ELLIPSOID ALGORITHM – PART 2

30/11/2020

Goal: find $x \in P = \{x \in \mathbb{R}^n \mid Cx \le d\}$

Assumptions: • P is bounded, i.e., $\exists K > 0$ s.t. $P \subseteq \{x \in \mathbb{R}^n \mid -K \le x_i \le K, i = 1,\dots,n\}$

• if non-empty, then P has full dimension

1. Find $E_0 = E(z_0, C_0)$ s.t. $P \subseteq E_0$; $k = 0$
2. While $z_k \notin P$

   Find $E_{k+1} = E(z_{k+1}, C_{k+1})$ s.t. • $P \subseteq E_{k+1}$

   • $\mathrm{vol}(E_{k+1}) \le \underbrace{\varrho}_{\text{constant} < 1} \cdot \mathrm{vol}(E_k)$

   $k = k+1$

   if k "too large", output "P is empty", STOP
3. OUTPUT z_k

Lemma: If $P = \{x \mid Cx \le d\} \subseteq \mathbb{Q}^n$ is bounded and C and d are integral, then all vertices of P are contained in a ball centered at origin and radius $R = \sqrt{n} \cdot 2^{\langle C,d\rangle - n^2}$.

$\Rightarrow E_0 = E(0, R \cdot I)$

where $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

How to get E_{k+1} from E_k?

Sketch: find an inequality in $Cx \le d$ that is violated by z_k: $c z_k > d_i$

consider a "half ellipsoid"

$E(z_k, C_k) \cap \{x \in \mathbb{R}^n \mid cx \le c z_k\}$

E_{k+1} … a small ellipsoid containing the intersection

Simple case: $E_k = E(0, I)$ unit ball at origin

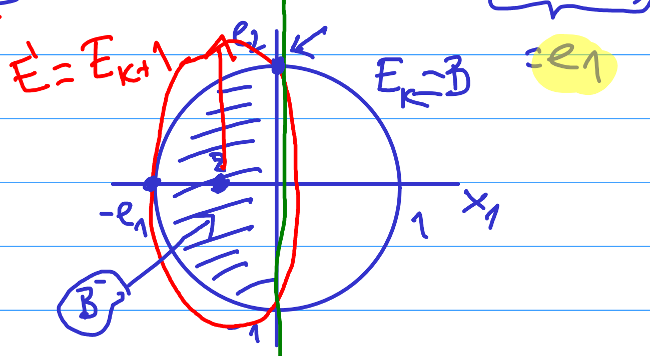
$H_k = \{x \in \mathbb{R}^n \mid x_1 \le 0\}$ our halfspace, i.e., $\underbrace{c = (1, 0, \dots, 0)}_{= e_1}$

let $B^- = \{x \in \mathbb{R}^n \mid x \cdot x \le 1, x_1 \le 0\}$

Aim: find E'

s.t. $B^- \subseteq E'$

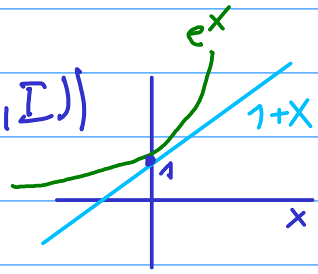
$\mathrm{vol}(E') \le \varrho \cdot \mathrm{vol}(E(0, I))$



Recall: • For $T(x) = \underbrace{A^{1/2}}_{Q}x + a$, $E(a,A) = T(E(0,I))$

$\Rightarrow$ • $vol(E(a,A)) = \boxed{|\det A^{1/2}|} \cdot vol(E(0,I))$

• $\underbrace{1 + x \le e^x}$



As B^- is symmetrical around the axis x_1, we are looking for an ellipsoid E' in the form

$$E' = \{x \in \mathbb{R}^n \mid \underbrace{(x-z)^T Z^{-1} (x-z)} \le 1\} \quad \text{for } \boxed{z = t \cdot e_1}, \; t < 0$$

due to the symmetry of B^-, we want Z^{-1} to be diagonal:

$$Z^{-1} = \begin{pmatrix} \frac{1}{p^2} & & & 0 \\ & \frac{1}{d^2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{d^2} \end{pmatrix} \qquad \text{for } \frac{1}{p^2} > 1, \quad \frac{1}{d^2} < 1$$

