LECTURE 7 12/11/2020

MAX SAT

INPUT: n boolean variables $x_1, x_2, \dots, x_n$

m clauses $C_1, C_2, \dots, C_m$

each C_j is a conjunction of several variables & their negations

e.g. $C_7 = x_2 \vee \bar{x}_3 \vee x_5$

each C_j has a weight $w_j \geq 0$

OUTPUT: assignment true/false to $x_1, \dots, x_n$

OBJECTIVE: max sum of satisfied clauses

Terminology: $x_i, \bar{x}_i$ ... literals

x_i ... positive literals

$\bar{x}_i$ ... negative literals

Assume: no literal repeats in clause

at most one of x_i and $\bar{x}_i$ appears in any clause

RAND SAT

For each i: independently at random set $x_i := $ true (Prob $\frac{1}{2}$) / false (Prob $\frac{1}{2}$)

THM: RAND SAT is a 2-approximation alg.

Proof: for each clause C_j ... indicator variable Y_j

$$Y_j = \begin{cases} 1 & \dots C_j \text{ satisfied} \\ 0 & C_j \text{ NOT satisfied} \end{cases}$$

For each clause with k literals the probability that C_j is NOT satisfied is exactly $\frac{1}{2^k}$

$$\Rightarrow \mathbb{E}[Y_j] = \Pr[C_j \text{ is satisfied}] = \boxed{1 - \frac{1}{2^k}} \geq \frac{1}{2}$$

$$\mathbb{E}\left[\sum_{j=1}^{m} w_j Y_j\right] \geq \frac{1}{2} \sum_{j=1}^{m} w_j \geq \frac{1}{2} \text{OPT} \qquad \sum_{j=1}^{m} w_j \geq \text{OPT}$$

linearity of expectation & previous bound

Remarks: if each clause ... exactly 3 literals
$\Rightarrow$ apr. ratio is $\frac{8}{7}$.

It holds: $\forall \varepsilon > 0$, it is NP-hard to approximate MAX-3SAT within $\frac{8}{7} - \varepsilon$.

BIASED SAT $\quad \overset{w_j}{C_j} = x_i \qquad \overset{w_{j'}}{C_{j'}} = \bar{x}_i$

Assumption (*) $\quad \forall i \quad \sum_{j: C_j = x_i} w_j \geq \sum_{j: C_j = \bar{x}_i} w_j$

For each i: set independ., random $x_i = \begin{cases} \text{true} & \text{with } p > \frac{1}{2} \\ \text{false} & 1-p \end{cases}$

Consider C_j

- C_j is positive unite clause $\qquad$ e.g. $C_j = x_7$

$$\Pr[C_j \text{ is satisfied}] = p$$

- C_j is a clause of lenght ≥ 2 $\qquad$ e.g. $C_j = x_2 \vee \bar{x}_4 \vee x_5$
  - $a = \#$ negative literals in C_j. $\qquad a = 1$
  - $b = \#$ positive literals in C_j. $\qquad b = 2$

$$\Pr[C_j \text{ is satisfied}] = 1 - p^a (1-p)^b \geq 1 - p^{a+b} \geq 1 - p^2$$

We aim at $p = 1 - p^2 \longrightarrow p = \frac{\sqrt{5}-1}{2}$ Golden ratio

Let U be the set of all clauses excluding the unit negative clauses

$$OPT \leq \sum_{j \in U} w_j$$

Proof: by assumption (*) $\square$

For C_j --> indicator variable $Y_j := \begin{cases} 1 & C_j \text{ satisf.} \\ 0 & \text{NOT} \end{cases}$

$$E\left[\sum_{j=1}^{m} w_j \cdot Y_j\right] \underset{\text{Lin. of expectation}}{=} \sum_{j=1}^{m} w_j \cdot E[Y_j] \geq \sum_{j \in U} w_j \Pr[C_j \text{ satisfied}] \geq$$

$$\geq p \cdot \sum_{j \in U} w_j \geq p \cdot OPT \qquad \frac{1}{\sim 0{,}618}$$

THM: BIASED SAT is $\frac{2}{\sqrt{5}-1}$ - approxim. algorithm.

