Directed Cycle Double Covers Cut-Obstacles and Robust ear Decompositions

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Abstract

A directed cycle double cover (DCDC) of a graph G is a family of cycles of G, each provided with an orientation, such that every edge of G is covered by exactly two oppositely directed cycles. Obvious obstructions to the existence of a directed cycle double cover in a graph are bridges. Jaeger [3] conjectured that bridges are actually the only obstructions. We study potential obstructions to extending a given set of orientations of edges into a DCDC and characterize such obstructions for short ears. Then, we formulate a conjecture on graph connections whose validity follows by the successful avoidance of one cut-type obstruction that we call cut-obstacles. The main result of this work claims that our *cut-obstacles avoidance conjecture* already implies Jaeger's directed cycle double cover conjecture.

Keywords: Directed cycle double cover, generalized bridge, short ear decompositions

1 Introduction, outline of methods and main results.

A directed cycle double cover (DCDC) of a graph G is a family of cycles of G, each provided with an orientation, such that every edge of G is covered by exactly two oppositely directed cycles. Jaeger's directed cycle double cover conjecture [3] asserts that for every 2-connected graph G a DCDC exists. Jaeger's conjecture trivially holds in the class of cubic bridgeless planar graphs. And it has certainly been positively settled for some more classes of graphs, as for example, graphs that admit a nowhere-zero 4-flow [4] and 2-connected projective-planar graphs [2]. We kindly invite the reader interested in more details about the development of the directed cycle double cover conjecture and related problems to consult [3, 5, 6].

In this paper we consider the following scenario: a set of orientations of edges of a cubic graph is given and we have to extend this set or to declare that no extension to a DCDC exists. In Subsection 1.1 we formalize this situation and in Subsection 1.2 we describe the contribution of this paper.

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1.1 Mixed graphs, reductions and cut-obstacles

A way of starting the construction of a DCDC of a graph $G = (V, E)$ which is cubic and bridgeless is to select a vertex $v \in V$ and *wire* its incident edges $\{v, x\}$, $\{v, y\}$, $\{v, z\}$. This means to deleted the vertex v and to add three new arcs to G, say $(x, y), (y, z)$ and (z, x). Here, the arc (x, y) represents the directed path (x, v, y); analogously for (y, z) and (z, x). As a result, we obtain a *mixed* graph with vertex set $V - \{v\}$ and edge set $E - \{\{v, x\}, \{v, y\}, \{v, z\}\}$, along with a set $\{(x, y), (y, z), (z, x)\}$ of arcs and pairs of arcs which are forbidden to belong to the same cycle of a directed cycle double cover. We can continue this procedure by sequentially selecting a vertex u in $V - \{v\}$ and wiring its incident edges and/or arcs. *If* we could continue this procedure until every edge is wired, we *might* be able to show the existence of a directed cycle double cover. This approach motivates the definitions of mixed graphs, reductions and cut-obstacles.

A *mixed graph* is a 4-tuple (V, E, A, R), where V is a vertex set, E is an edge set, A is a set of directed edges (arcs), and R is a subset of $A \times A$, that is, a set of pairs of arcs. It is require that in the graph (V, E), that is, the graph on vertex set V and edge set E, each vertex has degree at least one and at most three, and that, in (V, E, A, R), each vertex of degree one (resp. two) in (V, E) is the tail of exactly two arcs (resp. one arc) and the head of exactly two arcs (resp. one arc); note that directed loops are allowed. We refer to the elements in R as *forbidden pairs*.

Throughout this paper, $\{u, v\}$ denotes the (non-directed) edge with end vertices u, v and (u, v) denotes the arc directed from u to v; e usually denotes an undirected edge and $\vec{e}$ denotes an arc.

Let (V, E, A, R) be a mixed graph and $U \subseteq V$. A *reduction* of U on (V, E, A, R) is a procedure that outputs a new mixed graph (V', E', A', R') and a list $\mathcal{S}$ of directed paths and cycles (here, a directed loop is considered a directed cycle) so that the following items hold:

i) $V' = V - U$.

ii) $E' = \{\{u, v\} \in E : \{u, v\} \cap U = \emptyset\}$.

iii) Let $\tilde{A}$ denote the set of arcs obtained by replacing each edge in E incident to a vertex of U by two arcs oppositely directed and $A(U)$ be the subset of A that contains all directed edges incident to a vertex of U. The list $\mathcal{S}$ is the result of partitioning $A(U) \cup \tilde{A}$ into reducible paths and reducible cycles. A cycle or a path is *reducible* if no pair of its arcs is an element of R, and if it is not a 2-cycle composed of only arcs from $\tilde{A}$; in particular, a directed loop is a reducible cycle. In addition, a reducible path must have both end vertices in $V - U$.

iv) Let A'' be the set of arcs obtained by replacing each reducible path $P \in \mathcal{S}$ by a new directed edge, say $\vec{e}_P$, with both end vertices in $V - U$ and such that $\vec{e}_P$ has the orientation of P. We let $A' = \{(x, y) \in A : \{x, y\} \cap U = \emptyset\} \cup A''$.

v) Finally, $R' = \{\{(x, y), (x', y')\} \in R : \{x, y, x', y'\} \cap U = \emptyset\} \cup R'' \cup \tilde{R}$ where

The set R'' consists of all pairs $\{\vec{e}_P, \vec{e}_{P'}\}$ such that P and P' have a vertex in common, or there are arcs $\vec{e} \in P$ and $\vec{g} \in P'$ such that $\{\vec{e}, \vec{g}\} \in R$. The set $\tilde{R}$ consists of all pairs $\{\vec{e}_P, \vec{g}\}$ such that P contains an arc $\vec{e}$ and $\{\vec{e}, \vec{g}\} \in R$.

An illustration of a reduction is provided in Figure 1, while an example of a reduction is given in Observation 1.

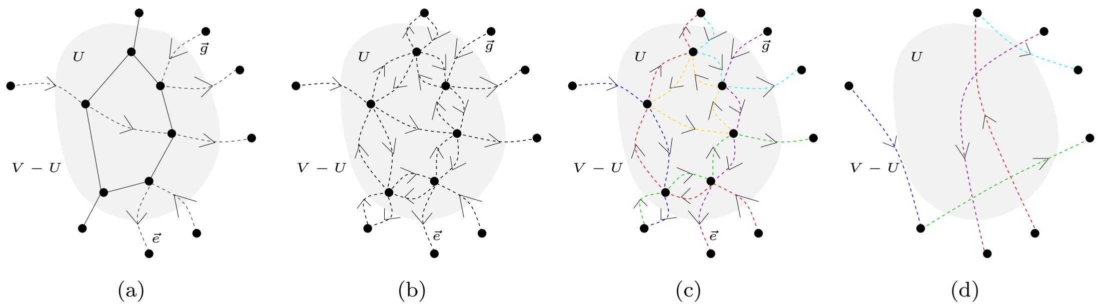


Figure 1: Reduction of $U \subset V$: (a) elements from A are represented by dotted lines and the pair $(\vec{e}, \vec{g})$ of arcs is not in R, (b) replace edges of E with an end vertex in U by two arcs oppositely directed, (c) partition of $A(U) \cup \tilde{A}$ into reducible paths and cycles and (d) resulting structure.

Observation 1. *Each mixed graph (V, E, A, R) with (V, E) a triangle admits a reduction of V.*

Proof. Let $V = \{x, y, z\}$. Without loss of generality, (z, x), (y, z) and (x, y) are in A. A reduction of V is given by the reducible cycles (x, z, x), (y, z, y), (x, y, x) containing (z, x), (y, z) and (x, y) in A, respectively and the reducible cycle (x, y, z, x). □

Let G be a cubic graph and $V_0, V_1, \ldots, V_k$ be a partition of $V(G)$. Let $j \in \{0, 1, \ldots, k\}$. A *consecutive reduction* of $V_k, V_{k-1}, \ldots, V_j$ is a sequence of $k - j + 1$ reductions such that for each $i \in \{k, \ldots, j\}$, the reduction of $V_i \subset W_i$ on (W_i, E_i, A_i, R_i) outputs $(W_{i-1}, E_{i-1}, A_{i-1}, R_{i-1})$; where $(W_k, E_k, A_k, R_k) = (V(G), E(G), \emptyset, \emptyset)$ and $(W_{-1}, E_{-1}, A_{-1}, R_{-1}) = (\emptyset, \emptyset, \emptyset, \emptyset)$. The following straightforward observation relates the notion of consecutive reductions to DCDC.

Observation 2. *Let $V_0, V_1, \ldots, V_k$ be a partition of the vertex set of a cubic graph G. Then each consecutive reduction of $V_k, V_{k-1}, \ldots, V_0$ constructs a DCDC of G.*

There are many potential obstructions to the existence of reductions. One of them is a cut-obstacle.

Definition 1 (Cut-obstacle)**.** *Let (V, E, A, R) be a mixed graph and $U \subseteq V$. We denote by C_U the subset of $E \cup A$ that contains all edges and arcs with exactly one end vertex in U. We say that there is a* cut-obstacle *at U in (V, E, A, R) if the number of edges in C_U is strictly less than twice the number of arcs in C_U.*

For an illustration of a cut-obstacle see Figure 2(a). Cut-obstacles are potential obstacles for the existence of reductions in the following sense: if we assume that all pairs of arcs in C_U belong to R, then there is not reduction of U on (V, E, A, R).

1.2 Contribution of this paper

In this paper we study ear decompositions of cubic graphs that admit reductions without creating cut-obstacles.

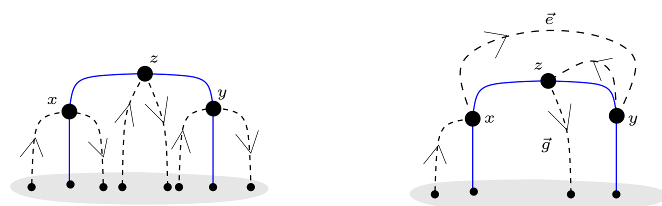
An *ear* is a path on at least 3 vertices or a star on 4 vertices; in particular, a graph consisting of one edge is not an ear. For each ear L, we denote by $I(L)$ the set of its internal vertices and if L is a path we say that L is a k-ear, provided $|I(L)| = k$. As usual, *adding* ear L to graph $G = (V, E)$ means identifying the leaves of L with distinct vertices of G. An ear L is called *short* if $|I(L)| \leq 3$.

Let $H = (V, E, A, R)$ be a mixed graph and L be a short ear. If L is a subgraph of (V, E) and the degree of each vertex in $I(L)$ is the same in L and in (V, E), then we say that L is a short ear of H.

