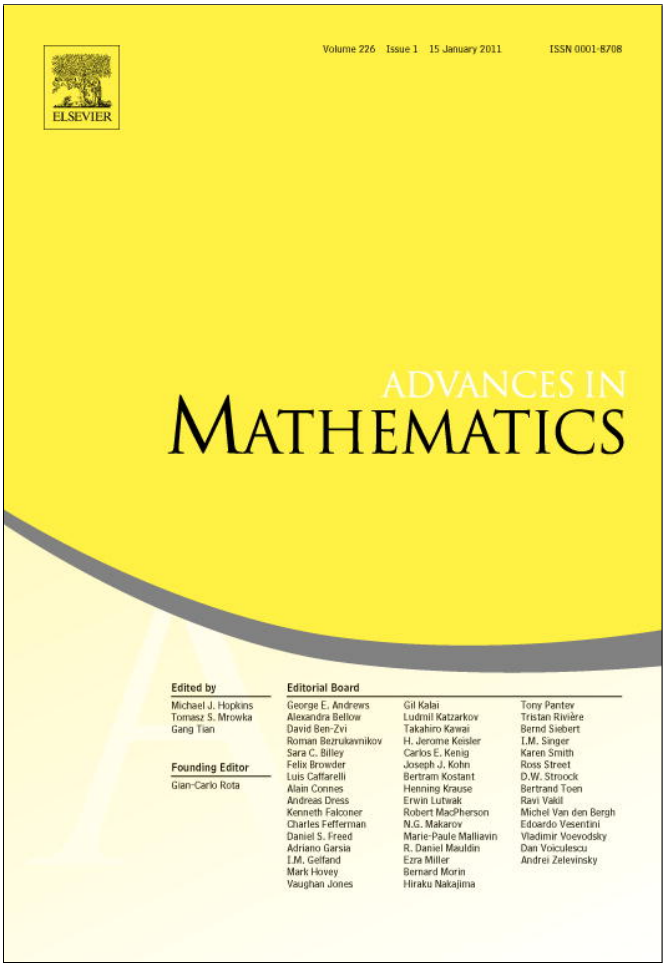
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On the optimality of the Arf invariant formula for graph polynomials

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Abstract

We prove optimality of the Arf invariant formula for the generating function of even subgraphs, or, equivalently, the Ising partition function, of a graph.

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Keywords: Ising partition function; Arf invariant formula; Graph embedding; Pfaffian; Complexity

Contents

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1. Introduction

Let $G = (V(G), E(G))$ be a finite un-oriented graph (loop-edges and multiple edges are allowed). We say that $E' \subset E(G)$ is *even* if the graph $(V(G), E')$ has even degree (possibly zero) at each vertex. By abuse of language, E' is also called an *even subgraph* of G. We say that $M \subset E(G)$ is a *perfect matching* if the graph $(V(G), M)$ has degree one at each vertex. Let $\mathcal{E}(G)$ denote the set of all even subgraphs of G, and let $\mathcal{P}(G)$ denote the set of all perfect matchings of G.

We assume that an indeterminate x_e is associated with each edge e, and define the generating polynomials for even sets and for perfect matchings, $\mathcal{E}_G$ and $\mathcal{P}_G$, in $\mathbb{Z}[(x_e)_{e\in E(G)}]$, as follows:

$$\mathcal{E}_G(x) = \sum_{E'\in\mathcal{E}(G)} \prod_{e\in E'} x_e,$$

$$\mathcal{P}_G(x) = \sum_{M\in\mathcal{P}(G)} \prod_{e\in M} x_e.$$

Knowing the polynomial $\mathcal{E}_G$ is equivalent to knowing the partition function Z_G^{Ising} of the Ising model on the graph G. This is explained later in the introduction.

Assume the vertices of G are numbered from 1 to n. If D is an orientation of G, we denote by $A(G, D)$ the *skew-symmetric adjacency matrix* of D defined as follows: the diagonal entries of $A(G, D)$ are zero, and the off-diagonal entries are

$$A(G, D)_{ij} = \sum \pm x_e,$$

where the sum is over all edges e connecting vertices i and j, and the sign in front of x_e is 1 if e is oriented from i to j in the orientation D, and -1 otherwise. As is well known, the Pfaffian of this matrix counts perfect matchings of G *with signs*:

$$\operatorname{Pfaf} A(G, D) = \sum_{M\in\mathcal{P}(G)} \operatorname{sign}(M, D) \prod_{e\in M} x_e,$$

where $\operatorname{sign}(M, D) = \pm 1$. We use this as the definition of the sign of a perfect matching M with respect to an orientation D.

We denote the polynomial $\operatorname{Pfaf} A(G, D) \in \mathbb{Z}[(x_e)_{e\in E(G)}]$ by $F_D(x)$ and call it the *Pfaffian* associated to the orientation D. The following result is well known.

Theorem 1. *(See Kasteleyn [9], Galluccio and Loebl [6], Tesler [15], Cimasoni and Reshetikhin [4].) If G embeds into an orientable surface of genus g, then there exist 4^g orientations D_i ($i = 1, \dots, 4^g$) of G such that the perfect matching polynomial $\mathcal{P}_G(x)$ can be expressed as a linear combination of the Pfaffian polynomials $F_{D_i}(x)$.*

The explicit expression for $\mathcal{P}_G(x)$ will be given in Theorem 2.11. We call it the *Arf invariant formula*, as it is based on a property of the Arf invariant of quadratic forms in characteristic two. As far as we know, the relationship with the Arf invariant was first observed in [4].

Let $c_{\text{match}}(G)$ be the minimal number of orientations D_i of G so that $\mathcal{P}_G(x)$ is a linear combination of the Pfaffian polynomials $F_{D_i}(x)$. We think of $c_{\text{match}}(G)$ as a kind of complexity of the graph G. Since every graph embeds into some surface, $c_{\text{match}}(G)$ is finite. Norine [13] conjectured that $c_{\text{match}}(G)$ is always a power of 4. He also showed that $c_{\text{match}}(G)$ cannot be equal to 2, 3, or 5. However, Miranda and Lucchesi [12] recently disproved Norine's conjecture by exhibiting a graph G with $c_{\text{match}}(G) = 6$.

The main result of the present paper is that, contrary to the case of perfect matchings, an analogue of Norine's conjecture is true for the even subgraph polynomial $\mathcal{E}_G(x)$ (or, equivalently, the Ising partition function of G). To explain the statement, we first need to recall how the Arf invariant formula for $\mathcal{P}_G$ can be used to obtain a similar formula for $\mathcal{E}_G$, using the following slight modification of a construction of Fisher. (This formula for $\mathcal{E}_G$ follows from [7, Theorem 2.3].) Although the construction may seem a little bit un-natural at first sight, it is justified by Proposition 1.2 and Theorem 2 below. We'll briefly comment on a different construction by Kasteleyn in Remark 1.4.

