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Bass' identity and a coin arrangements lemma

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ABSTRACT

A lemma on coin arrangements is an important trick in Sherman's proof of Feynman's conjecture on the two dimensional Ising model. Here, we show that the coin arrangements lemma is equivalent to Bass' identity.

1. Introduction

There is no doubt that the Riemann zeta function is a function of great significance in number theory because of its connection to the distribution of prime numbers. The Ihara zeta function was first introduced by Ihara [5]. This zeta function is associated with a finite graph and aperiodic closed walks in it. Bass' identity [1] shows that the Ihara zeta function is a determinant. It is shown in the book [12] that using this identity and some other ideas from linear algebra one can come up with the graph-theoretical analogue of the prime number theorem. Therefore, it seems that Bass' identity plays an essential role in the theory of the Ihara zeta function of a finite graph. Since the original work of Bass [1], his identity has been proven several times in an inspiring way. Let us mention for instance [3] and a lecture of Xavier Viennot at the Newton Institute, Cambridge University in 2008.

Independently, an evaluation of the Ihara zeta function of planar graphs and a determinantal formula for it appeared implicitly in the work of Kac, Ward, Feynman and Sherman on the 2-dimensional Ising problem [6,10]. This work is thus an important special case of Bass' identity. This combinatorial approach to the Ising problem has been less popular among combinatorialists and physicists than the Pfaffian method (see [7] for an introduction to both the Feynman and the Pfaffian methods as well as the relation with the Ihara zeta function and Bass' identity). Quite recently, however Feynman's approach reappeared in studies of the criticality of the Ising problem on closed Riemann surfaces of a positive genus [8]. New studies of the Feynman method as well as comparisons with the Pfaffian method appear (see e.g. [2]).

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In our short paper, we first show how one can derive from Bass' identity a new proof of an identity, due to Witt, that deals with the Lyndon words (see [4]). Basically, Witt's identity is presented as Bass' identity for one-vertex graphs.

Witt's identity is equivalent to a lemma on coin arrangements [10] and this lemma is one of the key tools in the proof of Feynman's conjecture by Sherman [10]. We follow Sherman's approach to the proof of Feynman's conjecture and show that it leads to a new proof of Bass' formula as well.

2. Witt's identity and the coin arrangements lemma

First, we briefly recall some definitions from the combinatorics of words. The reader may consult the book [9]. Let A be a finite ordered set, as our alphabet. The set of all finite sequences of elements (letters) of A will be denoted by $A^\star$. Each element of $A^\star$ is a *word* over alphabet A. In particular, the empty sequence of letters is also an element of $A^\star$ which is called the *empty word* and is denoted by 1. The length of a word w is defined as the number of its letters and is denoted by $|w|$. The set of all nonempty words over A will be denoted by A^+:

$$A^+ = A^\star - 1.$$

A word v is called a *right factor* of a word w if there exists a third word u such that $w = uv$. If u is a nonempty word, then the right factor v is called a *proper* right factor of the word w. Recall that two words x and y are said to be *conjugate* if there exist words $u, v \in A^\star$ such that

$$x = uv, \qquad y = vu.$$

This is an *equivalence relation* on $A^\star$ since x is the conjugate of y if and only if y can be obtained by a *cyclic permutation* of the letters of x. This partitions the set $A^\star$ into disjoint conjugacy classes. A word is said to be *primitive* if it is not a power of another word. We also recall that a *Lyndon word* is a primitive word that is minimal, with respect to the lexicographic order, in its conjugacy class. The set of all Lyndon words over the alphabet A will be denoted by $\mathcal{L}(A)$, or just by $\mathcal{L}$ if no confusions arise.

The following fundamental result is known as the *Lyndon factorization theorem* (see [9]).

Theorem 1. *Any word $w \in A^+$ may be written uniquely as a nonincreasing concatenation of the Lyndon words:*

$$w = l_1 l_2 \cdots l_n, \quad l_i \in \mathcal{L}, \quad l_1 \succeq l_2 \succeq \cdots \succeq l_n.$$

For example, word 2413412 is decomposed into the Lyndon words 24, 134, 12.