1.2.1 Obstructions for short ears

Our first contribution is that regardless the richness of obstacles for reductions, it is possible to characterize the potential obstacles for reductions of *short ears*.

Definition 2 (Inner obstacle)**.** *An inner obstacle at the internal vertices* $\{x, z, y\}$ *of a 3-ear in a mixed graph* (V, E, A, R) *is the configuration depicted in Figure 2(b) such that* $(\vec{e}, \vec{g}) \in R$.



(a) A cut-obstacle at $\{x, z, y\}$ (b) Inner Obstacle

Figure 2: Potential obstacles to performing reductions in a 3-ear.

Theorem 1. *Let* $H = (V, E, A, R)$ *be a mixed graph and* L *be a short ear of* H*. If* L *is not a 3-ear then there is always a reduction of* $I(L)$*. If* L *is a 3-ear and there is neither a cut-obstacle nor an inner obstacle at* $I(L)$*, then a reduction of* $I(U)$ *exists.*

In this work, we show that, in order to prove the DCDC conjecture, it suffices to perform consecutive reductions without creating cut-obstacles at 3-ears (Theorem 2). Roughly, for each cubic bridgeless graph G, we construct a graph H with a short ear decomposition with the following property. If there exists a consecutive reduction of H according to its short ear decomposition, then G admits a DCDC. Furthermore, the existence of such a consecutive reduction of H depends exclusively of creating no cut-obstacles at the 3-ears of the short ear decomposition of H. In the next Subsection we formalize this discussion.

1.2.2 Robust ear decompositions of trigraphs

A *trigraph* is a cubic graph that can be obtained from a cycle by sequentially adding *short ears*. We note that no trigraph can have a bridge.

Whenever H is a trigraph, the expression $(H_0, H_i, L_i)^n$ denotes a short ear decomposition of H where H_0 is the initial cycle of the ear decomposition, L_i is the i-th short ear, H_i is the intermediate graph obtained from H_0 by adding the first i short ears and $H_n = H$.

The following observation describes a natural obstacle for a short ear decomposition of a trigraph to admit a reduction without creating cut-obstacles.

Let L_i be a 3-ear. The *descendant* of $I(L_i)$ is the maximal subgraph H' of $H - V(H_i)$ such that for each component C of H' there exists an edge connecting C and $I(L_i)$.

Observation 3. *If H is a trigraph and there exists $i < n$ such that L_i is 3-ear and its descendant $I(L_i)$ is composed of at least 3 connected components, then any consecutive reduction of $I(L_n), I(L_{n-1}), \ldots, I(L_{i+1})$ creates a cut-obstacle at $I(L_i)$.*

This partly motivates next definition.

Definition 3 (Robust)**.** *Let H be a trigraph. We say that $(H_0, H_i, L_i)^n$ is robust if for each 3-ear, say L_i, the descendant of $I(L_i)$ is composed of at most 2 connected components. Moreover, if the descendant is composed of two connected components, then one of them is an isolated vertex adjacent to two vertices in $V(H_0)$.*

We next formally define short ear decompositions that admit a reduction without creating cut-obstacles.

Definition 4. *Let H be a trigraph with a robust short ear decomposition $(H_0, H_i, L_i)^n$. We say $(H_0, H_i, L_i)^n$ is* superb *if there exists a consecutive reduction of $I(L_n), I(L_{n-1}), \ldots, I(L_j)$ for some $j \in [n]$ such that the following properties hold:*

(i) $j = 1$ or $I(L_{j-1})$ cannot be reduced, and

(ii) a cut-obstacle at the internal vertices of a 3-ear in $\{L_{j-1}, L_j, \ldots, L_n\}$ is never created.

The second main result of this paper follows. Its proof is postponed to Section 3.

Theorem 2. *Given a 3-edge-connected cubic bridgeless graph G, one can construct a trigraph $H(G)$ with a robust ear decomposition $(H_0, H_i, L_i)^n$ which, if superb, encodes a DCDC of G.*

In order to understand Theorem 2, we note that due to Theorem 1 the potential obstacles to reducing a short ear are the cut-obstacle and the inner obstacle which can occur only at a 3-ear. Further, the construction of the trigraph $H(G)$ is such that no inner obstacles are ever created. Hence, Theorem 2 claims that the DCDC conjecture follows by showing that each robust ear decomposition of a trigraph is superb.

Problem 1. *Characterize robust ear decompositions of trigraphs that are superb.*

In the next subsection, Theorem 2 is strengthen.

1.2.3 Equivalent ear decompositions

In the proof of Theorem 2 we construct, for each cubic bridgeless graph G, a trigraph H and its robust ear decomposition $(H_0, H_i, L_i)^n$. This construction is ad-hoc. However, we prove that there exists the flexibility to change $(H_0, H_i, L_i)^n$, and even to change H itself while the conclusion of Theorem 2 remains valid. This is explained next.

Definition 5 (Surobust ear decomposition)**.** *Let H be a trigraph and let $(H_0, H_i, L_i)^n$ be a robust ear decomposition of H.*

- *We refer to a 2-ear or a 3-ear as a* base *if all its leaves are in H_0, as an* up *if one leaf belongs to H_0 and the other one to a base, as an* antenna *if exactly one leaf belongs to an up. If L is an antenna and b is its leaf which belongs to an up then the subpath of L of 3 vertices starting by b is called the* handle *of L.*
- *We say that $(H_0, H_i, L_i)^n$ is* surobust *if each 3-ear is either a base, or an up, or an antenna and moreover no 1-ear has both leaves in H_0 or one leaf in a base and second one in an up.*

The following notation helps with the next definition. Given a short ear decomposition $(H_0, H_i, L_i)^n$ of a trigraph, we say that a sequence $(L_{i_j})_{j\in[l]}$ of 2-ears is a *heel* if for every $j \in \{2, \ldots, l\}$ the leaves of L_{i_j} are exactly the internal vertices of $L_{i_{j-1}}$ and the sequence $(L_{i_j})_{j\in[l]}$ is maximal.

Definition 6 (Local exchange)**.** *Let H be a trigraph and let $\mathcal{H} = (H_0, H_i, L_i)^n$ be a surobust ear decomposition of H. A* local exchange *on the pair $H, \mathcal{H}$ is an operation that produces a pair $H', (H_0, H'_i, L'_i)^n$ of a trigraph with its short ear decomposition (see Figure 3) as follows: Let $L_{i_0} \in \{L_1, \ldots, L_n\}$ be a 3-ear antenna with vertices a, w_1, w_2, w_3, b. Let $(L_{i_j})_{j\in[l]}$ be a heel such that the leaves of L_{i_1} are w_2, w_3, and let u be the internal vertex of L_{i_l} that has the shortest distance (in the heel) to w_3. Further let L_k, L_m be the ears such that w_1 is a leaf of L_k and u is a leaf of L_m and assume $\{w_1, w'_1\}$, $\{u, u'\}$ are edges of L_k, L_m, respectively. Then, H', L'_k, L'_m are obtained by deleting the edges $\{w_1, w'_1\}$, $\{u, u'\}$ and adding the new edges $\{w_1, u'\}$, $\{u, w'_1\}$. The other ears do not change, namely $L'_i = L_i$ for all $i \notin \{k, m\}$. Note that $(H_0, H'_i, L'_i)^n$ is surobust and has the same set of handles as $\mathcal{H}$.*

We define an equivalence on the pairs $(H, \mathcal{H})$ where H is a trigraph and $\mathcal{H} = (H_0, H_i, L_i)^n$ is a surobust ear decomposition of H: $(H, \mathcal{H})$ is *equivalent* to $(H', \mathcal{H}')$, if the pair $(H', \mathcal{H}')$ is obtained from $(H, \mathcal{H})$ by a sequence of the following two operations: (1) a modification of the current ear decomposition to another one which is surobust and has the same set of the handles, and (2) a local exchange on the current trigraph.

Remark. *In this remark we explain the reason behind the definition of the local exchange operation. Assume that we are given a 3-edge-connected cubic graph G. In the proof of Theorem 2 we construct trigraph H and its robust ear-decomposition $\mathcal{H}$. Is $\mathcal{H}$ superb? Let us further assume that G has a DCDC. Each DCDC of G defines an embedding of G in an orientable surface with no dual loop. Such embedding induces an ear decomposition of G. Let $(H', \mathcal{H}')$ be the trigraph and its short ear decomposition constructed from the induced ear decomposition by the method of the proof of Theorem 2. Possibly H is not isomorphic to H', but we believe that $(H', \mathcal{H}')$ is equivalent to $(H, \mathcal{H})$ and $\mathcal{H}'$ is superb.*

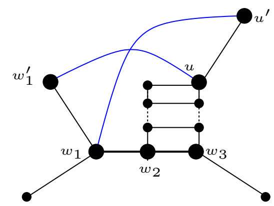


Figure 3: Illustration of a local exchange: edges $\{w_1, w'_1\}$, $\{u, u'\}$ are deleted, and new edges $\{w_1, u'\}$, $\{u, w'_1\}$ (the ones in blue) are added. The case depicted corresponds to the one that the last ear of the heel, namely L_{i_l}, is a 2-ear.

1.2.4 Main Contribution

In general, aside of cut-obstacles, there are many other obstacles for the existence of a reduction. The main contribution of this paper, is a strengthening of Theorem 2 which claims that *avoidance of cut-obstacle at 3-ears in a very restricted setting* is sufficient for proving the DCDC conjecture.

Conjecture 1 (Cut avoidance conjecture)**.** *Let H be a trigraph and let $\mathcal{H}$ be a surobust ear decomposition of H. Then there is an equivalent pair $(H', \mathcal{H}')$ such that $\mathcal{H}'$ is superb.*

The main result of this paper is the following.

Theorem 3. *If Conjecture 1 holds then the DCDC conjecture holds in general graphs.*

The following sections are devoted to the proofs of Theorem 1, Theorem 2 and Theorem 3.

2 Proof of Theorem 1

Lemmas 4 and 5 imply Theorem 1.