Definition 1.1. (See Fisher [5].) Let G be a graph. Let $\sigma = (\sigma_v)_{v \in V(G)}$ be a choice, for every vertex v, of a linear ordering of the half-edges incident with v. The *blow-up*, or Δ*-extension*, of (G, σ) is the graph G^σ obtained by performing the following operation one by one for each vertex v. Assume first that no edge incident with v is a loop-edge. Then σ_v is the same as a linear ordering of the edges incident with v. Let $e_1, \dots, e_d$ be this linear ordering and let $e_i = vu_i$, $i = 1, \dots, d$. We delete the vertex v and replace it with a path consisting of $6d$ new vertices $v_1, \dots, v_{6d}$ and edges $v_i v_{i+1}$, $i = 1, \dots, 6d - 1$. To this path, we add edges $v_{3j-2}v_{3j}$, $j = 1, \dots, 2d$. Finally we add edges $v_{6i-4}u_i$ corresponding to the original edges $e_1, \dots, e_d$. This definition can be extended naturally to the case where there are loop-edges, using that σ_v is a linear ordering on the set of half-edges incident with v.

The subgraph of G^σ spanned by the $6d$ vertices $v_1, \dots, v_d$ that replaced a vertex v of the original graph will be called a *gadget* and denoted by Γ_v. The edges of G^σ which do not belong to a gadget are in natural bijection with the edges of G. By abuse of notation, we will identify an edge of G with the corresponding edge of G^σ. Thus $E(G^\sigma)$ is the disjoint union of $E(G)$ and the various $E(\Gamma_v)$ ($v \in V(G)$).

It is important to note that different choices of linear orderings σ_v at the vertices of G may lead to non-isomorphic graphs G^σ. Nevertheless, one always has the following

Proposition 1.2. *(See Fisher [5].) There is a natural bijection between the set of even subsets of G and the set of perfect matchings of G^σ. More precisely, every even set $E' \subset E(G)$ uniquely extends to a perfect matching $M \subset E(G^\sigma)$, and every perfect matching of G^σ arises (exactly once) in this way.*

It follows that if we set the indeterminates associated to the edges of the gadgets equal to one in $\mathcal{P}_{G^\sigma}$, we get the even subgraph polynomial of our original graph G:

$$\mathcal{E}_G = \mathcal{P}_{G^\sigma}|_{x_e = 1\ \forall e \in E(G^\sigma) \setminus E(G)}. \tag{1}$$

If D is an orientation of G^σ, we define

$$F^\sigma_D(x) = \mathrm{Pfaf}\big(A\big(G^\sigma, D\big)\big|_{x_e=1\ \forall e \in E(G^\sigma)\setminus E(G)}\big).$$

Any polynomial obtained in this way will be called a *σ-projected Pfaffian*. Note that $F^\sigma_D(x)$ is a polynomial in the indeterminates associated to the edges of the original graph G.

Remark 1.3. Here, as before, we need to choose an ordering of the vertices of G^σ to define the adjacency matrix. We may, of course, take the ordering induced in the obvious way from an ordering of the vertices of G. In any case, permuting the ordering will only affect the sign of $F^\sigma_D(x)$: it gets multiplied by the sign of the permutation.

Now assume G is embedded into an orientable surface Σ of genus g. It is not hard to see that we can choose σ in such a way that G^σ also embeds into Σ. In view of (1), Theorem 1 implies the following result for $\mathcal{E}_G$:

Theorem 2. *(See Galluccio and Loebl [7].) If G embeds into an orientable surface of genus g, then for an appropriate choice of blow-up G^σ, there exist 4^g orientations D_i $(i = 1, \dots, 4^g)$ of G^σ such that the even subgraph polynomial $\mathcal{E}_G(x)$ can be expressed as a linear combination of the σ-projected Pfaffians $F^\sigma_{D_i}(x)$.*

Remark 1.4. As was pointed out to us by the referee, a similar result can also be obtained using a different and in some sense more natural blow-up construction discussed by Kasteleyn in [9, pp. 102–103], where every vertex of the original graph G is replaced by an even clique. In Kasteleyn's construction the correspondence between perfect matchings of the blown-up graph and even subgraphs of the original graph is not one-to-one, but many-to-one; however, with an appropriate choice of orientation of the edges of the clique, all but one of the perfect matchings corresponding to a fixed even subgraph cancel out when signs are taken into account. The drawback of Kasteleyn's construction is that the blown-up graph can in general not be embedded on the same surface as the original graph. Although the complications resulting from this problem can be dealt with, it is more appropriate for our purposes to use Fisher's construction where this problem does not arise.

It turns out that one can always choose the orientations D_i in Theorem 2 in such a way that the induced orientation on every gadget Γ_v is independent of i. (We will explain why this is so in Section 3, see Corollary 3.7.) This motivates the following definition.

Definition 1.5. Let $\Delta = (\Delta_v)_{v \in V(G)}$ be a choice of orientations of the gadgets Γ_v. An orientation D of G^σ is called *Δ-admissible* if D restricts to Δ_v on every gadget Γ_v.

Note that once Δ has been fixed, the set of Δ-admissible orientations of G^σ is in natural bijection with the set of orientations of the original graph G.

We now come to the main result of the paper, which is a lower bound for the number of orientations needed in the above expression for $\mathcal{E}_G$, provided we assume that the orientations in question are Δ-admissible.

Theorem 3 *(Main Theorem). Let G be a graph. Choose a blow-up G^σ and an orientation Δ of the gadgets that replaced the vertices of G in G^σ. Let $c_{\sigma,\Delta}(G)$ be the minimal cardinality of a set of Δ-admissible orientations D_i of G^σ such that the even subgraph polynomial $\mathcal{E}_G$ is a linear combination of the σ-projected Pfaffians $F^\sigma_{D_i}$. Then $c_{\sigma,\Delta}(G)$ is a power of* 4.

Let us denote by $c_{\mathrm{Ising}}(G)$ the minimum of the numbers $c_{\sigma,\Delta}(G)$, over all choices of σ and Δ. In view of the relationship of even subgraphs with the Ising model, we call $c_{\mathrm{Ising}}(G)$ the *Ising complexity* of G.

Theorem 4 *(Main Theorem (Cont'd)). For every graph G, the Ising complexity satisfies*

$$c_{\mathrm{Ising}}(G) = 4^g,$$

where the number g is the embedding genus of G.

Here, the *embedding genus* of G is the minimal genus of an orientable surface in which G can be embedded.

The proof of Theorems 3 and 4 will be given in Section 4.

Let us end the Introduction by pointing out some relations of our results with other topics.

