Types of perfect matchings in toroidal square grids

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Abstract

Let $T_{m,n}$ be toroidal square grid of size $m \times n$ and let both m and n be even. Let P be a perfect matching of $T_{m,n}$ and let $D(P)$ be the cycle-rooted spanning forest of P obtained by the generalized Temperley's construction. The types of P and $D(P)$ in the first homology group $H_1(\mathbb{T}, \mathbb{Z})$ of torus $\mathbb{T}$ with coefficients in $\mathbb{Z}$ has been extensively studied. In this paper we study the types of P and $D(P)$ in the first homology group $H_1(\mathbb{T}, \mathbb{F}_2)$ with the coefficients in $\mathbb{F}_2$. Our considerations connect two remarkable results concerning perfect matchings of toroidal square grids, namely Temperley's bijection and the Arf-invariant formula.

1 Introduction

A perfect matching P of a graph G is a collection of edges of G with the property that each vertex is adjacent to exactly one edge in P. This paper connects two remarkable results concerning perfect matchings of toroidal square grids, namely Temperley's bijection and the Arf-invariant formula. In separate sections we describe next each of the aforementioned two results.

1.1 Generalized Temperley's bijection

Temperley [Tem74] showed a seminal correspondence between perfect matchings and spanning trees of planar square grids. This correspondence was extended by Kenyon, Propp and Wilson [KPW00] to general planar graphs. The construction also applies to graphs on other surfaces, such as toroidal graphs, which we now sketch.

Let $G = (V, E)$ be a graph embedded in the torus, and let $G^* = (V^*, E^*)$ be its geometric dual, whose vertices are in bijection with the faces of G. There is a natural bijection between edges of G^* and edges of G: let f, f' be two vertices of G^* and let e be a common boundary edge of faces f and f'. Then, G^* has edge $e^* = \{f, f'\}$. Now, let $\mathcal{G} = \mathcal{G}(G)$ be the bipartite graph whose black vertices $V^B(\mathcal{G})$ are $V \cup V^*$ and whose white vertices $V^W(\mathcal{G})$ are in bijection with E (and hence with E^*). Edges in $\mathcal{G}$ are of two types: $\{v, \{v, v'\}\}$ where $\{v, v'\} \in E$, i.e., $\{v, v'\} \in V^W(\mathcal{G})$, and $\{f, \{f, f'\}\}$ where $\{f, f'\} \in E^*$ and thus also $\{f, f'\} \in V^W(\mathcal{G})$ by the bijective correspondence of the edges and the dual edges. Hence, each white vertex u of $\mathcal{G}$ has degree 4: two edges incident with u come from interpreting u as an edge of G, and the remaining two edges incident with u come from interpreting u as a dual edge of G. Graphs $\mathcal{G}$ obtained in this way are called *Temperleyan* (see [KPW00]). This paper studies toroidal square grids of size $m \times n$, which for the case where m and n are even positive integers can be easily seen to be Temperleyan.

A *cycle-rooted spanning forest (CRSF)* of G is a spanning subgraph of an orientation of G where each vertex has out-degree one. It follows that each component of a CRSF consists of exactly one oriented cycle, called *root cycle*, and oriented trees (arborescence) attached to it and oriented towards it. We say that a pair (F, F^*) is *dual* if F is a CRSF that spans G, F^* is a CRSF that spans G^* and each $w \in V^W(\mathcal{G})$ is crossed either by an edge of F or by an edge of F^* (but not by both).

There is a one-to-one correspondence, called *generalized Temperley's bijection*, between pairs (F, F^*) of dual CRSF's and perfect matchings of $\mathcal{G}$ (see [KPW00, DG15] for the construction). Given a perfect matching P of $\mathcal{G}$, the corresponding pair (F, F^*) is constructed as follows: we first orient each edge $e \in P$ from its black vertex to its white vertex, say w. Vertex w of $\mathcal{G}$ represents edge $e(w)$ of G (or of G^*), and edge e can be viewed as a 'half-edge' of $e(w)$. Let e' be the other 'half-edge' of $e(w)$. The white vertex of e' is w and we orient e' from w to its black vertex. This defines a CRSF of $\mathcal{G}$ which we denote by $D(P)$ (see Figure 1 for an illustration). We observe that since P is a perfect matching of $\mathcal{G}$, each white vertex has in- and out-degree 1 in $D(P)$, and if u and v are the other end-vertices of the edges of $D(P)$ incident with w then both u and v are either vertices of G or both are vertices of G^*. Hence, $D(P)$ uniquely decomposes into a a pair (F, F^*) of dual CRSF's.

It is mentioned in [DG15] that, if G is a toroidal graph, then each root cycle of each of its dual CRSF's (F, F^*) is non-contractible. We state this property for toroidal square grids formally now and postpone its proof to the next section.

Lemma 1 *Let $\mathcal{G}$ be a toroidal square grid of size $m \times n$, where both m and n are even. If P is a perfect matching of $\mathcal{G}$ and C is a root cycle of $D(P)$, then C is non-contractible.*

It is shown in [DG15], see also [Sun16], that for toroidal graphs the generalized Tem-
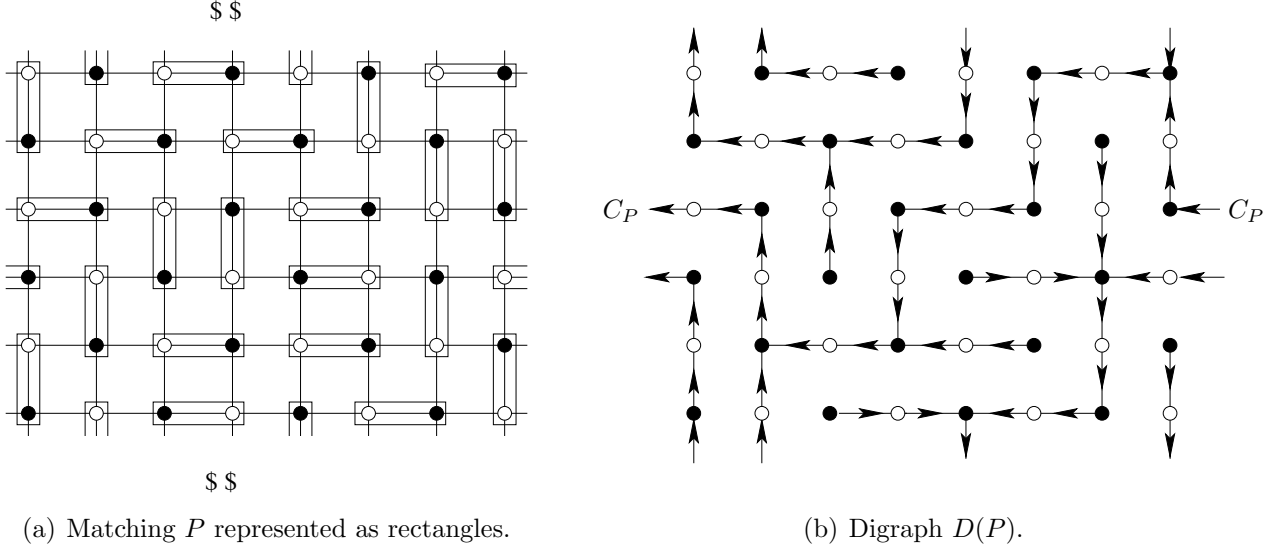
(a) Matching P represented as rectangles.