Now, we state Witt's identity which is an identity of two formal power series in the context of the Lyndon words [4]. For the sake of completeness, we also include a short proof, using the Lyndon factorization theorem.

If $A = \{a_1, \ldots, a_k\}$ is an alphabet, then we let $M(m_1, \ldots, m_k)$ be the number of the Lyndon words with exactly m_i occurrences of a_i, $i = 1, \ldots, k$.

Theorem 2 (*Witt's Identity*)**.** *Let $z_1, \ldots, z_k$ be commuting variables. Then we have on the level of the formal power series*

$$\prod_{m_1,\ldots,m_k \geq 0} \left(1 - z_1^{m_1} \cdots z_k^{m_k}\right)^{M(m_1,\ldots,m_k)} = 1 - z_1 - \cdots - z_k. \tag{1}$$

Proof. Using the geometric series formula and the Lyndon factorization theorem, we have

$$\frac{1}{1 - z_1 - \cdots - z_k} = \sum_{\omega \in \{z_1,\ldots,z_k\}^\star} \omega = \prod_{l \in \mathcal{L}} \frac{1}{1 - l}$$
$$= \frac{1}{\prod_{m_1,\ldots,m_k \geq 0} (1 - z_1^{m_1} \cdots z_k^{m_k})^{M(m_1,\ldots,m_k)}}. \quad \square$$

In Feynman's combinatorial solution of the two dimensional Ising problem, the following coin arrangements lemma [10] plays an important role.

Proposition 2.1. *Suppose we have a fixed collection of N objects of which m_1 are of one kind, m_2 are of second kind, $\ldots$, and m_k of k-th kind. Let $b_{N,i} = b(N, i; m_1, \ldots, m_k)$ be the number of exhaustive unordered arrangements of these objects into i disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and, none are periodic. Then, we have*

$$\sum_{i=1}^{N}(-1)^i b_{N,i} = 0, \quad (N > 1).$$

We note that $b(N, i; m_1, \ldots, m_k)$ is the number of sets of i Lyndon words in alphabet $\{1, \ldots, N\}$ containing in total m_1 occurrences of the letter 1, m_2 occurrences of the letter $2, \ldots,$ and m_k occurrences of the letter k.

There are many interesting proofs of the coin arrangements lemma [11]. Here, we present the original proof of Sherman [10] which uses Witt's identity. Indeed, it shows that the coin arrangements lemma and Witt's identity are equivalent.

Proof (*The Coin Arrangements Lemma*)**.** If we expand the left hand side of Witt's identity, we get

$$\prod_{m_1,\ldots,m_k \geq 0} (1 - z_1^{m_1} \cdots z_k^{m_k})^{M(m_1,\ldots,m_k)}$$
$$= 1 - \sum_{N\geq 1} \sum_{c_1,\ldots,c_k \geq 0: \sum c_i = N} \left(\sum_{i=1}^{N} (-1)^i b(N, i; c_1, \ldots, c_k) \right) z_1^{c_1} \cdots z_k^{c_k}.$$

By comparing the corresponding coefficients in both sides of Witt's identity, we obtain the desired identity. □

3. Primes in graphs and combinatorial zeta function

Definition 3.1. For a given graph $G = (V, E)$, which has no loop but may have multiple edges, we define its *symmetric digraph* $\vec{G}$ by replacing each undirected edge e with two arcs a_e and a_e^{-1} which are in opposite directions. We denote by $E(\vec{G})$ the edge-set of $\vec{G}$.

Definition 3.2. A *path* or a *walk* $C = a_{e_1} \cdots a_{e_t}$ in $\vec{G}$, where a_{e_i} is an oriented edge of G, is said to be *closed* if the starting vertex is the same as the terminal vertex. We say that two closed walks are *equivalent* if they differ by a cyclic permutation of the vertices.

Definition 3.3. A *prime* in a graph G is an equivalence class $[P]$ of the closed walk $P = a_{e_1} \cdots a_{e_t}$, $t > 0$, in $\vec{G}$, such that:

- it has no backtracking: which means no arc is followed immediately by the arc with the same ends but in the opposite direction, i.e., there is no $j = 1, \ldots, t-1$ with $a_{e_{j+1}} = a_{e_j}^{-1}$.
- it has no tail: which means the starting and the ending arcs are not two arcs with the same ends and just opposite directions, i.e., $a_{e_t} \neq a_{e_1}^{-1}$.
- it is primitive: which means there is no other closed walk P_0 with $P = P_0^l$ for some integer $l > 1$.