1.1. *Equivalence of $\mathcal{E}_G(x)$ and the Ising partition function*

The Ising partition function is defined by

$$Z_G^{\mathrm{Ising}}(\beta) = Z_G^{\mathrm{Ising}}(x)\big|_{x_e := e^{\beta J_e}\ \forall e \in E(G)}$$

where the J_e ($e \in E(G)$) are weights (coupling constants) associated with the edges of the graph G, the parameter β is the inverse temperature, and

$$Z_G^{\mathrm{Ising}}(x) = \sum_{\sigma: V(G) \to \{1,-1\}} \prod_{e=\{u,v\} \in E(G)} x_e^{\sigma(u)\sigma(v)}.$$

The theorem of van der Waerden [18] (see [10, Section 6.3] for a proof) states that $Z_G^{\mathrm{Ising}}(x)$ is the same as $\mathcal{E}_G(x)$ up to change of variables and multiplication by a constant factor:

$$Z_G^{\mathrm{Ising}}(x) = 2^{|V(G)|}\left(\prod_{e \in E(G)} \frac{x_e + x_e^{-1}}{2}\right)\mathcal{E}_G(z)\bigg|_{z_e := \frac{x_e - x_e^{-1}}{x_e + x_e^{-1}}}.$$

1.2. *Determinantal complexity*

For a polynomial $P(x_1, \ldots, x_n)$ with rational coefficients, let the *determinantal complexity* of P, denoted by $c_{\det}(P)$, be the minimum m so that, if A is the $m \times m$ matrix of variables $(x_{ij})_{i,j=1,\ldots,m}$, then P may be obtained from the determinant $\det(A)$ by a number of applications of the operation of replacing some variable x_{ij} by a variable x_k or by a rational constant. This concept was introduced by Valiant (see [17]) who also proved that $c_{\text{det}}(P)$ is at most $2c_{\text{form}}(P) + 2$; here $c_{\text{form}}(P)$ denotes the *formula size* of P, *i.e.* the minimum number of additions and multiplications one needs to obtain P starting from the variables $x_1, \dots, x_n$ and constants. The main problem in the area of algebraic complexity theory is to find lower bounds for $c_{\text{form}}(P)$. Lower bounds for $c_{\text{det}}(P)$ have recently been investigated extensively (see *e.g.* [11]). We suggest to study $c_{\text{det}}(\mathcal{E}(G))$ using the methods introduced in the present paper.

1.3. Pfaffian graphs

It follows from Theorem 4 that $c_{\text{Ising}}(G) = 1$ if and only if G is planar. This characterises the graphs for which the Ising partition function $Z_G(x)$ is equal to one Pfaffian, in the sense of Theorems 3 and 4. Note that this characterisation can be formulated in terms of excluded minors, by Kuratowski's theorem. It also provides a polynomial algorithm to recognize the graphs G for which $c_{\text{Ising}}(G) = 1$, since planar graphs can be recognized in polynomial time. We remark that it remains a longstanding open problem to characterise the *Pfaffian graphs*, *i.e.* graphs G satisfying $c_{\text{match}}(G) = 1$, in a way which yields a polynomial recognition algorithm (see [16]).

1.4. Even drawings

Norine [13] has considered drawings φ of a graph G on an orientable surface Σ such that the self-intersection number of every perfect matching M of G in this drawing is even. (Contrary to our definition of drawings (see Definition 2.1 below), he does not, however, allow edges to self-intersect.) Let us call a drawing satisfying Norine's definition *matching-even*. As pointed out by Norine, the Arf invariant formula for perfect matchings (Theorem 1) goes through if we have a matching-even drawing of a graph on a surface in place of an embedding. Moreover, Norine has shown that $c_{\text{match}}(G) = 1$ if and only if G has a matching-even drawing in the plane, and $c_{\text{match}}(G) = 4$ if and only if G has a matching-even drawing on the torus. It is, however, conceivable that, in general, the minimal genus of a surface supporting a matching-even drawing of G could be smaller than the embedding genus. We will point out that no such phenomenon can occur for even subgraphs (see Section 5 and Theorem 5 for a precise statement).

2. The Arf invariant formula for perfect matchings

In this section, we give a proof of the Arf invariant formula for the perfect matching polynomial. Other proofs can be found in [6,15,4].

Definition 2.1. A *drawing* of a finite graph G on a surface Σ is a continuous and piecewise smooth map φ from the topological realization of G (as a one-dimensional CW-complex) to Σ so that φ is injective except for a finite number of transverse double points, subject to the condition that for every double point $p = \varphi(x) = \varphi(x')$, none of the preimages x and x' is a vertex.

In other words, all intersections in the drawing happen in the interiors of edges. Note that we allow self-intersections of edges in this definition. A drawing without double points is called an *embedding*.

If a drawing φ of G is given, and $E' \subset E(G)$ is a collection of edges, we denote by $\kappa_\varphi(E')$ the number (mod 2) of double points of $\varphi(E')$. Note that if $E' = \{e_1, \dots, e_k\}$ is a collection of distinct edges, then

$$\kappa_\varphi\bigl(E'\bigr) = \sum_i \kappa_\varphi(e_i) + \sum_{i<j} \mathrm{cr}_\varphi(e_i, e_j) \pmod 2, \tag{2}$$

where $\mathrm{cr}_\varphi(e_i, e_j)$ is the number of intersections of the interiors of the edges e_i and e_j in the drawing. We emphasize that vertices of the graph never count as intersection points.

We will use the following result of Tesler.

Theorem 2.2. *(See Tesler [15].) Let G be a graph drawn in the plane. Then there is $\varepsilon_0 \in \{\pm 1\}$ and an orientation D_0 of G so that for every perfect matching M of G, its sign in* $\mathrm{Pfaf}(A(G, D_0))$ *satisfies*

$$\mathrm{sign}(M, D_0) = \varepsilon_0 (-1)^{\kappa_\varphi(M)}. \tag{3}$$

Definition 2.3 *(Tesler).* An orientation D_0 satisfying (3) is called a *crossing orientation*.

We now describe how an embedding of a graph in a surface can be used to make a planar drawing of that graph of a special kind. First, recall the following standard description of a genus g surface S_g with one boundary component. (We reserve the notation Σ_g for a closed surface of genus g.)

Definition 2.4. The *highway surface* S_g consists of a base polygon R_0 and bridges $R_1, \dots, R_{2g}$, where

- R_0 is a convex $4g$-gon with vertices $a_1, \dots, a_{4g}$ numbered clockwise;
- Each R_{2i-1} is a rectangle with vertices $x(i,1), \dots, x(i,4)$ numbered clockwise. It is glued with R_0 so that its edge $[x(i,1), x(i,2)]$ is identified with the edge $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ and the edge $[x(i,3), x(i,4)]$ is identified with the edge $[a_{4(i-1)+3}, a_{4(i-1)+4}]$;
- Each R_{2i} is a rectangle with vertices $y(i,1), \dots, y(i,4)$ numbered clockwise. It is glued with R_0 so that its edge $[y(i,1), y(i,2)]$ is identified with the edge $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ and the edge $[y(i,3), y(i,4)]$ is identified with the edge $[a_{4(i-1)+4}, a_{4(i-1)+5}]$. (Here, indices are considered modulo $4g$.)