(b) Digraph $D(P)$.

Figure 1: Perfect matching P of a 6×8 grid and associated $D(P)$.

perley's correspondence extends to homology considerations. Using the construction of Thurston [Thu90], one can associate each perfect matching P of G with its *height function* and with an element $[P]$ of the first homology group $H_1(\mathbb{T}, \mathbb{Z})$ of torus $\mathbb{T}$ with coefficients in $\mathbb{Z}$. Each pair (F, F^*) of dual CRSF's also naturally defines an element $[F, F^*]$ of $H_1(\mathbb{T}, \mathbb{Z})$ by considering the set of the root cycles. It is explained in [DG15] how $[P]$ and $[F, F^*]$ are related for a perfect matching P and the corresponding dual CRSF's (F, F^*).

1.2   Arf-invariant formula

In statistical physics a Perfect matching is called a *dimer arrangement*. Very often graph G comes with an edge-weight function $w : E \to \mathbb{Q}$. The statistics of the weighted perfect matchings is then given by the generating function

$$P(G, x) = \sum_{P \text{ perfect matching of } G} x^{\sum_{e \in P} w(e)},$$

also known as the *dimer partition function* on G. We note that if G is bipartite, with bipartition classes V_1 and V_2, then $P(G, x)$ equals the permanent of $A(G)$, where $A = A(G)$ is the $|V_1| \times |V_2|$ matrix given by $A_{uv} = x^{w(uv)}$.

If g denotes the minimum genus of an orientable surface $\mathcal{E}$ in which a graph G embeds, then the partition function of a dimer model on a graph G can be written as a linear combination of 2^{2g} Pfaffians of *Kasteleyn matrices*. These 2^{2g} Kasteleyn matrices are skew-symmetric $|V| \times |V|$ matrices determined by 2^{2g} orientations of G, called Kasteleyn (or Pfaffian) orientations. Kasteleyn himself proved this Pfaffian formula in the planar case and for toroidal square grids where all the horizontal edge-weights and all the vertical edge-weights are the same [Kas61], and stated the general fact [Kas67]. A complete combinatorial

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proof of this statement was first obtained much later by Gallucio and Loebl [GL99], and independently by Tesler [Tes00]. Cimasoni and Reshetikhin [CR07] provided a beautiful understanding of the formula in terms of the quadratic forms on $H_1(\mathcal{E}, \mathbb{F}_2)$ and since then the formula is known as the *Arf-invariant formula*.

1.3   Main results

The Arf-invariant formula requires to associate, to each perfect matching P of a graph G embedded in orientable surface $\mathcal{E}$, an element $[P]_2$ of the first homology group $H_1(\mathcal{E}, \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$; see e.g. Loebl and Masbaum [LM11] for details. This paper studies toroidal square grids. If T is a toroidal square grid then $[P]_2$ can be represented by a pair (p_1, p_2) defined as follows: let A be one layer of vertical edges of T and let B be one layer of horizontal edges of T. Then $p_1 = |P \cap A| \mod 2$ and $p_2 = |P \cap B| \mod 2$. We also say that (p_1, p_2) is the *type* of P. Let $D(P)$ be the CRSF of P defined above. Again, $D(P)$ naturally defines an element $[D(P)]_2$ of $H_1(\mathcal{E}, \mathbb{F}_2)$ by considering the root cycles: Let C be a root cycle of $D(P)$. Then we represent $[D(P)]_2$ by a pair (d_1, d_2) where $d_1 = |C \cap A| \mod 2$ and $d_2 = |C \cap B| \mod 2$. We also say that (d_1, d_2) is the *type* of $D(P)$. By Lemma 1 each root cycle of $D(P)$ is non-contractible. By construction the root cycles of $D(P)$ are disjoint and since T is a toroidal graph, the type of $D(P)$ is independent of the choice of root cycle C. We next observe that the type of $D(P)$ is never $(0,0)$. For the sake of contradiction, assume C is a cycle in $D(P)$ of type $(0,0)$. It follows that C must be the symmetric difference of boundaries of faces $F_1, \ldots, F_k$ of T. Since C is connected, it follows that C is contractible which contradicts Lemma 1. We thus have:

Lemma 2 *For every perfect matching P of T, $[D(P)]_2 \neq (0,0)$.*

We can now state the main result of this paper.

Theorem 3 *Let m and n be even positive integers and let $T_{m,n}$ denote the $m \times n$ toroidal square grid. If P is a perfect matching of $T_{m,n}$ such that $[P]_2 \neq (0,0)$, then $[P]_2 = [D(P)]_2$.*

We recall that $T_{m,n}$ with both m and n even positive integers is a Temperleyan graph. Our proof of Theorem 3 abstracts the properties of toroidal square grids that are needed for the argument to go through. It is natural to ask if Theorem 3 can be generalized to all toroidal Temperleyan graphs.

2   Preliminaries

Throughout this work, let m and n be two positive integers. We solely consider the case where both m and n are even and from now on, in order to avoid unnecessary repetitions, we omit this condition from the statement of all results. Say $m = 2m'$ and $n = 2n'$. Let $T_{m',n'}$ be the toroidal square grid with vertex set $\{0, \ldots, m'-1\} \times \{0, \ldots, n'-1\}$ and edges

with endpoints (i, j) and (i', j') of any of their edges differ in at most one coordinate and exactly by 1,[1] i.e.,

$$(i - i' = \pm 1 \wedge j = j') \vee (i = i' \wedge j - j' = \pm 1).$$

Note that for the standard embedding of $T_{m',n'}$ in the torus, we get that $T^*_{m',n'}$ is isomorphic to $T_{m',n'}$. Thus, if $\mathcal{G} = \mathcal{G}(T_{m',n'})$ is the Temperleyan graph associated to $T_{m',n'}$ as described in Section 1.1. then $\mathcal{G}$ is itself isomorphic to $T_{m,n}$. Without loss of generality we can think of the (i, j) nodes of $T_{m,n}$ where i and j are of distinct parity as those in one-to-one correspondence with the black vertices of $\mathcal{G}$ (equivalently, with $V(T_{m',n'}) \cup V(T^*_{m',n'})$), and thus henceforth refer them as black vertex. Analogously, we identify the (i, j) nodes of $T_{m,n}$ of equal parity with the white vertices of $\mathcal{G}$ (equivalently, with $E(T_{m',n'})$ or $E(T^*_{m',n'})$), and also henceforth refer to them as white vertex. From now on, we work directly with in $T_{m,n}$ in the understanding that via the isomorphism between $\mathcal{G}$ and $T_{m,n}$ our discussion indeed concerns $\mathcal{G}$.

