Proof.__ Edge-cuts are exactly sets of edges intersecting every spanning tree, i.e., basis of M_G. $\boxtimes$

Corollary G planar, G^* geometric dual. Then $M(G^*) = M^*(G)$.



Comparing rank functions

M minor of N if obtained by sequence of deletions and contractions

$$(M \setminus T)^* = M^* - T$$
$$(M - T)^* = M^* \setminus T$$

M <u>minor</u> of N iff M^* minor of N^*

M minor of N iff M obtained from N by deletion followed by contraction (contraction followed by deletion respectively)

Theorem G planar iff M_G is cographic.

<u>Proof</u>.
- $M(K_5)$, $M(K_{33})$ are not cographic.
- Minor of graphic is graphic.

Representable Matroids

$\boxed{\text{Representable over } \mathbb{F}}$ $\overset{def}{\equiv}$ isomorphic ~~of~~ to vectorial matroid over $\mathbb{F}$.

$\boxed{\text{Binary}}$ representable over $\mathbb{F}_2$

$\boxed{\text{Regular}}$ representable over any field → graphic are regular

(I/D) incidence matrix

Standard representation w.r.t. basis B $\begin{array}{c|c} \overset{B}{I} & A \end{array}$

$\boxed{\text{Thm}}$ $I|A$ standard representation of $M \Rightarrow A^T|I$ repres. of M^*

$\boxed{\text{Thm}}$ $\boxed{\text{Tutte}}$
- M binary iff U_2^4 is not minor of M $\left[U_2^4 : |X|=4, \text{ all } \leq 2 \text{ el. indep.} \right]$
- M regular iff binary and no F_7, F_7^* minor
- M graphic iff M regular and no $M(K_5)^*$, $M(K_{33})^*$ minor

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- Regular matroids $\equiv$ TU matrices