There is an orientation-preserving immersion Φ of S_g into the plane which is injective except that for each $i = 1, \dots g$, the images of the bridges R_{2i} and R_{2i-1} intersect in a square.

Now assume the graph G is embedded into a closed orientable surface Σ_g of genus g. We think of Σ_g as S_g union an additional disk R_∞ glued to the boundary of S_g. By an isotopy of the embedding, we may assume that G does not meet the disk R_∞, and that, moreover, all vertices of G lie in the interior of R_0. We may also assume that the intersection of G with any of the rectangular bridges R_i consists of disjoint straight lines connecting the two sides of R_i which are glued to the base polygon R_0. (This last assumption is not really needed, but it makes the proof of Proposition 2.6 below more transparent.) If we now compose the embedding of G into S_g with the immersion Φ, we get a drawing φ of G in the plane. A planar drawing of G obtained in this way will be called *special*. Observe that double points of a special drawing can only come from the intersection of the images of bridges under the immersion Φ of S_g into the plane. Thus every double point of a special drawing lies in one of the squares $\Phi(R_{2i}) \cap \Phi(R_{2i-1})$.

We now explain how a special drawing can be used to get a homological expression for the sign of a perfect matching. Let $H = H_1(\Sigma_g; \mathbb{F}_2)$ be the first homology group of Σ_g with coefficients in the field $\mathbb{F}_2$. We have canonical isomorphisms $H \cong H_1(S_g; \mathbb{F}_2) \cong H_1(S_g, R_0; \mathbb{F}_2)$. This gives us a basis

$$a_1, b_1, \dots, a_g, b_g \tag{4}$$

of H, where a_i corresponds to the class of the bridge R_{2i-1} in $H_1(S_g, R_0; \mathbb{F}_2)$, and b_i corresponds to the class of R_{2i}. Recall that H has a non-degenerate (skew-)symmetric bilinear form called the (mod 2) *intersection form*. (The name comes from the fact that this form can be defined using intersection numbers of closed curves on the surface.) We let $\cdot$ denote this form. In the basis (4), it is given by

$$a_i \cdot a_j = b_i \cdot b_j = 0, \tag{5}$$

$$a_i \cdot b_j = \delta_i^j \tag{6}$$

for all $i, j = 1, \dots, g$.

Definition 2.5. A *quadratic form on* $(H, \cdot)$ is a function $q : H \to \mathbb{F}_2$ so that

$$q(x + y) = q(x) + q(y) + x \cdot y \quad (x, y \in H). \tag{7}$$

We denote the set of such quadratic forms by Q. It follows from (7) that $q(0) = 0$ for all $q \in Q$. Also, (7) implies that a quadratic form $q \in Q$ is determined by its values on a basis of H, and these values can be prescribed freely in $\mathbb{F}_2$. It follows that Q has 4^g elements (the number of elements in $\mathbb{F}_2^{2g}$). Another way to see that Q has 4^g elements is to observe that the dual vector space $H^* = \mathrm{Hom}(H, \mathbb{F}_2)$ acts simply transitively on Q, where $\ell \in H^*$ acts on $q \in Q$ to give $q + \ell \in Q$.

The usefulness of special drawings and quadratic forms for studying perfect matchings comes from the following basic proposition. To state it, let $q_0 \in Q$ be the quadratic form on H whose value on each of the basis vectors a_i and b_i is zero.

Proposition 2.6. *Let φ be a special planar drawing of G obtained from an embedding of the graph G on the surface Σ_g. Then for every perfect matching $M \subset E(G)$ the number of double points $k_\varphi(M)$ satisfies*

$$k_\varphi(M) = q_0\big([M]\big) \pmod 2$$

where $[M]$ is the homology class of M.

Here, the homology class of a perfect matching is defined as follows. First, since all vertices of G lie in the base polygon R_0, every edge e of G defines a homology class $[e]$ in $H_1(S_g, R_0; \mathbb{F}_2)$. Since this group is canonically identified with H, we can think of $[e]$ as an element of H. If now M is a collection of distinct edges e_i, we let $[M]$ be the sum of the $[e_i]$.

Proof of Proposition 2.4. In view of (2) and (7), it is enough to show that

(i) For every edge e, the number of double points of $\varphi(e)$ is equal to $q_0([e])$ (mod 2).
(ii) For every pair of distinct edges e_1, e_2, the number $\mathrm{cr}_\varphi(e_1, e_2)$ is equal to $[e_1] \cdot [e_2]$ (mod 2).

Recall that every double point of a special drawing lies in one of the squares $\Phi(R_{2i}) \cap \Phi(R_{2i-1})$. To prove (i), assume that $e \cap R_{2i-1}$ consists of α_i straight lines, and $e \cap R_{2i}$ consists of β_i straight lines, for $i = 1, \dots, g$. Then the number of double points is

$$k_\varphi(e) = \sum \alpha_i \beta_i \pmod 2.$$

On the other hand, the homology class of e is $[e] = \sum_i \alpha_i a_i + \sum_j \beta_j b_j$. Using (5), (6), and (7), one has

$$\begin{aligned} q_0\big([e]\big) &= q_0\Big(\sum \alpha_i a_i\Big) + \Big(\sum \alpha_i a_i\Big) \cdot \Big(\sum \beta_j b_j\Big) + q_0\Big(\sum \beta_j b_j\Big) \\ &= \sum \alpha_i \beta_i \pmod 2, \end{aligned}$$

since $q_0(a_i) = 0 = q_0(b_j)$ for all i and j by the definition of q_0. Thus $k_\varphi(e) = q_0([e])$ (mod 2), as asserted. Statement (ii) is proved in a similar way. □

The following corollary is immediate from the definition of a crossing orientation.

Corollary 2.7. *Let φ be a special planar drawing of G obtained from an embedding of the graph G on the surface Σ_g. Let D_0 be a crossing orientation of G with respect to this drawing. Then there is $\varepsilon_0 \in \{\pm 1\}$ so that for every perfect matching M of G, its sign in* $\mathrm{Pfaf}(A(G, D_0))$ *satisfies*

$$\mathrm{sign}(M, D_0) = \varepsilon_0 (-1)^{q_0([M])}.$$

Thus the quadratic form q_0 controls the sign of any perfect matching in the orientation D_0. The following proposition says that, more generally, every $q \in Q$ controls the sign of perfect matchings in some orientation.