We remark that the graph G has a finite number of primes only if each pair of its cycles are vertex-disjoint. If G has two cycles C_1 and C_2 which have a vertex in common, then each closed walk of form $\left(\vec{C_1}\right)^n \left(\vec{C_2}\right)$, n a positive integer, is a prime of G.

Definition 3.4. The *size* (length) of a prime is defined, as follows

$$|P| = \text{ the number of arcs of } P.$$

Definition 3.5. The *Ihara zeta function* $\zeta(u, G)$ of a finite graph $G = (V, E)$ is a formal power series defined by

$$\zeta(u, G) = \prod_{[P]:prime} \left(1 - u^{|P|}\right)^{-1},$$

where u is a formal variable.

Definition 3.6. Let $G = (V, E)$ be a graph which has no loop but may have multiple edges. Let us denote the arcs of the symmetric digraph of G by $1, \dots, m$; hence $|E(\overrightarrow{G})| = m$. The *edge matrix* of G is matrix T of order $m \times m$ with (i, j)-entry equal to 1 if the terminal vertex of arc i is the starting vertex of arc j and $i \neq j^{-1}$. Otherwise the (i, j)-entry is 0.

We note that the edge matrix of graph G is really the matrix of the *transitions* between oriented edges of G. We are ready to state Bass' identity.

Theorem 3 (*Bass' Identity*)**.** *Let $G = (V, E)$ be a graph which has no loop but may have multiple edges. Let T be the edge matrix of G. Then*

$$\zeta^{-1}(u, G) = \det(I - uT).$$

There are many proofs of Bass' identity in the literature. In [3] three combinatorial proofs based on different aspects of the algebra of the Lyndon words have been given. In fact, in this paper we single out yet another relevant trick, the coin arrangement lemma. The combinatorial setting of [3] is very natural and it also leads to the *weighted* Bass' identity. The weighted Bass' identity is the one appearing naturally in recent studies of the critical Ising model [8]. In the proof of Witt's identity given in the next section, we also need the weighted Bass' identity.

In order to state the weighted Bass' identity we introduce first the weighted version of the Ihara zeta function of a finite graph; we note that it is called the *edge zeta function* in [12]. From now on, we also extend the graphs we consider.

Definition 3.7. From now on, we will allow *loops* in the edge-set of graphs. We assume each loop e has only one orientation a_e; hence, such a_e has no opposite orientation. In the symmetric digraph $\overrightarrow{G}$ we add a_e for each loop e to $E(\overrightarrow{G})$.

We also need to extend Definition 3.6.

Definition 3.8. Let G be a graph, loops and multiple edges allowed. Let us denote the arcs of the symmetric digraph of G by $1, \dots, m$; hence $|E(\overrightarrow{G})| = m$. The *weighted edge matrix* of G is matrix W of order $m \times m$ with (i, j)-entry equal to w_{ij} if the terminal vertex of arc i is the starting vertex of arc j and $i \neq j^{-1}$. Otherwise the (i, j)-entry is 0. Here, w_{ij} is a formal variable corresponding to the pair of arcs (i, j).

Definition 3.9. Let $C = a_1 \cdots a_m$ be a closed walk in a directed graph $D = (V, A)$; hence $a_1 \cdots a_m$ are arcs of D. We define the *edge norm* $N(C)$ of C, as follows

$$N(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{m-1} a_m} w_{a_m a_1}.$$

Now, we are ready to give the weighted version of the Ihara zeta function of a finite graph G.

Definition 3.10. The *weighted Ihara zeta function* $\zeta(W, G)$ of a finite graph $G = (V, E)$ is defined as follows

$$\zeta(W, G) = \prod_{[P]:\, prime} (1 - N(P))^{-1}.$$

Clearly, if we set $w_{ab} = u$ for every pair of arcs (a, b), we get the (standard) Ihara zeta function of G.

We arrive at the following weighted version of Bass' identity (see [3]).