Three parameters: p, d, t

i) E' and B touch at $-e_1$: note: $\underbrace{-e_1 - t \cdot e_1} = (-1-t, 0, \dots, 0)$

$$(-1-t, 0, \dots, 0)^T \cdot Z^{-1} \cdot (-1-t, 0, \dots, 0) = \boxed{\frac{(1+t)^2}{p^2} = 1}$$

ii) E' and B touch at e_2

$$(-t, +1, 0, \dots, 0)^T Z^{-1} (-t, +1, 0, \dots, 0) = \boxed{\frac{t^2}{p^2} + \frac{1}{d^2} = 1}$$

iii) we want the volume of E' to be as small as possible

$$vol(E') = \underbrace{\sqrt{|\det Z|}} \cdot vol(B)$$

$$Z = \begin{pmatrix} p^2 & & 0 \\ & d^2 & \\ & & \ddots \\ 0 & & & d^2 \end{pmatrix} \qquad \begin{cases} p^2 = (1+t)^2 \\ d^2 = \frac{(1+t)^2}{1+2t} \end{cases}$$

$$\sqrt{\det Z} = \frac{(1+t)^n}{\sqrt{(1+2t)^{n-1}}}$$

using math analysis:

$$\boxed{t = \frac{-1}{n+1}}$$

$$\frac{1}{D^2} = \frac{(n+1)^2}{n^2} \quad ; \quad \frac{1}{d^2} = \frac{n^2-1}{n^2}$$

the center of E': $z = \left(\frac{-1}{n+1}, 0, \dots, 0\right)$

$$Z^{-1} = \begin{pmatrix} \frac{(n+1)^2}{n^2} & & & 0 \\ & \frac{n^2-1}{n^2} & & \\ & & \ddots & \\ 0 & & & \frac{n^2-1}{n^2} \end{pmatrix}$$

HW: check that $B^- \subseteq E(z, Z)$.

Lemma (Volume reduction): $\frac{vol(E')}{vol(B)} \le e^{\frac{-1}{2(n+1)}} < 1$

Proof:

$$\frac{vol(E')}{vol(B)} = \frac{\sqrt{\det Z} \cdot vol(B)}{vol(B)} = \frac{n}{n+1} \cdot \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} \le$$

$$\le \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \le$$

$$\le e^{\frac{-1}{n+1}} \cdot e^{\frac{1}{n^2-1} \cdot \frac{n-1}{2}} = e^{\frac{-1}{n+1} + \frac{1}{2(n+1)}} = e^{\frac{-1}{2(n+1)}}$$