Proposition 2.8. *Let φ be a special planar drawing of G obtained from an embedding of the graph G on the surface Σ_g. Then there is $\varepsilon_0 \in \{\pm 1\}$, and a collection (D_q) of 4^g orientations of G indexed by quadratic forms $q \in Q$, such that for every perfect matching M of G one has*

$$\mathrm{sign}(M, D_q) = \varepsilon_0 (-1)^{q([M])}.$$

The following notation will be useful: if D is an orientation of a graph, and S is a set of edges, we write $D(S)$ for the orientation obtained from D by reversing the orientation of all edges in S.

Proof of Proposition 2.8. For $q = q_0$ we take D_q to be the crossing orientation D_0 which exists by Tesler's theorem (Theorem 2.2). Any other $q \in Q$ can be uniquely written as $q = q_0 + \ell$ where $\ell \in H^*$ is a linear form on H. We define $S_q \subset E(G)$ to be the set of edges e such that $\ell([e]) \neq 0 \in \mathbb{F}_2$, and define D_q to be the orientation $D_0(S_q)$. We have

$$\begin{aligned}\operatorname{sign}(M, D_q) &= \operatorname{sign}(M, D_0)(-1)^{|M\cap S_q|}\\ &= \varepsilon_0(-1)^{q_0([M])}(-1)^{|\{e\in M \mid \ell([e])\neq 0\}|}\\ &= \varepsilon_0(-1)^{q_0([M])}(-1)^{\ell([M])}\\ &= \varepsilon_0(-1)^{q([M])},\end{aligned}$$

as asserted. □

We now recall the definition of the *Arf invariant* [1] of a quadratic form $q \in Q$. Let $N_0 = 2^{g-1}(2^g+1)$, $N_1 = 2^{g-1}(2^g-1)$, and observe that $N_0 + N_1 = 4^g$ and $N_0 - N_1 = 2^g$. Recall that any $q \in Q$ is a function $H \to \mathbb{F}_2$.

Fact 2.9 *(Arf).* Any $q \in Q$ either takes N_0 times the value 0 (and hence N_1 times the value 1), or q takes N_1 times the value 0 (and hence N_0 times the value 1). We define $\operatorname{Arf}(q) \in \mathbb{F}_2$ to be equal to zero in the first case, and equal to one in the second case. Thus, for every $q \in Q$ one has

$$\sum_{x\in H}(-1)^{q(x)} = (-1)^{\operatorname{Arf}(q)}2^g. \tag{8}$$

For more about the Arf invariant, see for example Johnson [8], Atiyah [2]. We remark that there are N_0 quadratic forms of Arf invariant zero, and (hence) N_1 quadratic forms of Arf invariant one. In fact, the assignment $q \mapsto \operatorname{Arf}(q)$ is itself a quadratic form in an affine sense (see Theorems 2 and 3 of [2]).

The relevance of the Arf invariant for us comes from the following lemma, which is in some sense the dual statement to (8).

Lemma 2.10. *For every $x \in H$, one has*

$$\frac{1}{2^g}\sum_{q\in Q}(-1)^{\operatorname{Arf}(q)}(-1)^{q(x)} = 1. \tag{9}$$

We defer the proof to the end of this section. Combining Lemma 2.10 with Proposition 2.8, it follows that for every perfect matching M of G, we have

$$\frac{\varepsilon_0}{2^g}\sum_{q\in Q}(-1)^{\operatorname{Arf}(q)}\operatorname{sign}(M, D_q) = 1. \tag{10}$$

We refer to (10) as the *Arf invariant formula*. Thus, we have obtained the following more precise version of Theorem 1 stated in the introduction.

Theorem 2.11 *(Arf invariant formula for perfect matchings). Let the graph G be embedded into a closed orientable surface Σ_g of genus g. Then the perfect matching polynomial $\mathcal{P}_G$ can be written as a sum of 4^g Pfaffians associated to orientations D_q indexed by quadratic forms* *on* $H = H_1(\Sigma_g; \mathbb{F}_2)$:

$$\mathcal{P}_G = \sum_{q \in Q} \alpha_q \operatorname{Pfaf}\bigl(A(G), D_q\bigr),$$

where $\alpha_q = \varepsilon_0 (-1)^{\operatorname{Arf}(q)}/2^g$.

It remains to give the

Proof of Lemma 2.10. For $z \in H$, define $q_z : H \to \mathbb{F}_2$ by $q_z(x) = q_0(x) + z \cdot x$. Since $x \mapsto z \cdot x$ is a linear form on H, one has $q_z \in Q$. We claim that $\operatorname{Arf}(q_z) = q_0(z)$. Indeed, one has

$$\begin{aligned}\sum_{x \in H} (-1)^{q_z(x)} &= \sum_{x \in H} (-1)^{q_0(x) + z \cdot x} \\ &= (-1)^{q_0(z)} \sum_{x \in H} (-1)^{q_0(x+z)} = (-1)^{q_0(z)} \sum_{x \in H} (-1)^{q_0(x)} \\ &= (-1)^{q_0(z)} (-1)^{\operatorname{Arf}(q_0)} 2^g,\end{aligned}$$

where we have used (7) in the second equality and (8) in the last equality. But it is easy to check that $\operatorname{Arf}(q_0) = 0$. This proves the claim that $\operatorname{Arf}(q_z) = q_0(z)$ (again by the characterisation of the Arf invariant in (8)).

Now observe that the correspondence $z \mapsto q_z$ establishes a bijection $H \overset{\approx}{\to} Q$. This is because H^* acts simply transitively on Q, as already remarked above, and any linear form $\ell \in H^*$ is of the form $\ell(x) = z \cdot x$ for a unique $z \in H$ (because the intersection form is non-degenerate). Therefore we can prove (9) as follows:

$$\begin{aligned}\sum_{q \in Q} (-1)^{\operatorname{Arf}(q)} (-1)^{q(x)} &= \sum_{z \in H} (-1)^{\operatorname{Arf}(q_z)} (-1)^{q_z(x)} \\ &= \sum_{z \in H} (-1)^{q_0(z)} (-1)^{q_0(x) + z \cdot x} \\ &= \sum_{z \in H} (-1)^{q_0(z+x)} = \sum_{z \in H} (-1)^{q_0(z)} \\ &= (-1)^{\operatorname{Arf}(q_0)} 2^g = 2^g.\end{aligned}$$

This completes the proof of Lemma 2.10. $\square$

3. The Arf invariant formula for even subgraphs

Let G be a finite graph. Assume we have chosen a blow-up G^σ; recall that G^σ is determined by a choice $\sigma = (\sigma_v)$ of linear orderings of the half-edges at every vertex $v \in V(G)$. Assume we have also fixed a choice $\Delta = (\Delta_v)_{v \in V(G)}$ of orientations of the gadgets Γ_v.

In this section, we begin the proof of Theorem 3 by giving an upper bound for the number $c_{\sigma,\Delta}$ defined in the introduction. This is done by constructing an embedding of G into an orientable surface which is compatible with the choice of σ and Δ, and then proving an Arf invariant formula for $\mathcal{E}_G$ coming from this embedding.