Theorem 4 (*The Weighted Bass' Identity*)**.** *Let G be a graph, loops and multiple edges allowed. Let W be the weighted edge matrix of G. Then*

$$\zeta_E^{-1}(W, G) = \det(I - W).$$

4. Witt's identity via Bass' identity

We show that Witt's identity is the special case of the weighted Bass' identity when the graph G has only one vertex and k loops $e_1, \ldots, e_k$ attached to this vertex. Let a_i denote the directed loop a_{e_i} and let us associate the formal variable w_i with arc a_i, $i = 1, \ldots, k$. Primes in this graph are in one-to-one correspondence with the Lyndon words on the alphabet $\{a_1, \ldots, a_k\}$. Let us assign to the pair of arcs a_i, a_j the weight $w_{a_i a_j} = \sqrt{w_i w_j}$, $i, j \in \{1, \ldots, k\}$, and denote by W the corresponding weighted edge matrix.

Note that the left-hand side of Theorem 4 coincides with the left-hand side of Witt's identity. Consequently, the weighted Bass' identity reduces Witt's identity to the proof that $\det(I - W) = 1 - w_1 - \cdots - w_k$. Since

$$\det(I - W) = \prod_{\lambda:\ \text{eigenvalue of } W} (1 - \lambda), \tag{2}$$

the following Lemma finishes the proof of Witt's identity.

Lemma 4.1. *The only non-zero eigenvalue of W is* $(w_1 + \cdots + w_k)$.

Proof. For our one vertex graph G we have $W_{ij} = \sqrt{w_i w_j}$. Let $\sqrt{w}$ denote the vector $(\sqrt{w_1}, \ldots, \sqrt{w_k})$. We have $W = \sqrt{w}\sqrt{w}^T$. The dimension of the kernel of W is $k - 1$ and thus the multiplicity of the eigenvalue $\lambda_1 = 0$ is $k - 1$. Finally, for the last eigenvalue we have $W\sqrt{w} = (\sqrt{w}\sqrt{w}^T)\sqrt{w} = \sqrt{w}(\sqrt{w}^T\sqrt{w}) = \lambda\sqrt{w}$ where $\lambda = \sqrt{w}^T\sqrt{w} = (w_1 + \cdots + w_k)$. □

5. The weighted Bass' identity using the coin arrangements lemma

The first step is to rewrite the product form of the inverse of the weighted Ihara zeta function as an (infinite) sum, as follows

$$\begin{aligned}\zeta(W, G)^{-1} &= \prod_{g:prime} (1 - N(g)) \\ &= \sum_{\gamma \in \Gamma} (-1)^{|\gamma|} \prod_{g \in \gamma} N(g),\end{aligned}$$

where Γ is the set of all finite sets γ of primes in G.

In the next step, we divide Γ into Γ_1 and Γ_2 where Γ_1 is the finite set of all finite sets γ of primes so that each arc of $\vec{G}$ is traversed by γ *at most once*.

Γ_2 is thus the set of all finite sets γ of primes so that at least one of the arcs in $\vec{G}$ is traversed by γ *at least twice*. In the third step, we observe that

$$\sum_{\gamma\in\Gamma_1}(-1)^{|\gamma|}\prod_{g\in\gamma}N(g)$$

is equal to the determinant appearing in the weighted Bass' identity. To do this, we perform a standard useful trick.

Definition 5.1. Let $G=(V,E)$ be a graph, loops and multiple edges are allowed. The *line digraph* of the symmetric digraph of G, denoted by $L(\vec{G})$, is a digraph which has one vertex for each arc of $\vec{G}$. Two vertices representing the arcs xy and uv are connected by an arc from xy to uv in the line digraph when $y=u$, provided that $xy\neq uv^{-1}$. The *weight* of the arc from xy to uv is the weight $w_{(xz)(uv)}$ of the transition from xy to uv (see Definition 3.8).

Definition 5.2. Let a digraph D with weights on its arcs be given. The *weighted adjacency matrix* of D is the matrix with rows and columns corresponding to the digraph vertices, where a nondiagonal entry a_{ij} is the sum of the weights of the arcs from vertex i to vertex j, and the diagonal entry a_{ii} is the sum of the weights of the loops at vertex i.