OK now let us formulate the main Lemma. Here it is. It is the most important ingredient in the proof of the

Lemma 4. *Let $H = (V, E, A, R)$ be a mixed graph and L be a 2-ear of H. Then, there is a reduction of $I(L)$.*

Proof. Let $I(L) = \{x, y\}$ be the 2-ear and $u, v \in V - \{x, y\}$ be the vertices such that $\{x, u\}, \{y, v\} \in E$. Suppose at least one arc of A has both end vertices in $\{x, y\}$. Without loss of generality we can assume that $(x, y) \in A$. If $(y, x) \in A$, then a reduction of $\{x, y\}$ is given by the directed cycle (x, y, x) containing $(x, y) \in A$, the directed path (v, y, x, u) containing $(y, x) \in A$ and the directed path (u, x, y, v). If not, let x' and $\tilde{y}$ the ends of the arcs in A containing x and y, respectively. In this case, a reduction is given by the directed cycle (x, y, x) containing $(x, y) \in A$, the directed path (x', x, u) containing $(x', x) \in A$, the directed path $(v, y, \tilde{y})$ containing $(y, \tilde{y}) \in A$ and the directed path (v, y, x, u).

Now assume all arcs of A have exactly one end vertex in $\{x, y\}$. Let $x', y', \tilde{x}, \tilde{y}$ be the ends of the arcs in A such that $(x', x), (y', y), (x, \tilde{x}), (y, \tilde{y}) \in A$. In this situation, a reduction is given by the directed path (x', x, y, v) containing $(x', x) \in A$, the directed path (y', y, x, u) containing $(y', y) \in A$, the directed path $(u, x, \tilde{x})$ containing $(x, \tilde{x}) \in A$ and $(v, y, \tilde{y})$ containing $(y, \tilde{y}) \in A$. □

Lemma 5. *Let $H = (V, E, A, R)$ be a mixed graph and L be a 3-ear of H. If there is neither a cut-obstacle nor an inner obstacle at $I(L)$, then there is a reduction of $I(L)$.*

Proof. Let $I(L) = \{x, z, y\}$ and $u, v \in V - \{x, z, y\}$ such that $\{x, z\}, \{z, y\}, \{u, x\}, \{v, y\} \in E$. Further, for $a \in \{x, z, y\}$, let a', $\tilde{a} \in V - \{x, z, y\}$ be the vertices such that $(a', a) \in A$ and $(a, \tilde{a}) \in A$.

In order to prove the lemma, we distinguish the following three cases.

(i) There are no arcs in A incident with exactly one vertex of $\{x, z, y\}$. Without loss of generality, $(z, x), (y, z)$ and (x, y) are in A. In this situation, a reduction is given by the directed cycles (x, z, x) and (y, z, y) containing (z, x) and (y, z) in A, respectively, the directed path (u, x, y, v) containing (x, y) and the directed path (v, y, z, x, u).

(ii) There is exactly one arc in A with both end vertices in $\{x, z, y\}$; let us denote it by $\vec{e}$. Without loss of generality, $\vec{e} = (x, y)$ or $\vec{e} = (x, z)$.

In the case that $\vec{e} = (x, y)$, a reduction is given by the directed cycle (x, y, z, x) and the directed paths (x', x, u), $(u, x, z, \tilde{z}), (z', z, y, v), (v, y, \tilde{y})$. In the case that $\vec{e} = (x, z)$, a reduction is given by the directed cycle (x, z, x) and the directed paths (x', x, u), $(u, x, z, y, \tilde{y}), (y', y, v)$ and $(v, y, z, \tilde{z})$.

(iii) Exactly two arcs in A are incident with exactly one vertex of $\{x, z, y\}$. Hence, there are exactly two arcs in A with both end vertices in $\{x, z, y\}$.

First let us examine the case that both arcs connect the same pair of vertices. In the case that the arcs connect x, y, a reduction is given by the directed cycle (y, z, x, y) and the directed paths $u, x, z, \tilde{z}$, z', z, y, v and v, y, x, u. In the case that the arcs connect x, z (by symmetry for z, y), a reduction is given by the directed cycle (x, z, x) and the directed paths $u, x, z, y, \tilde{y}$, y', y, v and v, y, z, x, u.

Suppose now that the arcs connect distinct pair of vertices. The first case is that one arc connects x, z while the other connects z, y. Without loss of generality suppose $(x, z), (z, y) \in A$. A reduction is given by the set $\mathcal{S}$ consisting of the directed cycles (x, z, x), (z, y, z), and the directed paths $(u, x, z, y, v), (x', x, u)$ and $(v, y, \tilde{y})$. The second case is that one arc connects x, y while the other connects z, y. Without loss of generality $(x, y), (y, z) \in A$. Since there is no inner obstacle at $\{x, z, y\}$, the pair of arcs (x, y), $(z, \tilde{z})$ in A is not a forbidden pair in R. Hence, a reduction of $\{x, z, y\}$ is witness by the set consisting of the directed path $(u, x, y, z, \tilde{z})$ which contains $(x, y), (z, \tilde{z})$ from A (only) and the directed paths (x', x, z, y, v), (v, y, z, x, u) each containing at most one arc from A.

□

3 Construction and properties of the trigraph $H(G)$

It is a well-know fact that the DCDC conjecture holds if and only if it holds in the class of 3-edge-connected cubic graphs. Hence we concentrate on 3-edge-connected graphs in this section.

3.1 Ear decompositions of 3-edge-connected cubic graphs

Let G be a cubic bridgeless graph and $(G_0, G_i, P_i)^l$ be an ear decomposition of G. For each $t \leq l$, we say that $(G_0, G_i, P_i)^t$ is a partial ear decomposition of G. We first need to generalize the concept of descendant. Let $i \in [l]$ such that $|I(P_i)| \geq 3$ and $V(P_i) = \{\alpha_i, v_i^1, \ldots, v_i^k, \beta_i\}$, where $V(P_i) \cap V(G_{i-1}) = \{\alpha_i, \beta_i\}$. We say that $S \subset \{v_i^1, \ldots, v_i^k\}$ is a *segment* of P_i if $|S| \geq 3$ and S induces a connected subgraph of P_i. Let G'_i denote the graph obtained from G by deleting $V(G_i)$. For each segment S of P_i, the maximal subgraph G_i^S of G'_i such that $G[V(G_i^S) \cup S]$ is connected is called the *descendant of* S.

Definition 7 (connected ear decomposition)**.** *An ear decomposition, say $(G_0, G_i, P_i)^l$, of G is* connected *if the following conditions hold:*

(a) $G - V(G_0)$ is a connected graph, and

(b) for each $i \in [l]$ such that $|I(P_i)| \geq 3$ and for each segment S of P_i, the descendant of S is connected.

The next lemma (whose proof is in [1]) implies that each 3-edge-connected cubic graph admits a connected ear decomposition.

Lemma 6. *Let G be a 3-edge-connected cubic graph. There exist $e \in E(G)$ and an ear decomposition $(G_0, G_i, P_i)^l$ of $G - e$ such that P_l is a path of length 2, and for each $i \in \{0, \ldots, l-1\}$, denoting $V_i = V(G_0) \cup V(P_1) \cup \cdots \cup V(P_i)$, the graph $G[V - V_i]$ is connected.*

3.2 Initial step

Let G be a cubic graph and $(G_0, G_i, P_i)^k$ be an ear decomposition of G. Let v_0 be a fixed vertex of G_0. Let v_1 and v_2 denote the neighbours of v_0 in G_0. We obtain a cubic graph G' from G by subdividing edges $\{v_0, v_1\}$ and $\{v_0, v_2\}$ into $\{v_0, x_0\}$, $\{x_0, v_1\}$ and $\{v_0, y_0\}$, $\{y_0, v_2\}$, respectively, and adding the new edge $\{x_0, y_0\}$; this operation is known as a Y-Δ operation. Clearly, G is 3-edge-connected if and only if G' is, and G has a DCDC if and only if G' does so.

The cubic graph G' admits the ear decomposition starting at the triangle $T = (x_0, y_0, v_0)$, and with ears $P_0, P_1, \ldots, P_k$, where P_0 is the path obtained from the cycle G_0 by deleting v_0 and adding the edges $\{x_0, v_1\}$, $\{y_0, v_2\}$. If the ear decomposition $(G_0, G_i, P_i)^k$ of G is connected then so is the ear decomposition $(T, G'_i, P_i)_0^k$ of G'.

In the rest of this section we focus on such graphs G'. For each $i \in \{0, 1, \ldots, k\}$, the notation a_i, c_i stands for the end vertices of P_i, whenever P_i is a path; in particular $\{a_0, c_0\} = \{x_0, y_0\}$ is the set of the end vertices of P_0.

3.3 The construction of trigraph $H(G)$

Let H_0 be a cycle on $n(G)$ vertices, with $n(G)$ large enough. We choose 3 distinct vertices from $V_0 := V(H_0)$.

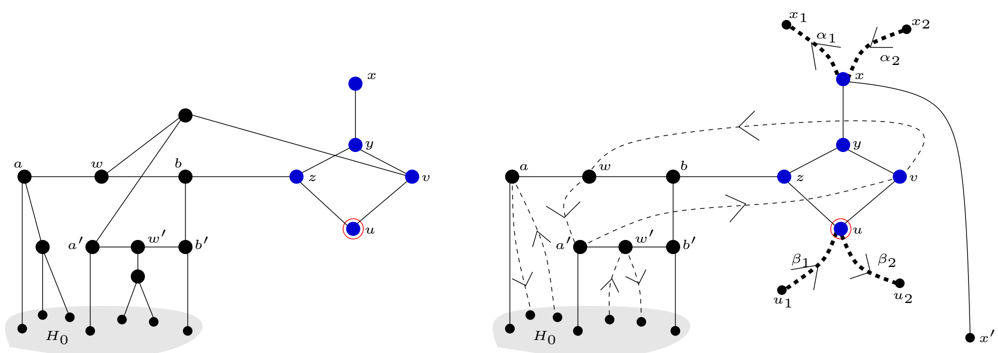
The following building block comes in handy to describe the construction of $H(G)$.

Definition 8 (Basic gadget)**.** *A* basic gadget $\mathcal{B} = \mathcal{B}(x, u)$ *is a sequence* $E_1, E_2, E_3', E_4, D_1, D_2, D_3$ *of short ears which, considering the vertices named according to Figure 4(a), are defined as follows:*

(i) $I(E_1) = \{a', w', b'\}$, $I(E_2) = \{a, w, b\}$, $V(E_3') = \{b, z, y, x\}$, $V(E_4) = \{z, u, v, y\}$, *and the end vertices of* E_1 *and* E_2 *belong to* H_0*,*

(iii) D_1, D_2, D_3 *are stars,* $\{a', w, v\}$ *is the set of leaves of* D_1*, and each star* D_2*,* D_3 *has two leaves in* H_0*. Moreover, vertex* w' *is a leaf of* D_2 *and vertex* a *is a leaf of* D_3*.*

Depending on the context, a basic gadget may also refer to the graph obtained by the union of the ears $E_1, E_2, E_3', E_4, D_1, D_2, D_3$*. In addition, we say that the path on vertex set* $\{b, z, y\}$ *is the* fixed path*,* u *is the* replica *and* x *is the* joint *of the basic gadget; whenever only the replica vertex* u *is specified, we denote by* x_u *the corresponding joint vertex.*



(a) Basic gadget $\mathcal{B}(x, u)$

(b) Gadget after reduction of its stars.