Recall the notion of a Kasteleyn orientation of a graph Γ which is embedded into the plane equipped with its standard clockwise orientation. We assume that when we walk around any bounded face of the embedding, we encounter each edge at most once. This property is satisfied for 2-connected graphs, but also for the embeddings of the gadgets Γ_v that we will consider.

Definition 3.1. An orientation D of Γ is *Kasteleyn* if every bounded face F of the embedding is *clockwise odd* with respect to D, meaning that the number of edges e of the boundary of F where the orientation of e in D coincides with the orientation of e as the boundary of F is odd.

If Γ is embedded into the interior of an oriented disk $\mathbb{D}$, the notion of Kasteleyn orientation is defined in the same way.

Proposition 3.2. *There exists an embedding of G^σ into a closed oriented surface Σ such that each gadget Γ_v is entirely contained in the interior of a closed disk $\mathbb{D}_v \subset \Sigma$ and the orientation Δ_v is Kasteleyn with respect to the embedding of Γ_v into the disk $\mathbb{D}_v$. Moreover, the disks $\mathbb{D}_v$ are pairwise disjoint.*

Proof of Proposition 3.2. Consider the gadget Γ_v with its chosen orientation Δ_v. Let $\{e'_1, \dots, e'_d\}$ be the edges of G^σ corresponding to the original (half-)edges of G incident with v, and let Γ'_v be the subgraph of G^σ consisting of Γ_v and these edges. The vertices of Γ'_v are those of Γ_v union one vertex, say u_i, for each of the edges e'_i ($i = 1, \dots, d$). We claim that Γ'_v can be embedded into an oriented disk $\mathbb{D}$ so that

- Γ_v lies in the interior of the disk,
- the vertices u_i lie on the boundary of the disk, and
- the orientation Δ_v is Kasteleyn with respect to the embedding restricted to Γ_v.

To see this, first embed Γ_v into the interior of the disk so that Δ_v is Kasteleyn, and then add the edges e'_i one after the other so that they never cross each other.

Thus we obtain a cyclic ordering of the vertices u_i, coming from the orientation of the boundary of the disk. It corresponds to a cyclic ordering c_v of the half-edges incident with v in the original graph G. It is important to observe that this cyclic ordering only depends on σ_v and Δ_v. The collection $c = (c_v)_{v \in V(G)}$ of cyclic orderings is sometimes called a rotation system on the graph G. As is well known, c gives G the structure of a ribbon graph. It means that G naturally embeds into an oriented surface $S(G, c)$ obtained as follows: take one oriented d-gon P_v for every d-valent vertex and one oriented rectangle $I_e \times [0, 1]$ for every edge e (here I_e is an interval). Then glue $I_e \times 0$ and $I_e \times 1$ to the boundary of the disjoint union of the polygons in the way prescribed by the structure of the graph G and the cyclic orientations c_v. The surface $S(G, c)$ has boundary, so we let $\Sigma(G, c)$ be the closed surface obtained from $S(G, c)$ by gluing disks to the boundary components of $S(G, c)$.

By construction, the blow-up G^σ of G also embeds into $\Sigma(G, c)$, *via* an embedding such that each gadget Γ_v is contained in the interior of the polygon P_v. The polygon P_v plays the role of the disk $\mathbb{D}_v$ in the statement of the proposition, and Δ_v is Kasteleyn in $\mathbb{D}_v$. This completes the proof. $\square$

The genus of the surface $\Sigma(G,c)$ is called the *genus* of the ribbon graph (G,c) and denoted by $g(G,c)$. It is the minimal genus of a closed orientable surface in which the ribbon graph (viewed as the surface $S(G,c)$) embeds.

Definition 3.3. We define $g(G,\sigma,\Delta)$ to be $g(G,c)$ where c is constructed from (σ,Δ) as in the proof of Proposition 3.2. It is the minimal genus of a closed orientable surface in which G^σ embeds so that the gadgets Γ_v are contained in disjoint disks $\mathbb{D}_v$ and the orientations Δ_v are Kasteleyn with respect to the embeddings $\Gamma_v \subset \mathbb{D}_v$.

Let $g = g(G,\sigma,\Delta)$ be the genus of the surface $\Sigma = \Sigma_g$ obtained in the previous proposition. We now apply the machinery of the previous section to the embedding of G^σ into Σ_g. Decompose Σ_g into base polygon R_0, bridges R_i, and additional disk R_∞, as described in Section 2. Perform an isotopy of the embedding to make G^σ disjoint from R_∞ and to move all gadgets Γ_v entirely into R_0. (This is possible because every gadget Γ_v is contained in its own disk $\mathbb{D}_v$.) Let φ denote the special planar drawing of G^σ obtained using the immersion Φ of the highway surface $S_g = \Sigma_g \backslash R_\infty$ into the plane. Note that in this drawing, the subgraph consisting of the disjoint union of the Γ_v is planarly embedded, and the orientation Δ of this subgraph is Kasteleyn in the sense of Definition 3.1.

Lemma 3.4. *The orientation $\Delta = (\Delta_v)_{v\in V(G)}$ of the union of the gadgets Γ_v can be extended to a crossing orientation D_0 of G^σ with respect to the drawing φ.*

Proof. This follows easily from the construction of a crossing orientation in [15, Section 6]. In fact, the following more general statement is true: if we remove from a planar drawing of a graph all edges involved in crossings, then any Kasteleyn orientation (as defined in Definition 3.1) of the remaining planar graph can be extended to a crossing orientation of the original graph. $\square$

Let $H = H_1(\Sigma_g;\mathbb{F}_2)$ and let Q be the set of quadratic forms on $(H,\cdot)$ where $\cdot$ is the intersection form on H. Let $D_q = D_0(S_q)$ be the orientations indexed by quadratic forms $q \in Q$ which were constructed in Proposition 2.8 starting with the crossing orientation D_0. Recall that D_0 corresponds to the quadratic form q_0.

Proposition 3.5. *Each D_q is a Δ-admissible orientation.*

Proof. Recall that D_q differs from D_0 precisely on the set of edges S_q defined as follows: write $q = q_0 + \ell$ where $\ell \in H^*$, then $e \in S_q$ if and only if $\ell([e]) \neq 0 \in \mathbb{F}_2$. But the edges of the gadgets Γ_v are zero in homology, since the gadgets are entirely contained in the base polygon R_0. Thus $S_q \cap E(\Gamma_v) = \emptyset$ for all $v \in V(G)$. Hence D_q coincides with D_0 on the gadgets. Since D_0 is Δ-admissible by construction, so is every D_q. $\square$

Here is, then, the main result of this section.