We note that the weighted edge matrix W of a graph G is the weighted adjacency matrix of the digraph $L(\vec{G})$ (see Definition 3.8). Clearly, a prime in G that does not visit twice the same arc of the symmetric digraph of G is just a directed cycle of $L(\vec{G})$. Hence we have

$$\sum_{\gamma\in\Gamma_1}(-1)^{|\gamma|}\prod_{g\in\gamma}N(g)=\sum_{\gamma'\in\Gamma_1'}(-1)^{|\gamma'|}\prod_{g'\in\gamma'}N'(g'),$$

where Γ_1' is the finite set of all finite sets γ' which are the vertex disjoint union of directed cycles g' in $L(\vec{G})$, and $N'(g')=\prod_{uv\text{ arc of }g'}w_{uv}$.

By a well-known graph theoretical interpretation of the determinants, using again the observation that the weighted edge matrix W of a graph G is the weighted adjacency matrix of the digraph $L(\vec{G})$, we get

Observation 5.3.

$$\sum_{\gamma\in\Gamma_1}(-1)^{|\gamma|}\prod_{g\in\gamma}N(g)=\det(I-W).$$

This finishes the third step in the proof of the weighted Bass' theorem.

Finally, the last step of the proof is to show that

$$\sum_{\gamma\in\Gamma_2}(-1)^{|\gamma|}\prod_{g\in\gamma}N(g)=0.$$

Let a_1 be a directed edge of $\vec{G}$. We define A_1 as the set of all primes p such that a_1 appears in p. Then, we claim that:

Observation 5.4. $\prod_{p\in A_1}(1-N(p))=1-\sum_\alpha w_{a_1\alpha}d_{1,\alpha}$,

where α is a directed edge of $\vec{G}$ and $d_{1,\alpha}$ is a formal (possibly infinite) sum of monomials, none of which has $w_{a_1\beta}$, β arbitrary, as a factor.

Proof. Each prime p can be uniquely decomposed into closed walks $S_1, \ldots, S_r$ each of which starts with a_1 and has no other appearance of a_1. Now, let $S(p)$ denote the multiset with elements $S_1, \ldots, S_r$. We call these walks *stones*. In the sum below, Ω denotes a finite set of primes in A_1.

$$\begin{aligned}\prod_{p \in A_1} (1 - N(p)) &= \sum_{\Omega} (-1)^{|\Omega|} \prod_{p \in \Omega} N(p) \\ &= \sum_{\Omega} (-1)^{|\Omega|} \prod_{p \in \Omega} \prod_{S \in S(p)} N(S) \\ &= \sum_{\mathfrak{S}} \alpha(\mathfrak{S}) \prod_{S \in \mathfrak{S}} N(S),\end{aligned}$$

where $\mathfrak{S}$ is a finite multiset of stones and $\alpha(\mathfrak{S}) = \sum_R (-1)^{|R|}$ where the sum is over all unordered arrangements R of the stones of $\mathfrak{S}$ into collections of distinct primes. This is the same as arranging the elements of $\mathfrak{S}$ into disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and, none are periodic. Now, it follows form the coin arrangements lemma (Proposition 2.1) that $\alpha(\mathfrak{S}) = 0$ whenever $|\mathfrak{S}| > 1$. If $|\mathfrak{S}| = 0$ then $\alpha(\mathfrak{S}) \prod_{S \in \mathfrak{S}} N(S) = 1$ and if $|\mathfrak{S}| = 1$ then

$$\alpha(\mathfrak{S}) \prod_{S \in \mathfrak{S}} N(S) = w_{a_1 \alpha} d_{1,\alpha},$$

where α is a directed edge of $\vec{G}$ and $d_{1,\alpha}$ is a formal (possibly infinite) sum of monomials, none of which has $w_{a_1\beta}$, β arbitrary, as a factor. □

Observation 5.4 shows that

$$\sum_{\gamma \in \Gamma_2} (-1)^{|\gamma|} \prod_{g \in \gamma} N(g) = 0$$

since the arc a_1 was chosen arbitrarily. This completes the proof of the weighted Bass' theorem.

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