∎

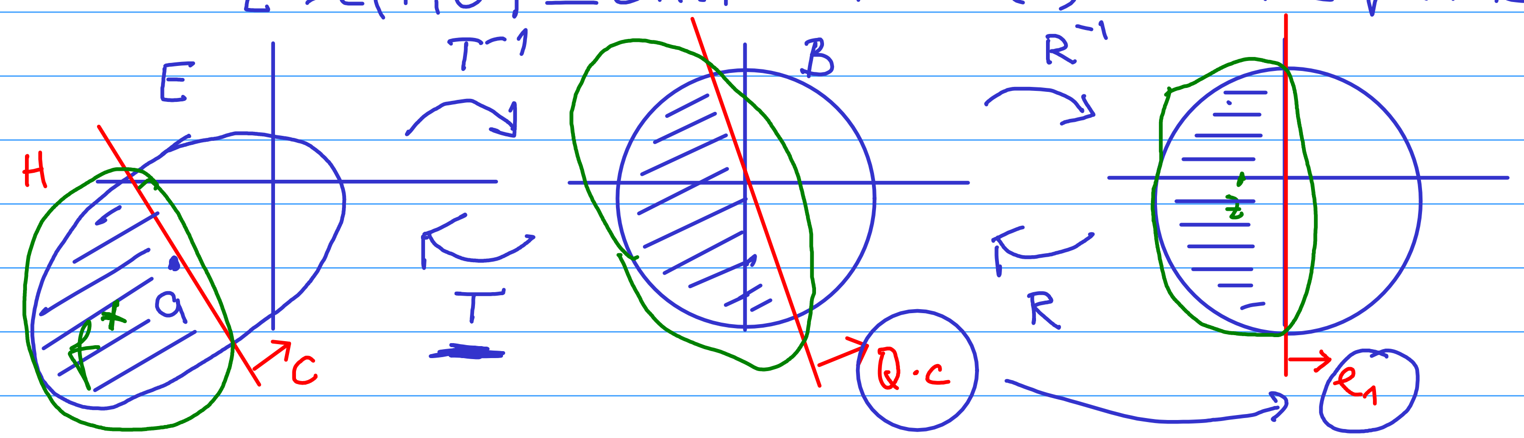
General case

Given: $E = E(a, A) = Q \cdot E(0, I) + a$ where $Q = A^{1/2}$

$H = \{x \in \mathbb{R}^n \mid c^T x \le c^T a\} = \{x \in \mathbb{R}^n \mid c^T (x - a) \le 0\}$

Looking for

$E' = E(f, C) \supseteq E \cap H$ with $vol(E')$ as small as possible



Note: in 2D : R is unique

in 3D : R is not unique

But it will suffice to use (see later)

Thus: $R^{-1}(T^{-1}(H \cap E)) \subseteq E\left(\frac{-e_1}{n+1}, Z\right)$

i.e. $\underbrace{H \cap E \subseteq T\left(R\left(E\left(\frac{-e_1}{n+1}, Z\right)\right)\right)}$

$T(y) = Q \cdot y + a$ $\qquad$ $T^{-1}(x) = Q^{-1}(x-a)$

$T^{-1}(H) = \{T^{-1}(x) \mid c^T(x-a) \leq 0\} = \{y \in \mathbb{R}^n \mid c^T(T(y) - a) \leq 0\}$

$= \{y \in \mathbb{R}^n \mid c^T Q y \leq 0\} = \{y \in \mathbb{R}^n \mid (Qc)^T y \leq 0\}$

Our constraints on the rotation R:

$$R^{-1}\left(\frac{Qc}{\|Qc\|}\right) = e_1 \quad \dots\to \quad R(e_1) = \frac{Qc}{\|Qc\|}$$

Lemma (Half ellipsoid lemma): For an ellipsoid $E' = E(f, C)$

where $f = a - \frac{1}{n+1} \cdot \frac{Ac}{\sqrt{c^T A c}}$

$C = \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1} \cdot \frac{Acc^T A^T}{c^T A c}\right)$

it holds:

- $E(a, A) \cap \{x \mid c^T(x-a) \leq 0\} \subseteq E(f, C)$
- $\frac{vol(E(f,C))}{vol(E(a,A))} \leq e^{\frac{-1}{2(n+1)}}$.

Proof: the center of the new ellipsoid

$$f = T(R(z)) = T\left(\underbrace{-\frac{1}{n+1} \cdot \frac{Qc}{\|Qc\|}}\right) = a - \underbrace{\frac{1}{n+1} \cdot \frac{Ac}{\sqrt{c^T A c}}}_{Q \cdot Q = A}$$

$z = \frac{-e_1}{n+1}$

$x - f = x - T(R(z)) = x - a - QRz = QR(R^{-1}Q^{-1}(x-a) - z)$