Theorem 3.6 *(Arf invariant formula for even subgraphs (abstract version)). Let G be a finite graph. Choose a blow-up G^σ and an orientation Δ of the gadgets which replaced the vertices of G in G^σ. Let $g = g(G,\sigma,\Delta)$ as defined in Definition* 3.3. *Then the even subgraph polynomial $\mathcal{E}_G(x)$ is a linear combination of the 4^g σ-projected Pfaffians $F^\sigma_{D_q}(x)$ associated to the*

Δ-admissible orientations D_q indexed by quadratic forms on $H = H_1(\Sigma_g; \mathbb{F}_2)$:

$$\mathcal{E}_G(x) = \sum_{q \in Q} \alpha_q F^{\sigma}_{D_q}(x),$$

where $\alpha_q = \varepsilon_0(-1)^{\mathrm{Arf}(q)}/2^g$, $\varepsilon_0 \in \{\pm 1\}$ is the universal sign coming with the crossing orientation D_0, and

$$F^{\sigma}_{D_q} = \mathrm{Pfaf}\, A\big(G^{\sigma}, D_q\big)\big|_{x_e=1\ \forall e \in E(G^{\sigma}) \setminus E(G)}.$$

Proof. This follows from formula (1) relating $\mathcal{E}_G$ to $\mathcal{P}_{G^{\sigma}}$, Theorem 2.11 applied to $\mathcal{P}_{G^{\sigma}}$, and Proposition 3.5. $\square$

The following corollary is a more precise version of Theorem 2 in the introduction.

Corollary 3.7 *(Arf invariant formula for even subgraphs (embedded version)). If G embeds into an orientable surface Σ of genus g, then one can choose the blow-up G^{σ} in such a way that there exist 4^g orientations D_i of G^{σ} such that the even subgraph polynomial $\mathcal{E}_G(x)$ can be expressed as a linear combination of the σ-projected Pfaffians $F^{\sigma}_{D_i}(x)$ $(i = 1, \ldots, 4^g)$. Moreover, for every $v \in V(G)$, each of the orientations D_i induces the same orientation on the gadget Γ_v.*

Proof. This follows from Theorem 3.6 using the fact that given an embedding of G into an orientable surface Σ of genus g, we can choose σ and Δ in such a way that $g(G, \sigma, \Delta) \leqslant g$. Here is a proof of this fact. Choose an orientation of the surface Σ. Since G is embedded in Σ, the orientation of Σ induces, at every vertex $v \in V(G)$, a cyclic ordering c_v of the half-edges incident with v. Now construct the graph G^{σ} by choosing a linear ordering σ_v at each vertex v which induces this cyclic ordering c_v. Then it is easy to see that G^{σ} also embeds into Σ, with each gadget Γ_v being embedded into a little disk neigborhood $\mathbb{D}_v$ of v in Σ. Next, choose the orientations Δ_v of Γ_v so that they are Kasteleyn with respect to the embeddings of the Γ_v into the oriented disks $\mathbb{D}_v$. Then the surface (with boundary) $S(G, c)$ constructed in the proof of Proposition 3.2 can be embedded into Σ. By the classification of surfaces, it follows that the genus of Σ is greater or equal to $g(G, \sigma, \Delta)$, since $g(G, \sigma, \Delta)$ is the genus of the closed surface $\Sigma(G, c)$ obtained by gluing disks to the boundary components of $S(G, c)$. $\square$

Remark 3.8. An even subgraph $E' \subset E(G)$ can naturally be viewed as a 1-cycle (mod 2) of G, and hence defines a homology class in $H_1(G; \mathbb{F}_2)$. Let $[E']$ be the image of this homology class in $H = H_1(\Sigma_g; \mathbb{F}_2)$ under the embedding of G into Σ_g constructed in Proposition 3.2. If now E' corresponds to a perfect matching M of G^{σ} under the bijection of Proposition 1.2, then the homology classes $[E']$ and $[M]$ in H coincide. (This is because every edge in $E(G^{\sigma}) \setminus E(G)$ is entirely contained in the base polygon R_0, and hence zero in homology.) Therefore, using Proposition 2.8, the σ-projected Pfaffian polynomial $F^{\sigma}_{D_q}$ can be written

$$F^{\sigma}_{D_q}(x) = \sum_{E' \in \mathcal{E}(G)} \mathrm{sign}\big(E', F^{\sigma}_{D_q}\big) \prod_{e \in E'} x_e,$$

where $\mathrm{sign}(E', F^{\sigma}_{D_q}) = \varepsilon_0(-1)^{q([E'])}$.

4. Optimality of the Arf invariant formula

We now give the proof of Theorem 3. Since Theorem 3.6 already gives an upper bound for $c_{\sigma,\Delta}(G)$, it remains only to prove the following.

Theorem 4.1. *Let G be a finite graph. Choose a blow-up G^σ and an orientation Δ of the gadgets which replaced the vertices of G in G^σ. Let $g = g(G, \sigma, \Delta)$ as defined in Definition* 3.3. *Assume there exists $k \geqslant 1$ and a collection of Δ-admissible orientations D_i and coefficients $\lambda_i \in \mathbb{Q}$ $(i = 1, \ldots, k)$ such that the even subgraph polynomial $\mathcal{E}_G(x)$ can be expressed as*

$$\mathcal{E}_G(x) = \sum_{i=1}^{k} \lambda_i F^\sigma_{D_i}(x).$$

Then $k \geqslant 4^g$.

Proof. A Δ-admissible orientation differs from the crossing orientation $D_0 = D_{q_0}$ only on edges of the original graph G. Let $S_i \subset E(G)$ be the set of edges where D_i differs from D_0. The sign of an even subgraph E' in $F^\sigma_{D_i}(x)$ is

$$\begin{aligned} \operatorname{sign}\bigl(E', F^\sigma_{D_i}\bigr) &= \operatorname{sign}\bigl(E', F^\sigma_{D_0}\bigr)(-1)^{|E' \cap S_i|} \\ &= \varepsilon_0 (-1)^{q_0([E'])} (-1)^{\ell_i(E')}, \end{aligned} \tag{11}$$

where we have defined

$$\ell_i\bigl(E'\bigr) = \bigl|E' \cap S_i\bigr| \pmod 2. \tag{12}$$

Now recall that any even subgraph $E' \subset E(G)$ can naturally be viewed as a 1-cycle (mod 2) of G, and every 1-cycle uniquely arises in this way. This establishes an identification

$$\mathcal{E}(G) \cong C_1(G; \mathbb{F}_2),$$

where $C_1(G; \mathbb{F}_2)$ is the space of 1-cycles on G. Hence $\mathcal{E}(G)$ is naturally endowed with the structure of an $\mathbb{F}_2$-vector space, called the *cycle space* of G in graph theory. Moreover, addition (mod 2) of 1-cycles corresponds to taking symmetric difference of even subgraphs. The function ℓ_i defined in (12) is a linear form ℓ_i on this vector space, since

$$\bigl|(E_1 \Delta E_2) \cap S_i\bigr| = |E_1 \cap S_i| + |E_2 \cap S_i| \pmod 2.$$

(Here, Δ denotes symmetric difference.)