The collection of nodes $(i, j) \in V(T_{m,n})$ for which $0 \leq j < n$ (respectively, $0 \leq i < m$) will be called the *i-th row* (respectively, *j-th column*), and i (respectively, j) will be called its index. Edges whose endvertices belong to the same row (respectively, column) will be called *horizontal* (respectively, *vertical*) edges. We let A be the complete layer of horizontal edges between vertices $(0, j)$ and $(m - 1, j)$, $j = 0, \ldots, n - 1$. Similarly, we let B be the complete layer of vertical edges between vertices $(j, 0)$ and $(j, n - 1)$, $j = 0, \ldots, m - 1$.

Paths and cycles, say S, will always be considered as subgraphs of whatever the graph or digraph we are dealing with. We thus rely on the standard notation $V(S)$ and $E(S)$ to denote S's vertices and edges, respectively. For a simple path or cycle S in whatever graph we consider, we say that vertex $v \in V(S)$ is a *corner* of S if one of the edges of S incident to v is a horizontal edge and the other one is a vertical edge.

We now establish a simple, yet key property, of certain type of cycles of $T_{m,n}$ (equivalently, of $\mathcal{G}$). The property concerns the number of vertices encircled by such cycles.

Lemma 4 *If C is a contractible simple cycle of $T_{m,n}$ all of whose corners are of the same color, then the disk encircled by C contains a single connected component of $T_{m,n} \setminus C$.*

Proof For the sake of contradiction, assume the disk encircled by C contains two connected components of $T_{m,n} \setminus C$. The only way in which this could happen is if there is a face of $T_{m,n}$ whose boundary contains two edges of C, say e and e', which do not share endvertices. Thus, e and e' must be either both horizontal or both vertical edges. Assume the former case happens (the latter case is handled similarly). Consider one of the two components (paths), say Q, obtained by removing edges e and e' from C. Since the endvertices of Q belong to two consecutive rows, then Q must contain an odd number of column edges, but then, due to $T_{m,n}$'s grid like structure, the path Q can not have all of its corners of the same color. $\square$

Lemma 5 *If C is a contractible simple cycle of $T_{m,n}$ all of whose corners are of the same color, then the disk encircled by C contains an odd number of vertices of $T_{m,n} \setminus C$.*

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[1] Here and throughout this work, unless said otherwise, arithmetic involving vertex labels is always modulo m and n over the first and second coordinates, respectively

Proof By Lemma 4 we know that C encircles one connected component of $T_{m,n} \setminus C$. Let Q be the shortest segment of C (ties broken arbitrarily) whose endvertices are corners of C. By minimality of Q, all vertices of Q must belong to either the same column or the same row. Without loss of generality, assume that the former case holds and that Q's vertices are $(i_1, j), (i_1 + 1, j), \ldots, (i_2, j)$. Let R_1 (respectively, R_2) be the rectangle (cycle with four corners) of $T_{m,n}$ that contains Q as a segment and has as opposite corners vertices (i_1, j) and $(i_2, j+2)$ (respectively, $(i_2, j-2)$). Because of the hypothesis concerning C's corners and the grid structure of $T_{m,n}$, either R_1 or R_2 must encircle an area of the torus also encircled by C. Without loss of generality, assume R_1 is such a rectangle and denote by C_1 the cycle that encircles it. Either $C = C_1$ or by minimality of Q the situation is one of the three depicted in Figure 2. By hypothesis, there is an odd number of vertices between any two consecutive corners of C_1, thence C_1 encircles an odd number of vertices. The desired conclusion follows by induction on the one or two simple cycles obtained by taking the symmetric difference between C and C_1. $\square$
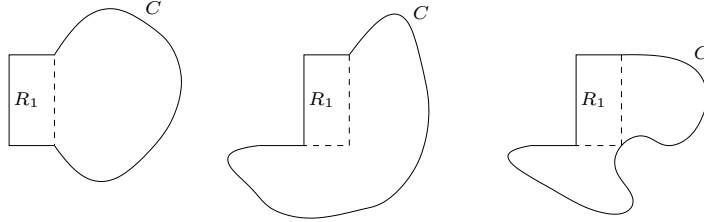

Figure 2: Possible relations between R_1 and C when $C \neq R_1$.

Consider now a perfect matching P of $T_{m,n}$ (in contrast to our view of paths and cycles as subgraphs, we consider matchings as subsets of edges). We now formally define the di-graph denoted by $D(P)$ in Section 1. Since P is a perfect matching of $T_{m,n}$, we denote it by $D_{m,n}(P)$ or simply by $D_{m,n}$. Di-graph $D_{m,n} = D_{m,n}(P)$ has vertex set U and edge set D, hence $D_{m,n} = (U, D)$. The vertex set of $D_{m,n}$ equals the vertex set of $T_{m,n}$, that is $U = V(T_{m,n})$. The edge set D is such that (see Figure 1):

- If v is black, then $vw \in D$ if and only if vw is an edge of P.

- If v is white, then $vw \in D$ provided: (1) vw is not an edge of P, and (2) if uv is the edge of the matching P incident to v, then u, v, and w all lie in the same row or the same column of $T_{m,n}$.

We are now ready to prove Lemma 1.

Proof [of Lemma 1] If C is a dicycle of $D_{m,n}(P)$, then $P \setminus C$ is a perfect matching of $T_{m,n} \setminus C$. By definition of $D_{m,n}(P)$ only black nodes can be corners, so all corners of C are of the same color. By Lemma 5, it follows that C cannot be contractible since otherwise the set of nodes $S \subseteq V(T_{m,n})$ encircled by C would be of an odd cardinality set covered by $P \setminus C$. $\square$

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3   Consistent Toroidal Systems

This section is dedicated to proving Theorem 3. If P is a perfect matching of $T_{m,n}$ then we denote by C_P an arbitrary cycle of $D_{m,n}$. Before precisely describing the abstract setting we will work in, we establish the following result, which essentially says that we can restrict our attention to certain "better behaved" perfect matchings. Say that a perfect matching P of $T_{m,n}$ is *well behaved*, if C_P neither contains a horizontal edge (respectively, vertical edge) both of whose endvertices are in row 0 nor both in row $m-1$ (respectively, both of whose endvertices are in column 0 nor both in column $n-1$).

Lemma 6 *If P is a perfect matching of $T_{m,n}$, then there is a well behaved perfect matching P' of $T_{m+4,n+4}$ for which the types of P and P' are the same, and the types of C_P and $C_{P'}$ are the same.*

Proof We explicitly build P' with the desired properties (see Figure 3 for an illustration of the construction). First, for each edge in $P \setminus (A \cup B)$, say between vertices (i,j) and (i',j') add an edge into P' between $(i+2, j+2)$ and $(i'+2, j'+2)$.
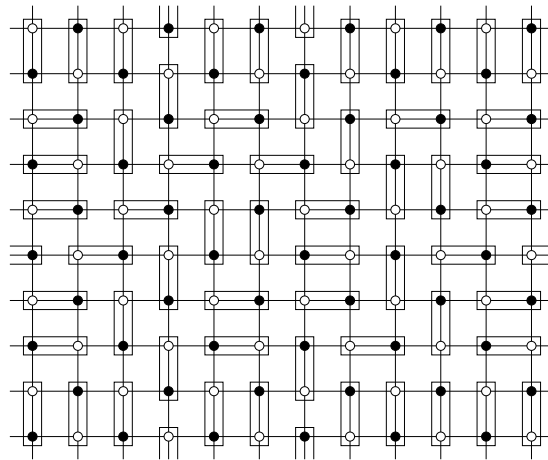

Figure 3: Well behaved matching P' in the 10×12 toroidal grid derived from the matching P of Figure 1(a).