Figure 4

We now describe the recursive construction of $H(G)$, along with a function Γ that maps the set of the vertices and edges of $G' \setminus T$ into a subset of the vertices and edges of $H(G)$. In addition, we define a short ear decomposition of $H(G)$ that we call *canonical*.

First, the canonical ear decomposition of $H(G)$ starts with the cycle H_0.

Set $V_0 = \{\Gamma(x_0), \Gamma(y_0), \Gamma(v_0)\}$. According to the following rules, for each $i \in \{0, 1, \ldots, k\}$, we obtain H_{t_i} from $H_{t_{i-1}}$ and the list $\mathcal{L}_{t_i}$ of short ears of the canonical ear decomposition of $H(G)$ that generates H_{t_i} from $H_{t_{i-1}}$; under this notation, $H_{t_{-1}} = H_0$.

- Case that P_i is a star with three leaves. Let s denote the center of P_i and let x, y, z denote its leaves. Graph H_{t_i} is obtained from $H_{t_{i-1}}$ by adding the new vertex u, and the new edges $\{u, \Gamma(x)\}$, $\{u, \Gamma(y)\}$ and $\{u, \Gamma(y)\}$. We let $\Gamma(s) = u$. For $w \in \{x, y, z\}$ let $\Gamma(\{s, w\}) = \{\Gamma(s), \Gamma(w)\}$. Further, $\mathcal{L}_{t_i}$ contains only one element: the star on vertex set $\{u, \Gamma(x), \Gamma(y), \Gamma(y)\}$.

- Case that P_i is a path. Let $a_i, b_{i_1}, \ldots, b_{i_l}, c_i$, with $l \geq 1$, be the sequence of vertices of path P_i. In order to obtain H_{t_i} from $H_{t_{i-1}}$, we first add *substantial path* Q_i, which is defined by the sequence of vertices $\Gamma(a_i), x_1, \ldots, x_l, \Gamma(c_i)$, where $x_1, \ldots, x_l$ are l new vertices. Then to each x_i, we connect a basic gadget graph $\mathcal{B}_i = \mathcal{B}(x_i, u_i)$; consequently, each of them is referred to as a *basic gadget of* $H(G)$. Set $\Gamma(b_{i_j}) = u_j$, for each $j \in \{1, \ldots, l\}$. Let ϕ denote the natural isomorphism between P_i and Q_i that maps a_i to $\Gamma(a_i)$ and c_i to $\Gamma(c_i)$. For each edge $\{u, v\} \in E(P_i)$, let $\Gamma(\{u, v\}) = \{\phi(u), \phi(v)\}$. Note that, under this setting, $\Gamma(\{u, v\}) \neq \{\Gamma(u), \Gamma(v)\}$. Finally, $\mathcal{L}_{t_i} = (R_1, \ldots, R_l)$, where for each $1 \leq j < l$, R_j is the list of ears

$$E_1(\mathcal{B}_j), E_2(\mathcal{B}_j), E_3, E_4(\mathcal{B}_j), D_1(\mathcal{B}_j), D_2(\mathcal{B}_j), D_3(\mathcal{B}_j), \tag{1}$$

where E_3 is the path obtained by the union of $E_3'(\mathcal{B}_j)$ and $\{x_j, x_{j-1}\}$; for $j = 1$ we let $x_0 = \Gamma(a_i)$. The last list R_l consists of the ears

$$E_1(\mathcal{B}_l), E_2(\mathcal{B}_l), F, E_3'(\mathcal{B}_l), E_4(\mathcal{B}_l), D_1(\mathcal{B}_l), D_2(\mathcal{B}_l), D_3(\mathcal{B}_l), \tag{2}$$

where F is the path on vertex set $\{x_{l-1}, x_l, \Gamma(c_i)\}$.

The following two observations are straightforward.

Observation 4. *If the ear decomposition of G' is connected, then the canonical ear decomposition of $H(G)$ is surobust.*

Observation 5. *Let $\mathcal{H}$ be the canonical ear decomposition of $H = H(G)$ and let $(H', \mathcal{H}')$ be equivalent to $(H, \mathcal{H})$. Then basic gadgets of H are disjoint subgraphs of H', and $G' \setminus T$ is obtained from $H' \setminus H_0$ by contracting each basic gadget of H to a single vertex.*

Observation 5 implies that the definition of function Γ can be extended from a canonical ear decomposition to every equivalent ear decomposition. This is formalized in the next definition. Note that Definition 9 is consistent with the definition of function Γ for canonical ear decompositions.

Definition 9. *Let $\mathcal{H}$ be the canonical ear decomposition of $H = H(G)$ and let $(H', \mathcal{H}')$ be equivalent to $(H, \mathcal{H})$. We define function Γ from the set of vertices and edges of G' to the set of vertices and edges of H' as follows.*
(1) $\Gamma(T)$ is the same as for H (since H_0 is the initial cycle of $\mathcal{H}'$).
(2) If $e \in E(G' \setminus T)$, then $\Gamma(e)$ is the edge of H' which corresponds to e after the contraction of all the basic gadgets of H in H'.
(3) Let $v \in V(G') \setminus T$ and let W be the subset of vertices of H' such that the contraction of W to a single vertex corresponds to v. If W is a single vertex, say $W = \{w\}$, then $\Gamma(v) = w$. Otherwise, W is the vertex set of a basic gadget, and we let $\Gamma(v) = u$, where u is the replica vertex of the basic gadget.

3.4 Gadget analysis

Let $(H_0, H_i, L_i)^n$ be a short ear decomposition of a trigraph. A subsequence $\mathcal{G}$ of $(L_1, \ldots, L_n)$ is called *gadget* if

$$\mathcal{G} = E_1(\mathcal{B}), E_2(\mathcal{B}), E_3, E_4(\mathcal{B}), D_1(\mathcal{B}), D_2(\mathcal{B}), D_3(\mathcal{B})$$

for some basic gadget $\mathcal{B} = \mathcal{B}(x, u)$; as in (1), E_3 is the path obtained by the union of $E'_3(\mathcal{B})$ and $\{x', x\}$ with x' a neighbour of x not in $V(\mathcal{B})$. Naturally, we use the terminology defined for basic gadgets on the gadgets as well. We consider the vertices of $\mathcal{G}$ named according to Figure 4(b).

A *reduction process* of gadget $\mathcal{G}$ is a consecutive reduction of a sequence

$$\mathcal{G}' = I(D_3), I(D_2), I(D_1), I(E_4), \ldots, I(E_j)$$

for some $1 \leq j \leq 4$ such that (i) $j = 1$ or (ii) $j > 1$ and $I(E_{j-1})$ does not have reduction or $I(E_{j-1})$ is a cut-obstacle. If (i) holds, we say that the reduction process is *complete*. Otherwise, we refer to it as *j-incomplete*.

From now on let $(H_0, H_i, L_i)^n$ be a short ear decomposition of a trigraph H and let $\mathcal{G}$ be a gadget of H. Let us assume that $D_3 = L_m$ and we make the following assumption.

Assumption 1. *There exists a consecutive reduction of $I(L_n), \ldots, I(L_{m+1})$ on H.*

Let H' denote the mixed graph obtained by a consecutive reduction of $I(L_n), \ldots, I(L_{m+1})$. Let us recall that $V(H_0), I(L_1), \ldots, I(L_m)$ is a partition of the vertex set of H'. By the definition of gadget, we have that the vertices x and u (recall that we are considering vertices named according to Figure 4(b)) have degree 2 in H', and thus, each of them is the tail of one arc and the head of one arc. Let us denote such arcs, according to Figure 4(b), by $\alpha_1 = (x, x_1)$, $\alpha_2 = (x_2, x)$ and $\beta_1 = (u_1, u), \beta_2 = (u, u_2)$. Our aim is to prove the following statement.

Theorem 7. *Each reduction process of $\mathcal{G}$ on H' satisfies exactly one of the following statements.*

(i) It is complete and either $\alpha_1 = \beta_1$, or $\alpha_2 = \beta_2$.

(ii) It is j-incomplete and $I(E_{j-1})$ is a cut-obstacle for some $2 \leq j \leq 4$.

In addition, if Statement (i) holds and $\alpha_1 = \beta_1$ (resp. $\alpha_2 = \beta_2$), then α_2 and β_2 (resp. α_1 and β_1) belong to the two distinct reducible paths (defined by the reduction process) that contain (x, x') and (x', x).

Proof. Let us first suppose that a complete reduction process of $\mathcal{G}$ has been performed on H'. In order to prove the statement of the theorem we need to show that either $\alpha_1 = \beta_1$, or $\alpha_2 = \beta_2$. For that, let us examine the single reductions involved in the consecutive reduction on $\mathcal{G}$ that witness the existence of the considered complete reduction process. By definition of complete reduction process, no cut-obstacles at 3-ears are created. Without loss of generality, by symmetry of the stars, we can assume that the local configuration depicted in Figure 4(b) is the one generated by the reduction of $I(D_3), I(D_2)$ and $I(D_1)$. Let $\tilde{H}$ denote the obtained mixed graph. Moreover, let $e = (v, w)$ and $e' = (a', v)$ in $\tilde{H}$; recall again that we consider vertices named according to Figure 4(b). We claim that the following holds.

