INTERIOR POINT METHODS – part 3 4/1/2021

PRIMAL-DUAL AFFINE SCALING POTENTIAL REDUCTION ALGORITHM

RECAP: (P) $\min c^T x$ s.t. $Ax = b$, $x \geq 0$ (D) $\max y^T b$ s.t. $A^T y + s = c$, $s \geq 0$ $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $L = \langle A, b \rangle$ input size

dual solution (y, s) uniquely represented by s

- Duality gap: $c^T x - y^T b = s^T x$
- Potential function: for $x > 0$, $s > 0$

$$G(x,s) = (n + \sqrt{n}) \ln x^T s - \sum_{i=1}^{n} \ln (x_i s_i)$$

We proved:

If $G(x,s) \leq -2\sqrt{n} L$, then $x^T s < e^{-2L}$ ... we are almost done

- Scaling: for $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) > 0$, let $\bar{X} = \mathrm{diag}(\bar{x}_1, \dots, \bar{x}_n)$

Then an affine mapping given by $x' = \bar{X}^{-1} x$ moves $\bar{x}$ to $e = (1, \dots, 1)^T$ that is far from boundaries

⇝ transformed variables $x' = \bar{X}^{-1} x$, $s' = \bar{X} s$

(P') $\min \bar{c}^T x'$, $\bar{A} x' = b$, $x' \geq 0$ (D') $\max b^T y$, $\bar{A}^T y + s' = \bar{c}^T$, $s' \geq 0$ where $\bar{c}^T = c^T \bar{X}$, $\bar{A} = A \cdot \bar{X}$

it holds: $x^T s = x'^T s'$, $G(x,s) = G(x', s')$

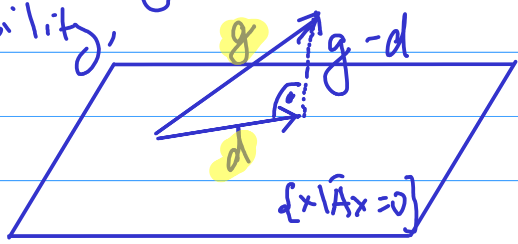
⇝ we can assume that our current primal solution is $e = (1, 1, \dots, 1)^T, s'$

- Iteration: rough idea:

compute the gradient $g = \nabla_x G(x,s)$ at (e, s')

change the primal solution in the direction $-g$

refined idea: to maintain feasibility, use the projection d of g to the hyperplane $\{x \mid \bar{A} x = 0\}$



PRIMAL STEP: if $\|d\| \geq 0{,}2$

set $\boxed{\tilde{x} = e - \frac{1}{4\|d\|} d \quad ; \quad \tilde{s} = s'}$

Lemma: $\tilde{x}$ is strictly feasible and

$$G(\tilde{x}, \tilde{s}) - G(e, s') \leq \frac{-1}{120}.$$

DUAL STEP: if $\|d\| < 0{,}2$

first idea: change the dual solution in the direction h, where $h = \nabla_s G(x,s)$ at (e, s')

refined idea: use g instead of h, as both point to the same ortant

more refined idea: to maintain feasibility of the new dual solution, use direction $g-d$ instead of g

set $\boxed{\tilde{s} = s' - (g-d)\mu \;;\; \tilde{x} = e}$ where $\mu = \frac{e^T s'}{n+\sqrt{n}}$

Lemma A: $\tilde{s}$ is strictly feasible.

It remains to show:

Lemma B: $$G(\tilde{x}, \tilde{s}) - G(e, s') \leq -\frac{3}{\rho}.$$

So far we proved:

Ü 3: $$\frac{e^T \tilde{s}}{e^T s'} \leq \frac{n + 0{,}2\sqrt{n}}{n+\sqrt{n}}$$

> Recall
> $$g = \frac{n+\sqrt{n}}{e^T s'} s' - e$$

We proceed with a few auxiliary lemmas:

Lemma 1: $\tilde{s} = \frac{e^T s'}{n+\sqrt{n}}(d+e)$.

Lemma 2: $$\sum_{j=1}^{n} \ln \tilde{s}_j \geq n \cdot \ln \frac{e^T \tilde{s}}{n} - \frac{1}{40}.$$

Proof of Lemma 1:

by definition $$\tilde{s} = s' - \frac{e^T s'}{n+\sqrt{n}}(g-d) = s' - \frac{e^T s'}{n+\sqrt{n}}\Big(\underbrace{\frac{n+\sqrt{n}}{e^T s'} s' - e}_{=g} - d\Big)$$

$$= \frac{e^T s'}{n+\sqrt{n}}(d+e) \qquad \square$$

Lemma 1

Proof of Lemma 2:

$$\sum_{j=1}^{n} \ln \tilde{s}_j - n \cdot \ln \frac{e^T \tilde{s}}{n} = \sum_{j=1}^{n} \ln\left(\frac{e^T s'}{n+\sqrt{n}}(1+d_j)\right) - n \ln\left(\frac{e^T s'}{n+\sqrt{n}}\left(1+\frac{e^T d}{n}\right)\right)$$

$$= \sum_{j=1}^{n} \ln(1+d_j) - n \ln\left(1+\frac{e^T d}{n}\right) \geq \sum_{j=1}^{n}\left(d_j - \frac{d_j^2}{2(1-0{,}2)}\right) - \cancel{n}\frac{e^T d}{\cancel{n}}$$

bounds on $\ln$: $-x - \frac{x^2}{2(1-a)} \leq \ln(1-x) \leq -x$ if $|x| \leq a < 1$

$$= -\frac{\sum_{j=1}^{n} d_j^2}{\frac{16}{10}} \geq -\frac{\frac{4}{100}}{\frac{16}{10}} = -\frac{1}{40}$$