Now, if there is a vertical edge in $P \cap A$ between $(0,j)$ and $(m-1,j)$, then include in P' the vertical edge between $(0,j)$ and $(m+3,j)$. Note that for each column j of $T_{m+4,n+4}$, $2 \le j \le n+1$, in which uncovered vertices remain (there are exactly four such vertices in each column) they can be covered in a (uniquely determined) way by two vertical edges.

Similarly, if there is an edge in $P \cap B$ between $(i,0)$ and $(i,n-1)$, include in P' an horizontal edge between $(i,0)$ and $(i,n+3)$, and proceed as in the previous paragraph but considering rows instead of columns and horizontal edges instead of vertical ones.

The vertices located in rows 0, 1, $m+2$ and $m+3$, which also belong to one of the columns 0, 1, $n+2$ and $n+3$, form 4 groups of 4 uncovered vertices. Cover each such group of 4 vertices by two vertical edges and include them in P'.

It is not hard to see that the described construction is sound, i.e., P' is a perfect matching of $T_{m+4,n+4}$. Moreover, C_P can be naturally identified with $C_{P'}$, the latter of which can be easily seen to be well behaved.

Finally, let A' and B' be defined as A and B but with respect to $T_{m+4,n+4}$ instead of $T_{m,n}$. It is straightforward to verify that $|P \cap A| = |P' \cap A'|$, $|P \cap B| = |P' \cap B'|$, $|C_P \cap A| = |C_{P'} \cap A'|$ and $|C_P \cap B| = |C_{P'} \cap B'|$, thus establishing the lemma's statement. $\square$

We now introduce the abstract setting in which we will henceforth work. Let T denote the surface of a torus. We say that $(R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$ is a *toroidal system* if the following holds (see Figure 4):
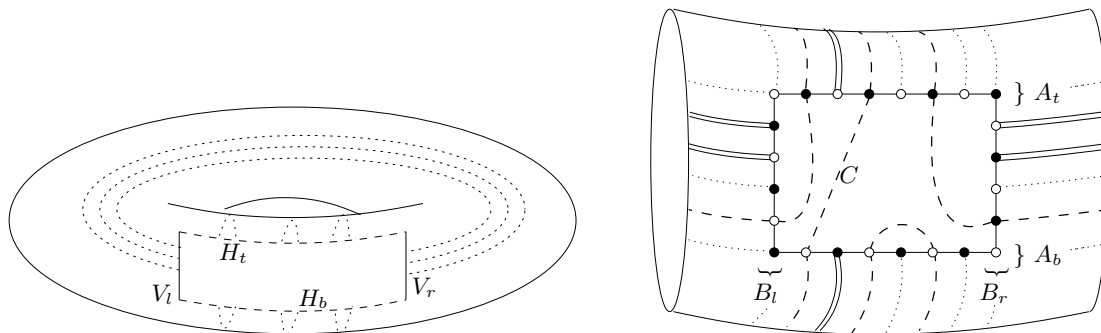
- **Rectangle:** R is a rectangle of the torus T. The *sides* of R are denoted clockwise H_t, V_l, H_b, V_r.[2] We call H_t, H_b the *horizontal sides* and V_l, V_r the *vertical sides* of R.

- **Vertices:** N is a set of vertices. An even number of *vertices* are located in each of the four boundary sides of R. We denote by A_t, A_b (respectively, B_l, B_r) the vertices located in H_t, H_b (respectively, V_l, V_r). It must hold that $|A_t| = |A_b|$ and $|B_l| = |B_r|$. Moreover, one vertex is located in each corner of R. Hence, e.g., $|A_t \cap B_l| = 1$. These vertices are called *corner vertices*.

  Vertices are bi-colored black and white alternately along the boundary of R. We henceforth assume that the corner vertex located at the intersection of V_l and H_t is colored white and hence completely fix the coloring of A_t, B_l, A_b, B_r. Traversing H_t, H_b starting from V_l, (respectively, V_l, V_r starting from H_t) one can naturally pair the vertices of A_t, A_b (respectively, B_l, B_r) in order of their appearance. We call *opposite* vertices each such pair and note that opposite vertices have different color.

- **Edges:** Sets A and B are collections of disjoint closed curves on T. Elements of $A \cup B$ are called *edges*. Each edge in A (respectively, B) connects two opposite vertices of A_t, A_b (respectively, B_r, B_l) and is otherwise disjoint with R. Each edge of $C \cap A$ (respectively, $C \cap B$) has its black vertex in A_t and its white vertex in A_b (respectively, black vertex in B_r and its white vertex in B_l).

- **Outer Matching:** $\mathcal{P}_{A,B}$ is a collection of edges of $A \cup B$ referred to as *matching edges*.

- **Toroidal cycle:** C is a non-intersecting non-contractible toroidal loop which is a subset of $R \cup A \cup B$ and avoids every corner vertex of R. Moreover, each point of C belonging to a side of R is a vertex of R.

- **Cycle Matching:** $O(C)$ is a collection of disjoint closed intervals of $C \cap R$ such that each element of $O(C)$ is disjoint with each edge of $\mathcal{P}_{A,B}$, has a vertex at one of its endpoints and the other endpoint is a point of the interior of R. Moreover, each vertex in C is covered by $\mathcal{P}_{A,B} \cup O(C)$.

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[2] Here t, l, b and r stand for top, left, bottom and right, respectively.

- **Inner Matching**: $O(R)$ is a collection of disjoint closed curves contained in R which cover each vertex uncovered by $\mathcal{P}_{A,B} \cup C$ and such that each curve o of $O(R)$ satisfies: (1) o is disjoint with $\mathcal{P}_{A,B} \cup C$, and (2) o is the segment of a side of R between two neighboring vertices, or o intersects the boundary of R in exactly one point, which is then a vertex.

(a) Depiction in the torus T of rectangle R (horizontal sides H_t and H_b shown as dashed lines, and vertical sides V_l, V_r shown as solid lines). Dotted lines correspond to edges $A \cup B$.

(b) Depiction of a slice of the torus. Bi-colored vertex sets A_t, B_r, A_b, B_l located in the solid straight lines representing the boundary of R. Dotted curves represent edges in $A \cup B$. Double solid lines correspond to edges in $\mathcal{P}_{A,B}$. Cycle C is represented by the dashed closed curve.