Observation 6. *In each consecutive reduction of $I(E_4), I(E_3), I(E_2), I(E_1)$ on $\tilde{H}$ that creates no cut-obstacles at $I(E_3)$, $I(E_2)$ and $I(E_1)$, the arc e belongs to the reducible path that contains (b, z) and e' belongs to the reducible path that contains (z, b).*

Proof of Observation 6. The hypothesis that no cut-obstacles at $I(E_3)$, $I(E_2)$ and $I(E_1)$ are created, implies that the consecutive reduction of $I(E_4), I(E_3)$ generates at least one arc with both end vertices in $I(E_2) = \{a, w, b\}$, otherwise we have that $I(E_2)$ is a cut-obstacle. In order to get such an arc, the consecutive reduction of $I(E_4), I(E_3)$ is so that e belongs to the reducible path that contains (b, z). Now, for the sake of contradiction, we assume that in a consecutive reduction of $I(E_4), I(E_3)$ which does not create cut-obstacles, the directed edge e' does not belong to the reducible path that contains (z, b). Therefore, without loss of generality, we can assume that the configuration locally depicted in Figure 7(b) is obtained by the consecutive reduction of $I(E_4), I(E_3)$; up to some different location of $\{u_1, x_1, x'\}$. On the obtained mixed graph there exists a reduction of $I(E_2)$ that does not create a cut-obstacle at $I(E_1)$. If so, the reduction of $I(E_2)$ is so that the arc (w, a') belongs to the reducible path that contains (b', b). Because of the fact that (b, w), (w, a') is a forbidden pair of arcs, the reducible path (b', b, w, a') exists and does not contain the arc (b, w). Moreover, since the pair (b, w), (u_1, b) is also forbidden, there exists the reducible path (u_1, b, b'). But then, it holds that the arc (b, w) and the remaining arc (w, b) (arising from the edge $\{w, b\}$) belong to the same reducible path or cycle. However, it does not correspond to a reduction since (a, w) and (w, a) are forced to be in the same cycle. □

It is a routine to check that the following list of reductions (encoded by their reducible paths and cycles) correspond to all possible 4 distinct reductions of $I(E_4)$.

Reduction 1: (y, v, w), (a', v, u, z), (z, u, u_2) and (u_1, u, v, y). See Figure 5(a).

Reduction 2: (z, u, v, w), (a', v, y), (y, v, u, u_2) and (u_1, u, z). See Figure 5(b).

Reduction 3: (z, u, v, y), (y, v, w), (u_1, u, z) and (a', v, u, u_2). See Figure 5(c).

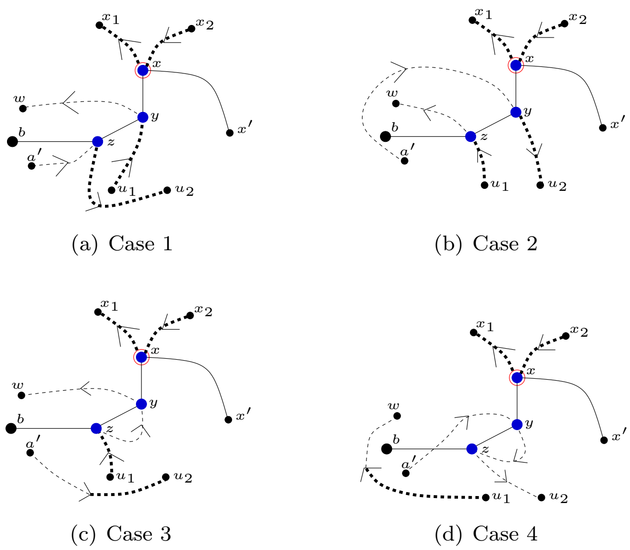
Reduction 4: $(y, v, u, z), (a', v, y), (z, u, u_2)$ and (u_1, u, v, w). See Figure 5(d)

Recall that we aim to prove that either $\alpha_1 = \beta_1$, or $\alpha_2 = \beta_2$.

We first study reduction 1 (the analysis of reduction 2 follows in an analogous way). Since we are examining the steps of a complete reduction process of $\mathcal{G}$ on H', we have that $I(E_3)$ is not a cut-obstacle; thus, there exists $i \in \{1, 2\}$ such that $\alpha_i = \beta_i$. Hence the study of reduction 1 involves the following three cases: (i) $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$, (ii) $\alpha_1 \neq \beta_1$ and $\alpha_2 = \beta_2$, and (iii) $\alpha_1 = \beta_2$ and $\alpha_1 = \beta_2$.

By Observation 6, if case (i) holds, then a reduction of $I(E_3)$ takes the reducible path (b, z, y, w) and makes that e' belongs to the reducible path that contains (z, b). Therefore, in a reduction of $I(E_3)$ which does not create cut-obstacles the following holds: the arc (z, u_2) (α_2, respectively) is in the reducible path that contains (x', x) ((x, x'), respectively); thus the statement of Theorem 7 follows.

Now we study case (ii). Recall that reduction 1 is depicted in Figure 5(a); to obtain case (ii) we need to set $(z, x) = (z, u_2) = (x_2, x)$. Let us suppose that there exists a reduction of $I(E_3)$ that does not create cut-obstacles. By Observation 6, this implies that e and (b, z) belong to the same reducible path. Since the pair e, (z, x) is not a forbidden pair of arcs, then there are two potential reducible paths such that e and (b, z) could belong to (in case that the required reduction exists); namely e belongs to either (b, z, y, w), or to (b, z, x, y, w). If the reduction takes



(a) Case 1 (b) Case 2

(c) Case 3 (d) Case 4

Figure 5: Resulting configurations of all possible reductions of $I(E_4)$. The arc incident to w contains e and the arc incident to a' contains e'.

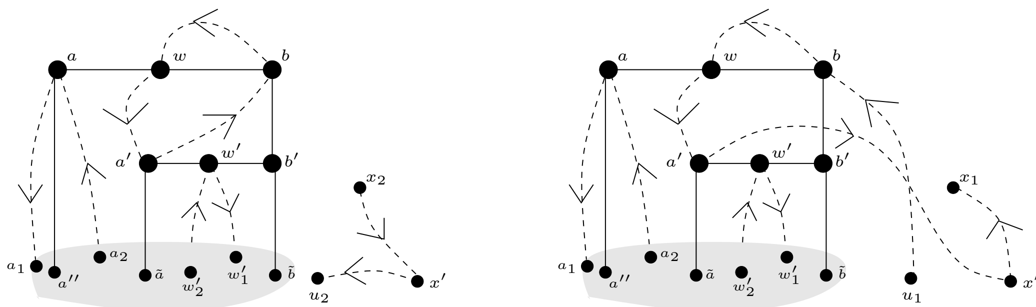
the reducible path (b, z, y, w), then by Observation 6 this reduction also takes the reducible path (a', z, b). We complete the proof using the same argument as for case (i); meaning, if a reduction of $I(E_3)$ exists then (u_1, y) (α_1, respectively) belongs to the reducible path that contains (x, x') ((x', x), respectively). We now suppose that the reduction takes the reducible path (b, z, x, y, w). This forces the reduction to consider the following reducible paths:

$$(u_1, y, z, b), (a', z, y, x, x'), (x', x, x_1).$$

However, this contradicts Observation 6, since e' and (z, b) must belong to the same reducible path.

We now move to study case (iii). We can obtain a local sketch of this case if we set $(x, y) = \alpha_1 = (u_1, y)$ and $(z, x) = \alpha_2 = (z, u_2)$, in Figure 5(a). First of all, by Observation 6, the reducible path (a', z, b) is created and therefore, the reducible path (b, z, y, w) exists as well. Hence, the reducible cycle (y, x, z, y) exists. It implies that the arcs (x, x'), (x', x) (arising from the edge $\{x, x'\}$) are in the same cycle; a contradiction to the definition of reduction.

Let us study reductions 3 and 4. By Observation 6, on the one hand, in the case of reduction 3 we have $(a', u_2) = (x_2, x)$ in Figure 5(c). On the other hand, in the case of reduction 4 we must have $(u_1, w) = (x, x_1)$ in Figure 5(d). Therefore, reductions 3 and 4 are analogous and it suffices to study one of them. We study reduction 3. In any reduction of $I(E_3)$ that does not create cut-obstacles, there exists only one possible reducible path, namely (b, z, y, w), such that Observation 6 is satisfied; because if there were a different reducible path, then this reducible path would contain (u_1, z) and would required that $(u_1, z) = (x, x_1)$, but such a path cannot contain (b, z) since the direction of (u_1, z) is opposite to the one of (b, z) in any potential reducible path. Moreover, since the pair (u_1, z), (z, y) is forbidden, then the reducible path (u_1, z, b) exists. Hence, (a', x) is not in



(a) A reduction of $V(E_3)$ in Case 1(i).

(b) A reduction of $V(E_3)$ in Case 1(ii).

Figure 6

the reducible path containing (z, b), a contradiction to the statement of Observation 6.

This conclude the first part of the proof of Theorem 7.

For the second part of the proof, we consider an j-incomplete reduction process of $\mathcal{G}$ for some $2 \leq j \leq 4$ and have to show that it implies the existence of a cut-obstacle at $I(E_{j-1})$. By definition of j-incomplete reduction process and Theorem 1, the desired result follows from the fact that no consecutive reduction of $I(D_3), I(D_2), I(D_1), I(E_4), \ldots, I(E_j)$ on H' creates an inner obstacle at $I(E_{j-1})$ (see Definition 2). A quick examination of Figure 4(b) show us that if $j \in \{2, 3\}$, then there is no inner obstacle at $I(E_{j-1})$; in the case that $j = 2$ (resp. $j = 3$), note that all arcs incident to w' (resp. a) are not incident to any other vertex of $I(E_1)$ (resp. $I(E_2)$). In the case that $j = 4$, if an inner obstacle at $I(E_3)$ were created, then it would be required that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, because in an inner obstacle there are exactly 2 arcs that do not have all its end vertices in $I(E_3)$. Moreover, it would be needed that the arc that connects x and z forms a forbidden pair with the arc that has exactly one-end vertex in $I(E_3)$ and this end vertex is y; however, this does never occur. □

The following theorem states that the condition either $\alpha_1 = \beta_1$, or $\alpha_2 = \beta_2$ is also sufficient for the existence of a complete reduction process of $\mathcal{G}$.

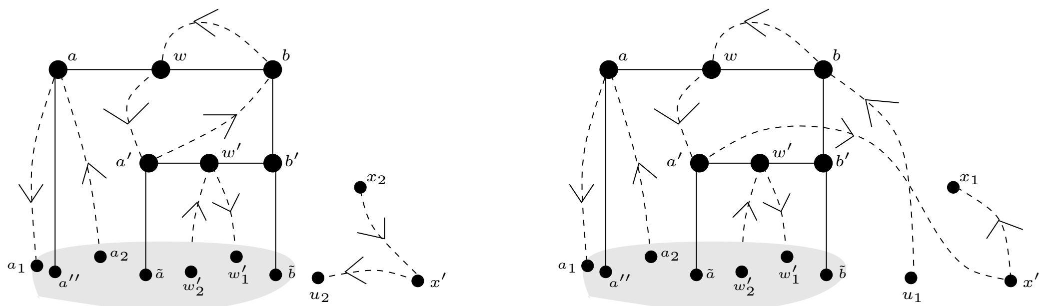
Theorem 8. *There exists a complete reduction process of $\mathcal{G}$ if and only if either $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$, or $\alpha_2 = \beta_2$ and $\alpha_1 \neq \beta_1$.*

Proof. The necessity of the condition is stated in Theorem 7. It remains to show that the condition is sufficient. Without loss of generality, $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$ can be assumed. We want to prove that a consecutive reduction of $\mathcal{G}^{-1}$ that creates no cut-obstacles at $I(E_3)$, $I(E_2)$ and $I(E_1)$ exists.