$\sum_{j=1}^{n} d_j^2 = \|d\|^2$

■

Proof of Lemma B:

$g = n + \sqrt{n}$

by def. of G

$$G(e,\tilde{s}) - G(e,s') = g \cdot \ln e^T \tilde{s} - \sum_{j=1}^{n} \ln \tilde{s}_j - g \cdot \ln e^T s' + \sum_{j=1}^{n} \ln s'_j$$

$$= g \cdot \ln \frac{e^T \tilde{s}}{e^T s'} - \sum_{j=1}^{n} \ln \tilde{s}_j + \sum_{j=1}^{n} \ln s'_j$$

arithmetic & geometric means inequality

$$\leq g \cdot \ln \frac{e^T \tilde{s}}{e^T s'} - n \ln \frac{e^T \tilde{s}}{n} + \frac{1}{40} + n \ln \frac{e^T s'}{n}$$

Lemma 2

$$= (g-n) \ln \frac{e^T \tilde{s}}{e^T s'} + \frac{1}{40}$$

$$\leq \sqrt{n} \ln \frac{n + 0{,}2\sqrt{n}}{n+\sqrt{n}} + \frac{1}{40}$$

$\frac{n + 0{,}2\sqrt{n}}{n+\sqrt{n}} = 1 - \frac{0{,}8\sqrt{n}}{n+\sqrt{n}}$

$\ln(1+x) \leq x$

(3)

$$\leq -\sqrt{n}\frac{0{,}8\sqrt{n}}{n+\sqrt{n}} + \frac{1}{40} \leq -\frac{0{,}8 \cdot \cancel{n}}{2\cancel{n}} + \frac{1}{40} \leq -\frac{3}{8}$$

$\sqrt{n} \leq n$

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This completes the description & analysis of the iterations.

HOW TO START THE INTERIOR POINT ALGORITHM

(P) $\min c^T x$ (D) $\max b^T y$ $A \ldots m \times n$

$$Ax = b \qquad\qquad A^T y + s = c$$

$$x \geq 0 \qquad\qquad s \geq 0$$

Move to higher dim: new variable

$[\,A\,] \to [\,A \mid b - Ae\,]$ $\uparrow$

(P') $\min c^T x + M x_{n+1}$

$$Ax + (b - Ae)x_{n+1} = b$$

$$x \geq 0, \quad x_{n+1} \geq 0$$

(D') $\max b^T y$

$$A^T y + s = c$$

$$(b - Ae)^T y \leq M \quad \ldots (b-Ae)^T y + s_{n+1} = 4^L$$

$$s \geq 0 \qquad s_{n+1} \geq 0$$

☺ The vector $(\underbrace{1, 1, \ldots, 1, 1}_{n+1})$ is a feasible solution of (P').

proof: think j-th constraint in (P)

$a_j x = b_j$ → new coefficient in the j-constraint is $b_j - a_j \cdot e$

jth row of A

$$a_j \cdot x + (b_j - a_j e) \cdot x_{n+1} = b_j \quad \blacksquare$$

set $M = 4^L$
where $L = \langle A, b, c \rangle$

Lemma: Assume that A, b, c are integral and that (D) has an optimal vertex solution.
Then $x_{n+1} = 0$ for all optimal solutions of (P').

Proof: let y be an optimal vertex solution of (D)
then $|y_i| < 2^L$ $\forall i$ (theory of polyhedra, cf. lecture 14/12/2020)

Consider the left hand side of the new constraint in (D'):

$$(b - Ae)^T y < 2^L \sum_{i=1}^m \left(b_i - \sum_{j=1}^n a_{ij}\right) \leq 2^L \underbrace{\left(\sum_{i=1}^m |b_i| + \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|\right)}_{\leq 2^L}$$

$$\leq 4^L \Rightarrow s_{n+1} = 4^L - (b - Ae)^T y > 0$$

As the duality gap equals $(x, x_{n+1})^T (s, s_{n+1}) = 0$

$\Rightarrow x_{n+1} \cdot s_{n+1} = 0 \quad \Rightarrow \quad x_{n+1} = 0 \quad \blacksquare$

$$G(x,s) = (n+\sqrt{n}) \ln x^T s - \sum_{j=1}^{n} \ln x_j s_j$$

Lemma: $(\overbrace{1,1,\dots,1,1}^{n+1})$ is a strictly feasible solution of (P').
If $c > 0$, then $M = 4^L$
$y = 0$, $s = c$, $s_{n+1} = M$ is a strictly feasible dual solution of (D')
and $G(\underbrace{(1,\dots,1}, s\, s_{n+1})) = O(n \cdot L)$.

Proof: $= (n+1+\sqrt{n+1}) \ln \left(\sum_{i=1}^{n} c_i + M \right) - \sum_{j=1}^{n} \ln c_j - \ln M$

$= O(nL)$ ∎

Corollary: The total number of iterations is $O(nL)$.

Proof:
- by the previous lemma, $G(x_0, s_0) = O(nL)$,
- by the description of the algorithm, $G(x,s) \le -2\sqrt{n}L$ for the last pair of feasible solutions,
- in each iteration, the potential decreases by $\Omega(1)$.

$\Rightarrow$ The algorithm runs in polynomial time.