Figure 4:

- **Segments**: Connected components of $C \setminus (A \cup B)$ (respectively, $C \setminus A$, $C \setminus B$) are called *segments* (respectively, *A-segments*, *B-segments*) of C (see Figure 6). The endpoints of a segment are the only one of its points that can belong to the boundary of R (it follows that segments are disjoint). We will say that a *segment is bi-colored* when located at its endpoints there are vertices of distinct color, otherwise we say the segment is *mono-chromatic*.

  We require that the vertices located at the endpoints of A-segments (respectively, B-segments) belong to A_t if they are black and to A_b if they are white (respectively, to B_r if they are black and to B_l if they are white).

We define $\mathcal{S}(P) = (R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$, for a well behaved perfect matching P, as follows. We let R equal the rectangle of the torus with sides the edges in rows 0 and $m-1$ of $T_{m,n}$, and columns 0 and $n-1$ of $T_{m,n}$. We let N be the set of vertices of $T_{m,n}$ located in the boundary of R. The set A (respectively, B) corresponds to the set of edges of $T_{m,n}$ between vertices in row 0 and $m-1$ (respectively, 0 and $n-1$). The matching $\mathcal{P}_{A,B}$ will be $P \cap (A \cup B)$ and the toroidal cycle C is set to C_P. The set $O(C)$ will consist of vertical edges of $C_P \setminus A$ incident to vertices in rows 0 or $m-1$ that are not covered by edges of $\mathcal{P}_{A,B}$ as well as horizontal edges of $C_P \setminus B$ incident to vertices in columns 0 or $n-1$

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also not covered by $\mathcal{P}_{A,B}$. Finally, the set of $O(R)$ will consist of all edges $P \setminus (A \cup B)$ not in $O(C)$ which are incident to vertices in rows 0 or $m-1$, or to vertices in columns 0 and $n-1$. The following result may be easily verified.
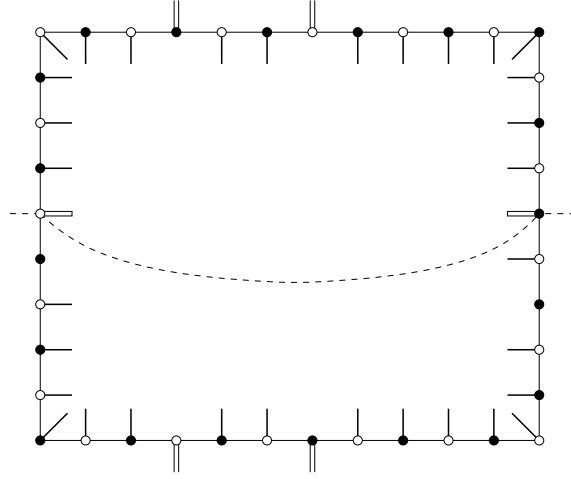

Figure 5: Toroidal system $\mathcal{S}(P)$ associated to the well behaved matching P' of Figure 3 (boundary of rectangle R drawn as solid line, cycle C shown as dashed line, elements of $O(R)$ shown as thick lines, elements of $\mathcal{P}_{A,B}$ depicted as double lines outside R and elements of $O(C)$ depicted also as double lines but inside R).

Proposition 7 *If P is a well behaved perfect matching of $T_{m,n}$, then $\mathcal{S}(P)$ is a toroidal system.*

In fact, we will soon see that the toroidal system guaranteed by the preceding proposition satisfies additional properties. Below, we first introduce some new terminology and then the properties of toroidal systems that we will be concerned with.

An interval I of the loop H_t' (respectively, H_b', V_l', V_r') consisting of H_t (respectively, H_b, V_l, V_r) and the edge between the endvertices of H_t (respectively, H_b, V_l, V_r) will be called *loop interval.*

For a segment we distinguish two situations (see again Figure 6):

- **Ears:** if the endpoints of the segment belong to the same side of R (H_t, V_r, H_b, or V_l).

- **Lines:** if the endpoints of the segment belong to opposite sides of R (H_t and H_b, or V_l and V_r).

We analogously define A-ears, B-ears and A-lines, B-lines. Let D be a disc of T bounded by exactly two A-lines and two loop intervals belonging to opposite loops H_t' and H_b'. We call D A-regular and its boundary loop intervals are called *ends* of D. A toroidal cycle C is called A-regular toroidal cycle if all its A-segments are A-lines. We define B-regular disc and B-regular toroidal cycle analogously. For instance, in Figure 6, the cycle is B-regular
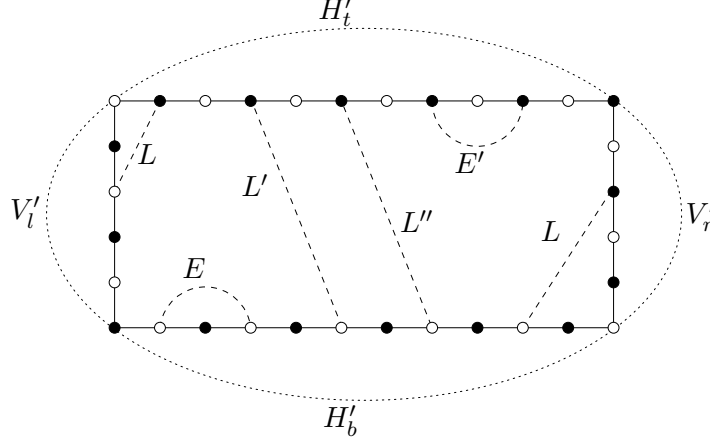
Figure 6: Illustration of $C \cap R$ for a toroidal cycle C and a rectangle R (the boundary of R is depicted as a solid line and the cycle C by dashed lines/curves): C has six segments, five A-segments (A-lines L, L' and L'' and ears E and E') and one B-segment. The curved dotted lines are the completions of H_t, V_r, H_b, V_l that form H'_t, V'_r, H'_b, V'_l, respectively.

but not A-regular (because there are two A-segments which are ears). The cycle C is called *regular*, if it is both A-regular and B-regular.

A toroidal system $(R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$ is called *consistent*, if the following properties hold:

Segment consistency (S1): Let S be a segment of C and let e and e' be the edges whose endpoints are incident to the endvertices of S. Then, S is bi-colored if and only if $|\{e, e'\} \cap \mathcal{P}_{A,B}|$ is even.

Line consistency (L1): If L is an A-line (respectively, B-line), then it has its black endvertex in A_t and its white endvertex in A_b (respectively, black endvertex in B_r and its white endvertex in B_l).

Ear consistency: Every ear E is mono-chromatic and satisfies:

Condition (E1): If E's endvertices are in A_t (respectively, B_r), then their color is black, otherwise their color is white.

Condition (E2): If D is the open disc of T bounded by the A-ear (respectively, B-ear) E and the loop-interval between E's endvertices, then E's endvertices are black if and only if D contains an odd number of endpoints of elements of $O(R) \cup O(C) \cup \mathcal{P}_{A,B}$.