By the proof of Theorem 7, it is possible to reduce $I(D_3), I(D_2), I(D_1), I(E_4)$ in such a way that we end up in Case 1 (depicted in Figure 5(a)). We consider the notation from Figure 5(a); recall that we have $(x, x_1) = (u_1, y)$. We reduce $I(E_3)$ in such a way that the following list of reducible paths and cycles determines the reduction:

$$(b, z, y, w), \quad (a', z, b), \quad (x', x, y, z, u_2), \quad (x_2, x, x'), \quad (x, y, x).$$

By this reduction we obtain the configuration depicted in Figure 7(a).



(a) A reduction of $I(E_3)$ in Case 1(i).

(b) A reduction of $I(E_3)$ in Case 1(ii).

Figure 7

After this reduction of $I(E_3)$, we complete the consecutive reduction by reducing $I(E_2)$ and $I(E_1)$, respectively by the reductions determined by the lists of reducible paths and cycles:

$$(b, w, b), (a', b, b'), (b', b, w, a, a_1), (a'', a, w, a'), (a_2, a, a'') \text{ and}$$

$$(a'', a', \tilde{a}), (\tilde{a}, a', w', w'_1), (w'_2, w', b', \tilde{b}), (\tilde{b}, b', a_1), (a', b', w', a'), \text{ respectively.}$$

□

3.5 Relatives of a gadget

Next we define and analyze modifications of the gadget which are needed later. First we note that, in the construction of the canonical ear decomposition of $H(G)$, there are two blocks of ears involved. In addition to the gadget, a slightly different block of ears arises (namely, block (2)); we shall refer to such subsequence as a *1-gadget**. In addition, we introduce the *2-gadget*, 1-gadget**, 2-gadget*** and the *double gadget*.

Let $(H_0, H_i, L_i)^n$ be an ear decomposition of a trigraph. A subsequence $\mathcal{G}$ of $(L_1, \ldots, L_n)$ is called *1-gadget** if

$$\mathcal{G} = E_1(\mathcal{B}), E_2(\mathcal{B}), F, E'_3, E_4(\mathcal{B}), D_1(\mathcal{B}), D_2(\mathcal{B}), D_3(\mathcal{B})$$

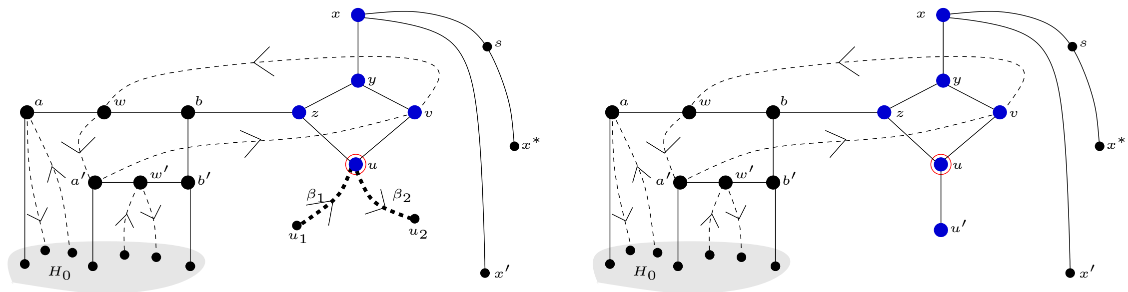
for some basic gadget $\mathcal{B} = \mathcal{B}(x, u)$; as in (2), F is the 1-ear with internal vertex x and the end vertices not in $V(\mathcal{B})$.

In the case that F is a 2-ear with internal vertices x, s and s is neither a joint vertex, nor a replica vertex of a basic gadget of H, we say that $\mathcal{G}$ is a *2-gadget** (see Figure 8(a)). We refer to 1-gadgets* and to 2-gadgets* as *gadgets**.

In addition to the gadgets*, we need to introduce *gadgets***. Each gadget** is obtained from a gadget* by removing the 2-ear E_4 and adding two new 1-ears instead. Hence a gadget**

$$\mathcal{G} = F, E_1(\mathcal{B}), E_2(\mathcal{B}), E_3(\mathcal{B}), F_1, F_2, D_1(\mathcal{B}), D_2(\mathcal{B}), D_3(\mathcal{B}),$$

where F is as described for gadgets* (accordingly we have 1-gadgets** and 2-gadgets**), F_2 is the 1-ear with vertex set $\{y, v, u\}$ and F_1 is the 1-ear with vertex set $\{z, u, u'\}$ with u' the neighbour



(a) 2-gadget* after reduction of its stars.

(b) 2-gadget** after reduction of its stars.

Figure 8: A 2-gadget* (resp. 2-gadget**) contains the 2-ear on vertex set $\{x', x, s, x^*\}$.

of u that is not in $V(\mathcal{B})$ (as described in Figure 8(b)). In the case of a 2-gadget* or a 2-gadget**, we say that vertex s is the *pending vertex*.

Let again $D_3(\mathcal{B}) = L_m$ and **Assumption 1** holds. If $\mathcal{G}$ is a gadget*, suppose that β_1 and β_2 are the arcs incident to the replica vertex of $\mathcal{G}$ with the orientations according to Figure 8(a). If $\mathcal{G}$ is a gadget**, let β_1 denote the arc (u', u) and β_2 the arc (u, u').

The gadget* and the gadget** is analyzed analogously as the gadget, and we get the theorem below. Its proof is the same as the proofs of Theorems 7 and 8 and we do not include it.

Theorem 9. *Let $\mathcal{G}$ be a gadget* or a gadget**. If there exists a consecutive reduction of $\mathcal{G}^{-1}$ that creates no cut-obstacles at a 3-ear from $\mathcal{G}$, then β_1 and β_2 belong to the two distinct reducible paths that contain (y, x) and (x, y), respectively. Moreover, such a consecutive reduction always exists.*

Next we introduce the last relatives of a gadget. We refer to them as *double gadgets* and illustrate them in Figure 9. A subsequence $\mathcal{D}$ of ears of $L_1, \ldots, L_n$ is called a *double gadget* if

$$\mathcal{D} = F', \mathcal{G}_{\mathcal{B}} - F(\mathcal{G}_{\mathcal{B}}), \mathcal{G}_{\mathcal{B}'} - F(\mathcal{G}_{\mathcal{B}'})$$

where $\mathcal{B} = \mathcal{B}(x, u)$, $\mathcal{B}' = \mathcal{B}'(x', u')$ are basic gadgets, $\mathcal{G}_{\mathcal{B}} - F(\mathcal{G}_{\mathcal{B}})$ and $\mathcal{G}_{\mathcal{B}'} - F(\mathcal{G}_{\mathcal{B}'})$ are obtained from gadgets* or gadgets** $\mathcal{G}_{\mathcal{B}}$ and $\mathcal{G}_{\mathcal{B}'}$ associated to $\mathcal{B}$ and $\mathcal{B}'$ by removing two ears $F(\mathcal{G}_{\mathcal{B}})$ and $F(\mathcal{G}_{\mathcal{B}'})$ and F' is a 2-path with internal vertices x, x' and end vertices not in $V(\mathcal{G}_{\mathcal{B}} \cup \mathcal{G}_{\mathcal{B}'})$. We also write $\mathcal{D} = \mathcal{D}(\mathcal{B}, \mathcal{B}')$.

As before we suppose that the last ear of the double gadget $\mathcal{D}$ is L_m and that the **Assumption 1** holds. Note that in $\mathcal{D}(\mathcal{B}, \mathcal{B}')$, when $\mathcal{G}_{\mathcal{B}}$ and $\mathcal{G}_{\mathcal{B}'}$ are gadgets*, the replica vertices of $\mathcal{B}$ and $\mathcal{B}'$ have degree 2 in the mixed graph obtained by the consecutive reduction of $I(L_n), \ldots I(L_{m+1})$ and therefore each of them is incident to two arcs, say β_1, β_2 in the case of the replica vertex of $\mathcal{B}$, and β_1', β_2' in the case of the replica vertex of $\mathcal{B}'$ (their directions are as in Figure 9). In the case that $\mathcal{G}_{\mathcal{B}}$ is a gadget**, we have u has degree 3. If u'' denotes the neighbour of u that does not belong to $V(\mathcal{B})$, we denote $\beta_1 = (u'', u)$ and $\beta_2 = (u, u'')$; analogously for $\mathcal{G}_{\mathcal{B}'}$. We point out that possibly $\beta_1 = \beta_2'$ or $\beta_2 = \beta_1'$.

As expected double gadgets have the same behavior with respect to the reduction process as gadgets do. Therefore the following statement for double gadgets follows.

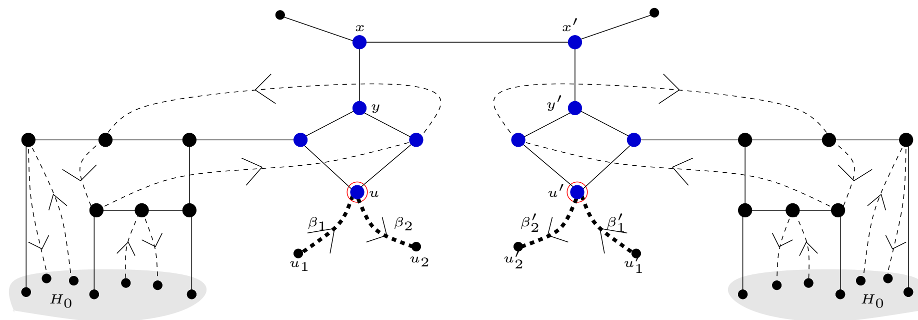


Figure 9: Example of a double gadget where $\mathcal{G}_{\mathcal{B}} - F(\mathcal{G}_{\mathcal{B}}), \mathcal{G}_{\mathcal{B}'} - F(\mathcal{G}_{\mathcal{B}'})$ are obtained from gadgets*.