Next, we observe that by the construction of the surface Σ_g in Proposition 3.2, the composite map

$$C_1(G; \mathbb{F}_2) \to H_1(G; \mathbb{F}_2) \to H_1(\Sigma_g; \mathbb{F}_2)$$

induced by the embedding of G into Σ_g, is onto. In other words, any homology class $x \in H = H_1(\Sigma_g; \mathbb{F}_2)$ can be realized by some even subgraph of G. Choose a sub-vector space C

of $C_1(G;\mathbb{F}_2)$ which maps isomorphically onto H. When we think of C as a subset of $\mathcal{E}(G)$, we denote C by $\mathcal{C}$. Clearly, the zero element of C corresponds to the empty subgraph $\emptyset$ as an element of $\mathcal{C} \subset \mathcal{E}(G)$.

For every $i = 1, \dots, k$, we get a linear form ℓ'_i on H defined as

$$\ell'_i(x) = \ell_i\big(E'_x\big) \quad (x \in H), \tag{13}$$

where $E'_x \in \mathcal{C}$ is the unique element of $\mathcal{C}$ which maps to x. Observe that the homology class $[E'_x] \in H$ is equal to x.

Define the quadratic form $q_i \in Q$ by $q_i = q_0 + \ell'_i$. Putting together (11) and (13), we have shown the following: for every $E' \in \mathcal{C}$, and for every $i = 1, \dots, k$, one has

$$\begin{aligned} \operatorname{sign}\big(E', F^{\sigma}_{D_i}\big) &= \varepsilon_0(-1)^{q_0([E'])}(-1)^{\ell_i(E')} \\ &= \varepsilon_0(-1)^{q_0([E'])}(-1)^{\ell'_i([E'])} \\ &= \varepsilon_0(-1)^{q_i([E'])}. \end{aligned} \tag{14}$$

We are now ready to prove Theorem 4.1. By hypothesis, there exists $\lambda_i \in \mathbb{Q}$ $(i = 1, \dots, k)$ such that

$$\sum_{i=1}^{k} \lambda_i \operatorname{sign}\big(E', F^{\sigma}_{D_i}\big) = 1$$

for all even sets $E' \in \mathcal{E}(G)$. Since every homology class $x \in H$ is realized by some E'_x belonging to the set $\mathcal{C}$ for which expression (14) is valid, it follows that

$$\sum_{i=1}^{k} \lambda_i \varepsilon_0 (-1)^{q_i(x)} = 1 \quad (\forall x \in H). \tag{15}$$

Now recall from Lemma 2.10 that

$$\sum_{q \in Q} \alpha_q (-1)^{q(x)} = 1 \quad (\forall x \in H), \tag{16}$$

where $\alpha_q = 2^{-g}(-1)^{\mathrm{Arf}(q)}$. The following Lemma 4.2 implies that $(\alpha_q)_{q \in Q}$ is the unique solution of (16). Since all $\alpha_q \neq 0$, every $q \in Q$ must appear in (15). It follows that $k \geqslant |Q| = 4^g$, as asserted. $\square$

Lemma 4.2. *One has*

$$\det\big((-1)^{q(x)}\big)_{(q,x)\in Q\times H} \neq 0.$$

Proof. Recall that any $q \in Q$ can be written $q = q_0 + \ell$ for a unique linear form $\ell \in H^*$. Thus we can describe the matrix in question as

$$\big((-1)^{q_0(x)+\ell(x)}\big)_{(\ell,x)\in H^*\times H}.$$

Multiplying this matrix on the right by the diagonal matrix with entries $(-1)^{q_0(x)}$ $(x \in H)$, we get

$$\big((-1)^{\ell(x)}\big)_{(\ell,x)\in H^*\times H}.$$

In this $4^g \times 4^g$ matrix, the scalar product of any two rows corresponding to linear forms ℓ and ℓ' is

$$\sum_{x\in H}(-1)^{\ell(x)+\ell'(x)} = \begin{cases} 4^g & \text{if } \ell = \ell', \\ 0 & \text{if } \ell \neq \ell'. \end{cases}$$

(Recall that a non-trivial linear form on an $\mathbb{F}_2$-vector space takes the value 0 as many times as it takes the value 1.) Thus the matrix is 2^g times an orthogonal matrix. Hence it is non-singular. □

This completes the proof of Theorem 4.1, and (hence) of Theorem 3.

To prove Theorem 4, it only remains to show that given a graph G, the minimal genus $g(G, \sigma, \Delta)$, over all choices of (σ, Δ), is equal to the embedding genus of G. But this was already shown in the proof of Corollary 3.7. Thus Theorem 4 is proved as well.

5. Even drawings don't help

Definition 5.1. A drawing φ of a graph G on a surface Σ (as defined in Definition 2.1) is called *even* if the number of double points $\kappa_\varphi(E')$ is even for every even subgraph E' of G.

It is easy to see that the proof of the Arf invariant formula for even subgraphs in Section 3 goes through if we start with an even drawing of the graph on a surface in place of an embedding. More precisely, we can replace in Corollary 3.7 the embedding of G with an even drawing of G, and the result still holds. However, even drawings cannot reduce the number of Δ-admissible orientations needed to express $\mathcal{E}_G$ as a linear combination of σ-projected Pfaffians. This can be deduced from Theorem 3. The underlying topological reason is stated in the next theorem.

Theorem 5. *Let G be a graph. The minimal genus of an orientable surface which supports an even drawing of G is equal to the embedding genus of G.*

Proof. One can prove this result by algebraic-topological arguments using non-degeneracy of the intersection form on closed surfaces. We omit that proof but remark that the main idea can be found in the proof of [3, Lemma 3]. (We thank M. Schaefer for this reference. M. Schaefer has informed us that Theorem 5 also follows from techniques in [14].)

Here is a proof of Theorem 5 in the spirit of the present paper. Assume we have a drawing φ of G on a surface Σ of genus g. The orientation of Σ induces a cyclic orientation c_v of the half-edges at every vertex $v \in V(G)$. As in the proof of Corollary 3.7, we can find (σ, Δ) inducing that cyclic orientation at every vertex. If we now assume the drawing is even, the proof of Theorem 3.6 goes through and we can express $\mathcal{E}_G$ as linear combination of 4^g σ-projected Pfaffians associated to orientations which are all Δ-admissible. But by our optimality statement in Theorem 4.1, we know that one needs at least $c_{\sigma,\Delta}(G) = 4^{g(G,\sigma,\Delta)}$ such orientations to do that. Thus $g \geqslant g(G, \sigma, \Delta)$. Since G can be embedded in the surface of genus $g(G, \sigma, \Delta)$ constructed in the proof of Proposition 3.2, it follows that g is greater or equal than the embedding genus of G, as asserted. □

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