Region consistency:

Condition (R1): If D is an open A-regular (respectively, B-regular) disc, then D contains an even number of endpoints of elements of $O(R) \cup O(C) \cup \mathcal{P}_{A,B}$.

Condition (R2): Let L be an A-line containing $k \geq 0$ edges of B. Let R' be obtained by concatenation of $k + 1$ copies of R by its vertical boundaries. This

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completely determines $N', A', B', \mathcal{P}'_{A',B'}, O(R')$ and $O(C')$. Let L' denote the cover of L drawn in R': we note that L' has no edge of B'. If D is an open disc bounded by L', an interval of H'_b (respectively, an interval of V'_l), the vertical side V'_r (respectively, horizontal side H'_t), and an interval of H'_t (respectively, V'_r), then D contains an even number of endpoints of elements of $O(R') \cup O(C') \cup \mathcal{P}'_{A',B'}$. We also require the analogous condition to hold for B-lines.

It is not hard to see that the following strengthening of Proposition 7 holds.

Proposition 8 *If P is a well behaved perfect matching of $T_{m,n}$, then $\mathcal{S}(P)$ is a consistent toroidal system.*

We say that C has *type* $(0,1)$ if C has an even number of edges of A and an odd number of edges of B. Analogously, we define types $(0,0),(1,0),(1,1)$. As usual, a type is also associated to the set $\mathcal{P}_{A,B}$ according to the parity of its intersection with A and B, respectively.

Our goal for the remaining part of this article is to prove the following result (which immediately implies Theorem 3:

Theorem 9 *Let $(R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$ be a consistent toroidal system. If neither the type of C nor the type of $\mathcal{P}_{A,B}$ is $(0,0)$, then the type of C is the same as the type of $\mathcal{P}_{A,B}$.*

Henceforth, we say that two toroidal systems $\mathcal{S} = (R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$ and $\mathcal{S}' = (R', N', \{A', B'\}, \mathcal{P}'_{A',B'}, C', O(C'), O(R'))$ are *equivalent* if the type of C is the same as the type of C' and the type of $\mathcal{P}_{A,B}$ is the same as the type of $\mathcal{P}'_{A',B'}$.

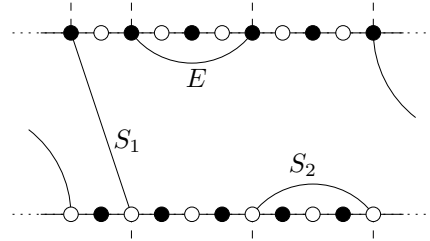Theorem 9 will be a direct consequence of the the next two results.

Proposition 10 *If $\mathcal{S} = (R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$ is a consistent toroidal system where neither the type of C nor the type of $\mathcal{P}_{A,B}$ is $(0,0)$, then either Theorem 9 holds for $\mathcal{S}$ or there is a regular consistent toroidal system $\mathcal{S}'$ which is equivalent to $\mathcal{S}$.*

Proof We describe how to iteratively construct a new A-regular consistent toroidal system, say $\mathcal{S}' = (R', N', \{A', B'\}, \mathcal{P}'_{A',B'}, C', O(C'), O(R'))$, equivalent to $\mathcal{S}$. The system $\mathcal{S}'$ is then analogously adjusted (arguing with respect to B instead of A) to yield the claimed consistent regular system.
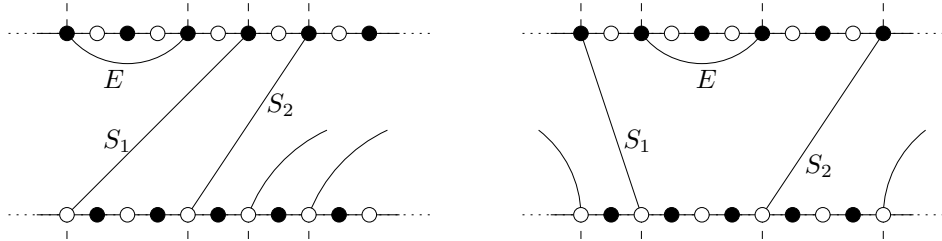
Let E be a black-endvertices-A-ear which is *minimal*, i.e., C does not intersect the loop interval between the endvertices of E. Let e_1 and e_2 be the edges of $A \cap C$ which are adjacent to E. Let the black-endvertex of e_i be denoted by x_i, and the white-endvertex by y_i, $i = 1, 2$. Denote by D the open disk with boundary E and the loop interval between x_1 and x_2. By the choice of E, the loop intervals between x_1, x_2 and between y_1, y_2 do not intersect C. Because of consistency conditions (S1) and (E1), there is exactly one edge of $\mathcal{P}_{A,B} \cap C$ incident with an endvertex of E: without loss of generality let it be e_1. Let us denote by S_1 the segment of C prolonging e_1 and by S_2 the segment of C extending e_2.

Next, we establish the following claim; the set of edges of $\mathcal{P}_{A,B}$ that are incident with the interior of the loop interval between x_1 and x_2, henceforth denoted $\mathcal{P}_{A,B}(E)$, has an even cardinality. Indeed, because (E2) holds for E, there is an odd number of endpoints of elements of $O(R) \cup O(C) \cup \mathcal{P}_{A,B}$ contained in D. By minimality of E, none of the elements of $O(C)$ is contained in D and each element of $\mathcal{P}_{A,B}$ is either contained in D or disjoint with it. Hence, the interior of the loop interval between x_1 and x_2 is incident with an odd number of curves of $O(R)$; let us denote by $R(E)$ the set consisting of these curves. All (the odd number of) vertices in the interior of the loop interval between x_1 and x_2 not covered by elements of $R(E)$ must be covered by elements of $\mathcal{P}_{A,B}(E)$. We conclude that $|\mathcal{P}_{A,B}(E)| + |R(E)|$ and $|R(E)|$ are both odd, and hence $|\mathcal{P}_{A,B}(E)|$ is even, thus proving the claim.
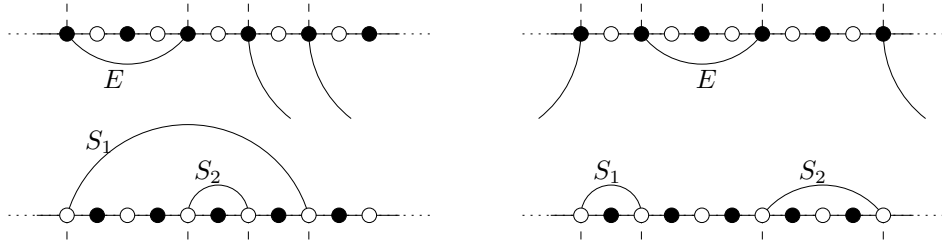
There are three possibilities for the types of S_1 and S_2: (1) S_1 is an A-line and S_2 is a white-endvertices-A-ear, (2) both S_1 and S_2 are A-lines, or (3) both S_1 and S_2 are white-endvertices-A-ears. Each possible case is illustrated in Figure 7(a), 7(b), and 7(c): note that for the last two cases there are two subcases depending on the relative positions of S_1 and S_2.
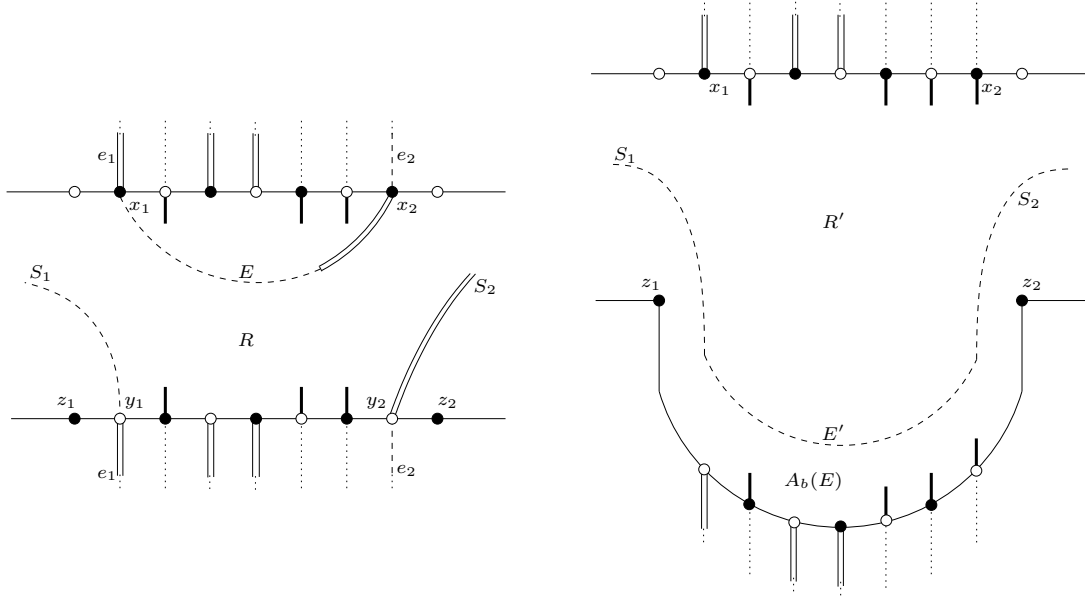