Theorem 10. *Let $\mathcal{D} = \mathcal{D}(\mathcal{B}, \mathcal{B}')$ be a double gadget such that $\{b, z, y\}$ (resp. $\{b', z', y'\}$) is the* fixed *path, u (resp. u') is the* replica *vertex and x (resp. x') is the* joint *vertex of $\mathcal{B}$ (resp. $\mathcal{B}'$). If there exists a consecutive reduction of $\mathcal{D}^{-1}$ that creates no cut-obstacle at a 3-ear from $\mathcal{D}$, then the arcs β_1, β_2 (resp. β'_1, β'_2) belong to the reducible paths that contain (y, x) and (x, y) (resp. (y', x') and (x', y')). Moreover, such a consecutive reduction always exists.*

4 Proof of the main results

In this section we show the proofs of Theorem 2 and Theorem 3.

Let G be 3-edge-connected, $H(G)$ and Γ as defined in Section 3.3. We recall that each ear decomposition equivalent to the canonical ear decomposition of $H(G)$ is surobust.

Definition 10 (Relevant ear decompositions)**.** *Let $\mathcal{H}$ be the canonical ear decomposition of $H = H(G)$ and $(H', \mathcal{H}')$, be equivalent to $(H, \mathcal{H})$. Set $\mathcal{H}' = (H_0, H'_i, L'_i)^n$. We say that $\mathcal{H}'$ is* relevant *if the sequence $L'_1, \ldots, L'_n$ (possibly after some reordering) can be partitioned into blocks of ears, with each block being either a star (on 3 or 4 vertices), or a gadget, or a gadget*, or a gadget**, or a double gadget.*

Note that the canonical ear decomposition is relevant since its sequence of building ears can be partitioned into blocks of ears with each block consisting of a star on 4 vertices, or of a gadget, or of a 1-gadget*.

The following observation follows directly from the construction of trigraph $H(G)$. We recall that, by definition, the pending vertex of 2-gadgets* and 2-gadgets** is neither a joint vertex, nor a replica vertex.

Observation 7. *Let $\mathcal{H} = (H_0, H_i, L_i)^n$ be a canonical ear decomposition of $H = H(G)$ and $(H', \mathcal{H}' = (H_0, H'_i, L'_i)^n)$be an equivalent pair such that $\mathcal{H}'$ is relevant. If $\mathcal{G}$ is a 2-gadget* of $\mathcal{H}'$ with x and s the internal vertices of $F(\mathcal{G})$ and x the joint vertex of $\mathcal{G}$ and s the pending vertex of $\mathcal{G}$ (see definition in Figure 8(b)), then s is the internal vertex of a star on 4 vertices of $\{L_1, \ldots, L_n\}$.*

Definition 11 (Reduction process — extension)**.** *Let $(H_0, H_i, L_i)^n$ be an ear decomposition of $H(G)$. A* reduction process *of $L_1, \ldots, L_n$ is a consecutive reduction of $I(L_n), \ldots, I(L_j)$ so that* *(i) either $j = 1$, or*

(ii) $j > 1$ and $I(L_{j-1})$ does not have reduction or $I(L_{j-1})$ is a cut-obstacle.

If (i) holds, we say that the reduction process is complete. *Otherwise, we refer to it as j-*incomplete.

In what follows, $\mathcal{H} = (H_0, H_i, L_i)^n$ denotes the canonical ear decomposition of $H = H(G)$. Lemma 11 is implied by Theorem 7, Theorem 9 and Theorem 10.

Lemma 11. *Let $(H', (H_0, H'_i, L'_i)^n)$ be equivalent to $(H, \mathcal{H})$. Each reduction process of $L'_1, \ldots, L'_n$ satisfies exactly one of the following two statements.*

- *The reduction process is complete.*
- *The reduction process is j-incomplete and $I(L'_{j-1})$ is a cut-obstacle.*

In Lemma 12 below we state that superb relevant ear decompositions encode directed cycles double covers. This immediately proves Theorem 2.

Lemma 12. *Let $(H', \mathcal{H}' = (H_0, H'_i, L'_i)^n)$ be equivalent to $(H, \mathcal{H})$ and let $\mathcal{H}'$ be a relevant ear decomposition. If $\mathcal{H}'$ is superb then it encodes a DCDC of G.*

Proof. Due to Lemma 11 and definition of superb, a consecutive reduction of $I(L'_n), \ldots, I(L'_1)$ that does not create cut-obstacle at any 3-ear exists. We show that such a consecutive reduction encodes a DCDC of G.

We recall that, by Observation 5, the cubic graph G' can be obtained from the trigraph H' by contracting each basic gadget of H into a single vertex. Since $\mathcal{H}'$ is relevant, its sequence of ears $L'_1, \ldots, L'_n$ can be partitioned into blocks of ears $\mathcal{E}_1, \ldots, \mathcal{E}_t$ where each $\mathcal{E}_i$ is either a star on 3 or 4 vertices, or a gadget, or a gadget*, or a gadget**, or a double gadget. This partition gives rise to a partition $W_1, \ldots W_t$ of the union of the internal vertices of the ears of G', where each W_i contains either a single vertex, or two vertices in the following way:

- if $\mathcal{E}_i$ is a star with center $\Gamma(s)$, then $W_i = \{s\}$,
- if $\mathcal{E}_i$ is a gadget or a 1-gadget*, or a 1-gadget** with replica vertex $\Gamma(b)$, then $W_i = \{b\}$,
- if $\mathcal{E}_i$ is a 2-gadget*, or a 2-gadget** with replica vertex $\Gamma(b)$ and pending vertex $\Gamma(s)$, then $W_i = \{b, s\}$,
- if $\mathcal{E}_i$ is a double gadget with replica vertices $\Gamma(b), \Gamma(b')$, then $W_i = \{b, b'\}$.

Recall that function Γ (see Definition 9) embeds the set of vertices and edges of G' into a subset of the vertices and edges of H'. Assume now that it is possible to perform a consecutive reduction of $W_t, \ldots W_1$. After performing this consecutive reduction of $W_t, \ldots W_1$ on G', we obtain a mixed graph on the vertex set V_0, where the underlying graph is a triangle. Moreover by Observation 1, we have that each mixed triangle has a reduction; thus, by Proposition 2, G' has a DCDC, and also G does. Thus, to conclude the proof of the lemma, it suffices to prove the following claim.

Claim 1. *A consecutive reduction of $\mathcal{E}_t^{-1}, \ldots, \mathcal{E}_1^{-1}$ induces a consecutive reduction of $W_t, \ldots W_1$.*

The rest of the proof is devoted to prove Claim 1. Let $i \in \{1, \dots, t\}$. We assume that a consecutive reduction of $\mathcal{E}_t^{-1}, \dots, \mathcal{E}_i^{-1}$ and the corresponding consecutive reduction of $W_t, \dots W_i$ is performed on H' and G', respectively. Let $M(i)$ and $M(G', i)$ denote the generated mixed graphs, with $M(t) = H'$, $M(G', t) = G'$. Without loss of generality, $\mathcal{E}_t^{-1}$ is a star, and hence the reduction of W_t is natural since W_t is the center of a star. Thus, it suffices to prove that the reduction of $\mathcal{E}_{i-1}^{-1}$ on $M(i)$ induces a reduction of W_{i-1} on $M(G', i)$.

The following statements, namely Properties 1 and 2, are satisfied for every $u, v \in V(G')$ and for $j = t$. We assume that they also hold for all $j \geq i$ and we argue that they are satisfied for $i - 1$.

Property 1. The arc (u, v) is in $M(G', j)$ if and only if there exists an arc $\vec{a}$ in $M(j)$ with its head in $\{x_{\Gamma(v)}, \Gamma(v)\}$ and its tail in $\{x_{\Gamma(u)}, \Gamma(u)\}$; in other words, after vertex-contraction of all basic gadgets of $M(i)$ into their replica vertices, the arc $(\Gamma(u), \Gamma(v))$ exists. Recall that $x_{\Gamma(u)}$ denotes the joint vertex of the basic gadget with replica $\Gamma(u)$; in case $\Gamma(u)$ is not part of a basic gadget, set $x_{\Gamma(u)} = \Gamma(u)$. By abuse of notation, we denote $\vec{a}$ by $\Gamma((u, v))$. Furthermore, note that if a pair of arcs is forbidden in $M(G', i)$, then their images under Γ form a pair of arcs forbidden in $M(i)$.

Property 2. Suppose that (u', v') is an arc that belong to the reducible path of the reduction of W_i on $M(G', i+1)$ which is replaced by arc (u, v) in order to obtain $M(G', i)$. Then, either (u', v') is an arc of $M(G', i+1)$, or $\{u', v'\}$ is an edge of $M(G', i+1)$. If (u', v') is an arc of $M(G', i+1)$, then the arc $\Gamma((u', v'))$ of $M(H(G), i+1)$ (defined in Property 1) is in the reducible path replaced by the arc $\Gamma((u, v))$. If (u', v') is an arc obtained by the orientation of edge $\{u', v'\}$ of $M(G', i)$, then the corresponding orientation of $\Gamma(\{u', v'\})$ is in the reducible path replaced by the arc $(\Gamma((u, v))$.

Assuming validity of both properties for i, it is a routine to check that it is also valid for $i - 1$ by using Theorems 7, 8, 9 and 10. This finishes the proof of the statement of Claim 1, and also of Lemma 12. □

Finally, Theorem 3 is a straightforward consequence of Lemma 12 and Lemma 13. We first make a crucial observation, and then formulate Lemma 13. As before, $\mathcal{H} = (H_0, H_i, L_i)^n$ denotes the canonical ear decomposition of $H(G)$.

Observation 8. *Let $(H', \mathcal{H}' = (H_0, H_i', L_i')^n)$ be equivalent to $(H(G), \mathcal{H})$. If $\mathcal{H}'$ is superb, then $\mathcal{H}'$ is a robust ear decomposition.*

Proof. Assume $\mathcal{H}'$ is not robust. First of all note that the descendant of $E_1(\mathcal{B}), E_2(\mathcal{B})$, for all basic gadgets $\mathcal{B}$ of H' consist of exactly 2 connected components and one of them is an isolated vertex adjacent to two vertices in $V(H_0)$, since it is exactly the center of the stars $D_2(\mathcal{B}), D_3(\mathcal{B})$, respectively. On the other hand, note that the descendant of every 3-ear $E_3(\mathcal{B})$ has at most 2 connected components, and there is no component corresponding to an isolated vertex adjacent to vertices in $V(H_0)$. Thus, by the assumption, for some gadget $\mathcal{G} = \mathcal{G}(\mathcal{B})$ of H', there exists a 3-ear $E_3(\mathcal{B}) = L'_m \in \{L'_1, \dots, L'_n\}$ such that its descendant has exactly 2 connected components and hence, there is no path in the descendant of $E_3(\mathcal{B})$ connecting u to x; we consider name of the vertices as in Figure 4(b). Note that for every consecutive reduction of $I(L'_n), \dots, I(L'_{m+1})$ we have $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. Therefore, by Theorem 7, each reduction process of $\mathcal{G}$ is j-incomplete and $I(E_{j-1}(\mathcal{G}))$ is a cut-obstacle for some $2 \leq j \leq 4$. This implies that $\mathcal{H}'$ is not superb, a contraction. □

4.1 Proof of Theorem 3

The theorem follows immediately from Lemma 12 and from the next lemma.