RANDOMIZED ROUNDING

for a boolean variable x_i: $\frac{\text{true}}{\text{false}}$ $\longrightarrow$ binary variable $y_i = \begin{cases} 1 \\ 0 \end{cases}$

for a clause C_j $\frac{\text{satisfied}}{\text{not satisfied}}$ $\longrightarrow$ binary variable $z_j = \begin{cases} 1 \\ 0 \end{cases}$

then $\boxed{\sum_{j=1}^{m} w_j \cdot z_j}$ .... sum of weights of satisfied clauses

$\forall$ clause C_j: e.g. $C_j = x_1 \vee \bar{x}_2 \vee x_3$

introduce linear inequality $z_j \leq y_1 + (1-y_2) + y_3$

$$\boxed{\sum_{i: x_i \in C_j} y_i + \sum_{i: \bar{x}_i \in C_j} (1-y_i) \geq z_j}$$

LP $\boxed{0 \leq z_j \leq 1 \quad \forall j, \qquad 0 \leq y_i \leq 1, \ \forall i}$

LP-SAT

1. Solve the LP $\longrightarrow (y^*, z^*)$ .. opt. solution of the LP relaxation
2. For each i: set $x_i := \begin{cases} \text{true} & \text{with prob. } y_i^* \\ \text{false} & 1-y_i^* \end{cases}$

THM: LP-SAT is a $\left(1-\frac{1}{e}\right)$-approximation algorithm. $\sim 0{,}63$

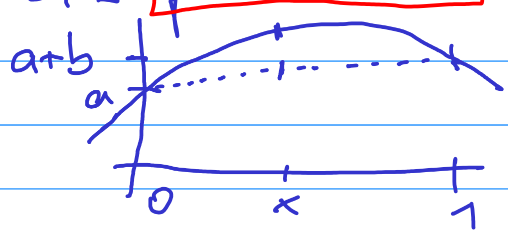
Fact A: For any non-negative $a_1, \dots, a_n$:

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$

Fact B: If a function $f(x)$ is concave on $[0,1]$, and $f(0)=a$, $f(1)=a+b$, then $\forall x \in [0,1]$: $f(x) \geq a+bx$.

Fact C:

$$\left(1-\frac{1}{n}\right)^n \leq \frac{1}{e}$$



Proof of THM: consider y^*, z^* ... an opt. of LP

Consider a clause C_j of length l_j

$$\Pr[C_j \text{ is NOT satisfied}] = \prod_{i: x_i \in C_j} (1-y_i^*) \prod_{i: \bar{x}_i \in C_j} y_i^* \le$$

Fact A

$$\le \left[\frac{1}{l_j}\left(\sum_{i: x_i \in C_j}(1-y_i^*) + \sum_{i:\bar{x}_i \in C_j} y_i^*\right)\right]^{l_j}$$

$$= \left[1 - \frac{1}{l_j}\left(\sum_{i: x_i \in C_j} y_i^* + \sum_{i:\bar{x}_i \in C_j}(1-y_i^*)\right)\right]^{l_j}$$

by LP

$$\le \left(1 - \frac{z_j^*}{l_j}\right)^{l_j} = f(z_j^*)$$

Note: $\boxed{f(0) = 0} = a$

$f(1) = \boxed{1-\left(1-\frac{1}{l_j}\right)^{l_j}} = b$

$$\Pr[C_j \text{ is satisfied}] \ge \underbrace{1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}}_{f}$$

check: $f''(x) \le 0$ for $x \in [0,1]$ $\Rightarrow$ concave

Fact B

$$\ge \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^* \overset{\text{Fact C}}{\ge} \left(1 - \frac{1}{e}\right) z_j^*$$

$$\mathbb{E}\left[\sum_{j=1}^m w_j \cdot Y_j\right] \underset{\text{Lin. of exp.}}{=} \sum_{j=1}^m w_j \Pr[C_j \text{ is satisfied}] \ge$$

$$\ge \sum_{j=1}^m w_j \cdot \left(1-\frac{1}{e}\right)\cdot z_j^* = \left(1-\frac{1}{e}\right) \sum_{j=1}^m w_j z_j^*$$

$$\ge \left(1-\frac{1}{e}\right) OPT. \qquad \blacksquare$$

Note: for $l_j = 1$ --- prob. C_j satisfied: z_j^*

$\Rightarrow$ LP SAT performs best on short (i.e., unit) clauses

while RAND SAT performs best on long clauses