(a) Case (1).

(b) Case (2).

(c) Case (3).

Figure 7:

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By consistency conditions (L1) and (E1), the vertices located at the endpoints of $S_1e_1Ee_2S_2$ are colored black and white, respectively, in case (1); are both black, in case (2); and, are both white, in case (3).

(a) Illustration of a portion of a toroidal system $\mathcal{S}$ containing a black-endvertices-A-ear labeled E. Cycle C shown as dashed line, edges in A shown as dotted lines, elements of $O(R)$ shown as thick lines, elements of $\mathcal{P}_{A,B}$ depicted as double lines outside R, and elements of $O(C)$ depicted also as double lines but inside R.

(b) Illustration of the portion of the toroidal system $\mathcal{S}'$ obtained by applying (to the system $\mathcal{S}$ illustrated in Figure 8(a)) the procedure described in the proof of Proposition 10. The components C', $\mathcal{P}'_{A',B'}$, and the elements of $O(R')$ and $O(C')$ are depicted following the same conventions as in Figure 8(a).

Figure 8:

Let z_i be the neighbor of y_i along H_b (see Figure 8(a)). We construct R' by attaching a new disc $A_b(E)$ to the loop interval between z_1 and z_2; in the interior of the outer boundary of $A_b(E)$ we place the same number of vertices as there are in the loop interval of R between z_1 and z_2 (see again Figure 8(a)). We then copy e_1Ee_2 to the just added disc and obtain C' from C as shown in Figure 8(b). Next, we remove vertices that end up in the interior of R'. This completely determines the new vertex set N'. Every A-edge e incident to vertices x_i' and y_i', x_i' between x_1 and x_2 (including them) and y_i' between y_1 and y_2 (also including them), is replaced by an edge e' with endvertices x_i' and the copy of y_i' that was placed in the outer boundary of $A_b(E)$. All other edges are preserved. This completely determines the new set of edges A' and B'. Note that there is a one-to-one correspondence between old A/B-edges and new A'/B'-edges. The set of new edges associated to edges in $\mathcal{P}_{A,B}$ determine $\mathcal{P}'_{A',B'}$. The curves of $O(C')$ are those elements of $O(C)$ contained in C' which are incident to the boundary of R' (see Figure 8(b)). Finally, we add to $O(R')$ all elements in $O(R)$ that

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are incident to the boundary of R' as well as curves that match all the not-yet-covered new vertices (see Figures 8(a) and 8(b)).

We need to show that the new system $\mathcal{S}'$ is consistent. First, note that $S = S_1 e_1 E e_2 S_2$ is the only new segment created. In case (1), S is a line whose endvertex in A_t is the corresponding vertex of S_1 (hence, by consistency condition (L1) of S_1, it is colored black) and whose other endvertex in A_b is the corresponding one of S_2 (hence, by (E1) of S_2, it is colored white). It follows that the line S satisfies condition (L1). In case (2), S is a black-endvertices-A-ear with endvertices in A_t; and, in case (3), S is a white-endvertices-A-ear with endvertices in A_b. Thus, for these latter two cases, condition (E1) is satisfied.

We show next that S satisfies consistency condition (S1). Let e'_1 be the edge incident to the endvertex of S_1 not covered by e_1. Similarly, let e'_2 be the edge incident to the endvertex of S_2 not covered by e_2. Because consistent lines are bi-colored and consistent ears are monochromatic, we have that S is bicolored if and only if case (1) holds. Thus, to establish that (S1) holds in $\mathcal{S}'$ we need to show that $|\{e'_1, e'_2\} \cap \mathcal{P}'_{A',B'}|$ is even if and only if case (1) holds. Since by condition (E1), the term $|\{e_1, e_2\} \cap \mathcal{P}_{A,B}|$ is odd, we get that the parity of $|\{e'_1, e'_2\} \cap \mathcal{P}'_{A',B'}|$ differs from the parity of $|\{e'_1, e_1\} \cap \mathcal{P}_{A,B}| + |\{e_2, e'_2\} \cap \mathcal{P}_{A,B}|$, which by consistency conditions (L1) and (E1) of $\mathcal{S}$ is odd if and only if case (1) occurs. This finishes the proof that $\mathcal{S}'$ satisfies (S1).

It is important to note that regarding B-ears and B-regular discs, we did not change the system. Hence, B-ears and open B-regular discs stay consistent.