Lemma 13. *Let $(H', \mathcal{H}' = (H_0, H_i', L_i')^n)$ be equivalent to $(H(G), \mathcal{H})$. Assume there exists a consecutive reduction, say $\mathcal{R}$, of $I(L_n'), \ldots, I(L_1')$ that makes $\mathcal{H}'$ superb. Then there is a pair $(H'', \mathcal{H}'' = (H_0, H_i'', L_i'')^n)$ equivalent to $(H(G), \mathcal{H})$ such that $\mathcal{H}''$ is a relevant ear decomposition and $\mathcal{R}$ induces a consecutive reduction $\mathcal{R}'$ of $I(L_n''), \ldots, I(L_1'')$ that makes $\mathcal{H}''$ superb.*

Proof. Note that by Observation 8, we have $\mathcal{H}'$ is a robust ear decomposition. In the following, we show how to obtain $H'', \mathcal{H}'' = (H_0, H_i'', L_i'')^n$ of Lemma 13 from $H', \mathcal{H}' = (H_0, H_i', L_i')^n$ by modifying $\mathcal{H}'$ only; meaning $H' = H''$.

Since $(H', \mathcal{H}')$ is equivalent to $(H(G), \mathcal{H})$, we get that $\mathcal{H}'$ is surobust (see Definition 5). By Observation 5, for each basic gadget $\mathcal{B}$ of $H(G)$, the ears

$$E_1(\mathcal{B}), E_2(\mathcal{B}), E_3^*(\mathcal{B}), D_1(\mathcal{B}), D_2(\mathcal{B}), D_3(\mathcal{B})$$

belong to $\{L_1', \ldots, L_n'\}$, where $E_3^*(\mathcal{B})$ is a 2-ear or a 3-ear that contains the fixed path of $\mathcal{B}$; recall that the ear $E_1(\mathcal{B})$ is a base, $E_2(\mathcal{B})$ is an up, $E_3^*(\mathcal{B})$ is an antenna. Since $\mathcal{H}'$ is surobust, the stars $D_1(\mathcal{B}), D_2(\mathcal{B}), D_3(\mathcal{B})$ cannot be modified. In this proof, for a given $\mathcal{B}$, we call the sequence of ears $E_1(\mathcal{B}), E_2(\mathcal{B}), E_3^*(\mathcal{B})$ the *core* of $\mathcal{B}$. Without loss of generality, from now on, we assume that the list of ears $L_1', \ldots, L_n'$ is ordered so that each core forms a block of consecutive ears. Therefore, we can consider a natural ordering, say $\prec$, of the basic gadgets of $H(G)$: $\mathcal{B} \prec \mathcal{B}'$ if $E_3(\mathcal{B}) = L_k$, $E_3(\mathcal{B}') = L_m$ and $k < m$.

We denote by $\mathcal{L}$ the set of the paths consisting of the cores of all basic gadgets of H. Let L be a 3-ear of $\mathcal{H}'$. L is either a base, or an up, or an antenna and $\mathcal{H}'$ is surobust, hence the corresponding path of $L \in \mathcal{L}$.

From now on, we refer to a gadget, a gadget*, a gadget** and a double gadget of $\mathcal{H}'$ as a *block-gadget*.

If every basic gadget of $H(G)$ is contained is some block-gadget of $\mathcal{H}'$, then we have that $\mathcal{H}'$ is relevant and we can put $\mathcal{H}'' = \mathcal{H}'$. Therefore, we assume that the set of basic gadgets of $H(G)$ that are not contained in some block-gadget of $\mathcal{H}'$ is not empty and we denote the set of such basic gadgets of $H(G)$ by $\Sigma(\mathcal{H}')$.

The following claim therefore proves Lemma 13.

Claim 2. *There exists a short ear decomposition $\mathcal{H}^*$ of H' such that $|\Sigma(\mathcal{H}^*)| < |\Sigma(\mathcal{H}')|$ and $\mathcal{R}$ induces a consecutive reduction, say $\mathcal{R}^*$, of $\mathcal{H}^*$. Moreover, each 3-ear in the set of building ears of $\mathcal{H}^*$ also belongs to $\{L_1', \ldots, L_n'\}$. Further, $\mathcal{H}^*$ is superb and $(H', \mathcal{H}^*)$ is equivalent to $(H(G), \mathcal{H})$.*

The rest of this proof is devoted to prove Claim 2.

We obtain $\mathcal{H}^*$ from $\mathcal{H}'$ by modifying the list $L_1', \ldots, L_n'$.

Let $\mathcal{B}$ be the first basic gadget, according to the order $\prec$, such that $\mathcal{B} \in \Sigma(\mathcal{H}')$ (namely, is not contained in a block-gadget of $\mathcal{H}'$). There are two possible scenarios: (1) either the ear $E_4(\mathcal{B})$ is in $\{L'_1, \ldots, L'_n\}$, (2) or not.

Case (1) $E_4(\mathcal{B})$ is in $\{L'_1, \ldots, L'_n\}$.

We assume the first ear of $\mathcal{B}$ is L'_r. If $E_4(\mathcal{B})$ is in $\{L'_1, \ldots, L'_n\}$, then E_3^* is a 2-ear, otherwise, $\mathcal{B}$ would belong to a gadget. Let x be the joint vertex of $\mathcal{B}$. Clearly, x is a leaf of E_3^*. This implies that there exists an ear $L \in \{L'_1, \ldots, L'_{r-1}\}$ that contains x as an internal vertex, with L a 1-ear or a 2-ear. If L was a 1-ear, then $\mathcal{B}$ would be contained in a 1-gadget*. Thus, L is a 2-ear. Since L is not contained in a 2-gadget*, we have that for $I(L) = \{x, x'\}$, the vertex x' is the joint or the replica vertex of a basic gadget $\mathcal{B}' \neq \mathcal{B}$. If x' was the replica vertex of $\mathcal{B}'$, then $\mathcal{B}'$ would not be contained in a block gadget (because $E_4(\mathcal{B}')$ would not be in $\{L'_1, \ldots, L'_n\}$), which contradicts the choice of $\mathcal{B}$.

Therefore, x' is a joint vertex of $\mathcal{B}'$. Since $\mathcal{B}$ is not contained in a double gadget, then neither $E_4(\mathcal{B}')$, nor both $F_1(\mathcal{G}_{\mathcal{B}'})$, $F_2(\mathcal{G}_{\mathcal{B}'})$ belong to $\{L'_1, \ldots, L'_n\}$, where $F_1(\mathcal{G}_{\mathcal{B}'})$, $F_2(\mathcal{G}_{\mathcal{B}'})$, denote the 1-ears of the gadget** $\mathcal{G}_{\mathcal{B}'}$. Therefore, $F_2(\mathcal{G}_{\mathcal{B}'}), F'_1 \in \{L'_1, \ldots, L'_n\}$, where F'_1 is a 2-ear that contains the edges of $F_1(\mathcal{G}_{\mathcal{B}'})$ and an extra edge e not in $E(\mathcal{G}_{\mathcal{B}'})$. Clearly, F'_1 is not contained in a block-gadget.

In this case we let the list of building ears of $\mathcal{H}^*$ be obtained by removing $F'_1, F_2(\mathcal{G}_{\mathcal{B}'})$ from $\{L'_1, \ldots, L'_n\}$ and adding two new ears $E_4(\mathcal{B}')$ and the 1-ear contained in F'_1 that contains e and has end vertex the replica vertex of $\mathcal{B}'$. Hence, $\mathcal{B}$ and $\mathcal{B}'$ belong to a double gadget of $\mathcal{H}^*$. As F'_1 is not contained in a block-gadget of $\mathcal{H}'$, we have that $\Sigma(\mathcal{H}^*) + 2 = \Sigma(\mathcal{H}')$. The result follows.

Case (2) $E_4(\mathcal{B})$ is not in $\{L'_1, \ldots, L'_n\}$.

Let $\{z, u, v, y\}$ be the vertex set of $E_4(\mathcal{B})$, where u is the replica vertex of $\mathcal{B}$ (as in Figure 4(a)). As $E_4(\mathcal{B})$ is not in $\{L'_1, \ldots, L'_n\}$, the 1-ear, say F_2, on vertex set $\{u, v, y\}$ is in $\{L'_1, \ldots, L'_n\}$ and a 1-ear or a 2-ear F_1 containing $\{z, u\}$ is in $\{L'_1, \ldots, L'_n\}$.

First let F_1 be a 1-ear. If $E_3^*(\mathcal{B})$ is a 2-ear, then the proof is the same as the proof of Case (1): just use $F_1(\mathcal{B}), F_2(\mathcal{B})$ instead of $E_4(\mathcal{B})$. If $E_3^*(\mathcal{B})$ is a 3-ear, then $\mathcal{H}'$ is not robust: to see this, note that the descendant of $E_3^*(\mathcal{B})$ has exactly 2 connected components — since v is a leaf of the star $D_1(\mathcal{B})$ and u has degree 3 in $F_1 \cup F_2$, there is no path in the descendant of $E_3^*(\mathcal{B})$ that connects the joint of $\mathcal{B}$ to a vertex from $\{u, v\}$ (see Figure 4(b)) — and none of them is an isolated vertex connected to two vertices in $V(H_0)$ (see Definition 3). Because of Observation 8, this contradicts the assumption that $\mathcal{H}'$ is superb.

If F_1 is a 2-ear, then we obtain a list of building ears $\mathcal{L}$ by replacing F_2, F_1 in $\{L'_1, \ldots, L'_n\}$, by $E_4(\mathcal{B})$ and the 1-ear $F_1 - \{z, u\}$. Therefore, either $\mathcal{B}$ is in a block-gadget of $\mathcal{L}$ (which implies $\Sigma(\mathcal{H}^*) + 1 = \Sigma(\mathcal{H}')$), or not. If not, the result follows by Case (1).

This finishes the proof of Claim 2. □

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