Next, we show that A-ears and regions of $\mathcal{S}'$ are also consistent. First, let $\widetilde{E}$ be an A-ear of $\mathcal{S}'$ which does not involve the new segment S. Clearly, $\widetilde{E}$ is an A-ear in $\mathcal{S}'$ as well. Observe that unless the disc $\widetilde{D}$ of the consistency condition (E2) of $\widetilde{E}$ intersects $A_b(E)$, condition (E2) is clearly satisfied. Thus, assume $\widetilde{D}$ intersects $A_b(E)$. Note that by minimality of E, $|R(E)|+1$ endpoints are removed from the interior of $\widetilde{D} \setminus A_b(E)$ and exactly the same number is added to the interior of $A_b(E)$. More precisely, $|R(E)|$ of the removed points are endpoints of elements of $O(R)$ incident to vertices between y_1 and y_2 and the remaining removed point is the endpoint of the element of $O(C)$ covering y_2. Similarly, $|R(E)|+1$ of the added points belong to elements of $O(R')$ covering the vertices of the outer boundary of $A_b(E)$. The same reasoning establishes conditions (R1) and (R2) in $\mathcal{S}'$ when the region's boundary does not contain the new segment S.

We next argue that if S forms an A-ear, then (E2) holds for S in $\mathcal{S}'$. In case (1), no new ear is created. In case (3), the new segment S is a white-endvertices-A-ear and the interior of its open disc $D(S)$ consists of the interior of the open disc $D(S_1)$ of S_1, the interior of the open disc $D(S_2)$ of S_2, and the points on elements of $O(R')$ covering the vertices of the outer boundary of $A_b(E)$; we argued above that their number is even. Hence, by consistency of S_1 and S_2, segment S satisfies (E2). In case (2), the new segment S is a black-endvertices-A-ear. Because (L1) and (E2) hold for $\mathcal{S}$, the difference between the number of endpoints of elements of $O(R) \cup O(C) \cup \mathcal{P}_{A,B}$ in its interior and in the open A-regular disc of $\mathcal{S}$ defined by S_1 and S_2 is exactly $|R(E)|$, which is odd. Hence, the new ear S also satisfies (E2).

It remains to check (R1) and (R2) for the regions which contain the new segment S as a line in their boundary. This situation only arises in case (1) when S_1 is a line and S_2

is a white-endvertices-A-ear in $\mathcal{S}$. Next, we determine the difference of the number of the inner points on each side of S with respect to S_1. On one side we add the even number of endpoints of elements of $O(R')$ covering the vertices of the outer boundary of $A_b(E)$. To the same side we add the points in the open disc $D(S_2)$ of S_2, whose number is even by (E2). On the other side we delete $|R(E)|$ (odd number) endpoints covered by elements of $O(R)$ incident with the interior of the loop interval between y_1 and y_2. We further delete an even number of endpoints in the interior of $D(S_2)$, and finally we delete exactly one endpoint in the interior of S_2 (the endpoint in the interior of R of the element of $O(C)$ incident to y_2). Summarizing, the aforementioned difference on each side of S with respect to S_1 is even and thus both conditions (R1) are (R2) are established. $\square$

Summing up, if there is a counterexample to Theorem 9, then there is a regular counterexample $\mathcal{S} = (R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$. Assume $\mathcal{S}$ is such that $|A \cap C| = k$ and $|B \cap C| = l$. Let $\mathcal{S}' = (R', N', \{A', B'\}, \mathcal{P}'_{A',B'}, C', O(C'), O(R'))$ be the system obtained by first concatenating l copies of $\mathcal{S}$ by the vertical boundaries of R, and then concatenating k copies of the resulting system by the horizontal boundaries of R. We claim that $\mathcal{S}'$ is a regular consistent toroidal system. Indeed, the regularity is clear since C' has exactly one edge of A' and exactly one edge of B'. Thus, conditions (E1) and (E2) vacuously hold for $\mathcal{S}'$. Moreover, conditions (S1) and (L1) are easily seen to be inhereted by $\mathcal{S}'$ from $\mathcal{S}$. Hence, the region consistency conditions (R1) and (R2) need to be checked and easily seen to follow from the consistency of $\mathcal{S}$.

If both k and l are odd, then the type of $\mathcal{P}'_{A',B'}$ is the same as the type of $\mathcal{P}_{A,B}$. Let exactly one of k, l, say k, be even. Furthermore, let $k > 0$. Since $\mathcal{S}$ is a counterexample to Theorem 9, it must be that $\mathcal{P}_{A,B}$ has an odd number of edges of A and thence $\mathcal{P}'_{A',B'}$ has an odd number of edges of A'. Clearly, $\mathcal{P}'_{A',B'}$ will have an even number of edges of B'. Since C' has exactly one edge in each of A' and B', it follows that $\mathcal{S}'$ would also be a counterexample to Theorem 9. Summarizing, if there is a counterexample to Theorem 9, then there is a counterexample $\mathcal{S}$ with both $A \cap C$ and $B \cap C$ having at most one edge.

Proposition 11 *If $\mathcal{S} = (R, N, \{A, B\}, \mathcal{P}_{A,B}, C, O(C), O(R))$ is a regular consistent toroidal system with both $A \cap C$ and $B \cap C$ consisting of at most one edge, then $\mathcal{S}$ satisfies Theorem 9.*

Proof First, we consider the case where $C \cap B = \emptyset$. Because C is not of type $(0, 0)$, it must be that $|C \cap A| = 1$. Let L be the only A-line. Let D be the open disc bounded by L, an interval of H_b, the vertical side of V_r, and an interval of H_t. By condition (R2), we have that D contains an even number of endpoints of elements of $O(R) \cup O(C) \cup \mathcal{P}_{A,B}$. However, only endpoints of $O(R)$ belong to D, and therefore $O(R)$ covers an even number of vertices of the boundary of D. The remaining (even number) of vertices must be covered by $\mathcal{P}_{A,B}$. Since each edge of $\mathcal{P}_{A,B} \cap A$ covers an even number of vertices of the boundary of D, it follows that $|\mathcal{P}_{A,B} \cap B|$ is even. Since $\mathcal{P}_{A,B}$ can not be of type $(0, 0)$, we get that $\mathcal{P}_{A,B}$ is of type $(1, 0)$, thus completing the analysis of the case being considered. The case that $C \cap A = \emptyset$ is analogous.

To conclude, let C have exactly one edge from both A and B. We note that $C \cap \mathcal{P}_{A,B}$ may consist of zero, one or two edges. We consider in detail the case where $|C \cap \mathcal{P}_{A,B}| = 2$. The

remaining cases can be dealt with similarly. Let L be the only A-line of C. As in condition (R2) for L, we concatenate two copies of R by its vertical side. Let A'-line L' be the cover of L in the concatenated rectangle R' and let D be one of the two open discs obtained by subtracting L' from the interior of R'.

By condition (R2) the disc D has an even number of endpoints of elements of $O(R')$. We also note that D has an even number of vertices in its boundary. It follows that $\mathcal{P}'_{A',B'}$ (see condition (R2)) covers an even number of vertices of D's boundary. Moreover, it is not hard to see that edges of $\mathcal{P}'_{A',B'}$ that are incident to exactly one vertex of the boundary of D can be placed in one-to-one correspondence with the edges of $\mathcal{P}_{A,B}$ that are incident to at least one vertex in the boundary of D. Hence, the cardinality of $\mathcal{P}_{A,B}$ must also be even. Since $\mathcal{P}_{A,B}$ is not of type $(0,0)$, it must be of type $(1,1)$. $\square$

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