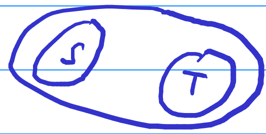
LECTURE 6 5/11/2020

DISJOINT PATH PROBLEM (EDP)

INPUT: $G=(V,E)$, k pairs $(s_1,t_1),\dots,(s_k,t_k)$

OUTPUT: $I \subseteq \{1,\dots,k\}$ together with a path P_i for each $i\in I$ s.t. P_i connects $s_i - t_i$ and each edge is used by at most one path

OBJECTIVE: maximize $|I|$

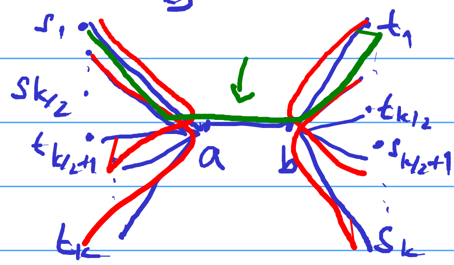


What does NOT work: $S=\{s_1,\dots,s_k\}$, $T=\{t_1,\dots,t_k\}$

find max flow between $S-T$

Note: max flow $S-T$: k

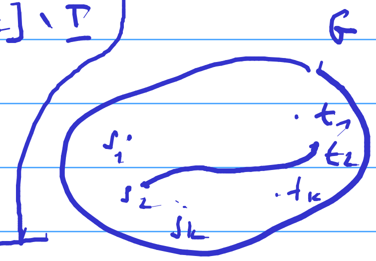
EDP -- 1



NP_hard

GREEDY ALG FOR EDP $[k]=\{1,\dots,k\}$

1) $I:=\emptyset$
2) for each $i\in[k]\setminus I$, let P_i be a shortest $s_i - t_i$ path
   find $j\in[k]\setminus I$ s.t. $|P_j|\le|P_i|$ $\forall i\in[k]\setminus I$
   if there is no such path, go to (4)
3) $I:=I\cup\{j\}$, $G:=G\setminus P_j$, go to (2)
4) OUTPUT: I and P_i for each $i\in I$



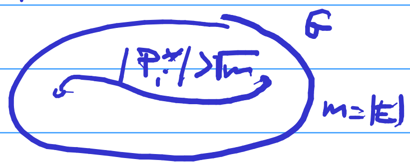
THM: the greedy alg. has approximation ratio $2\sqrt{m}+1$, $m=|E|$.

Proof: consider an optimal solution OPT

let P_i^* be the path connecting $s_i - t_i$, for $i\in OPT$

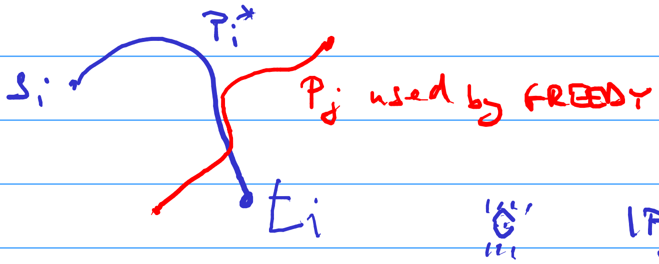
$OPT_e = \{ i\in OPT \mid |P_i^*| > \sqrt{m} \}$

$OPT_s = OPT \setminus OPT_e$



$|OPT_e| \le \sqrt{m}$

consider P_i^* for $i \in OPT_s \setminus I$



why did GREEDY not connect s_i–t_i by P_i^* ?

Some edge $e \in P_i^*$ used by P_j EARLY

$|P_j| \le |P_i^*|$

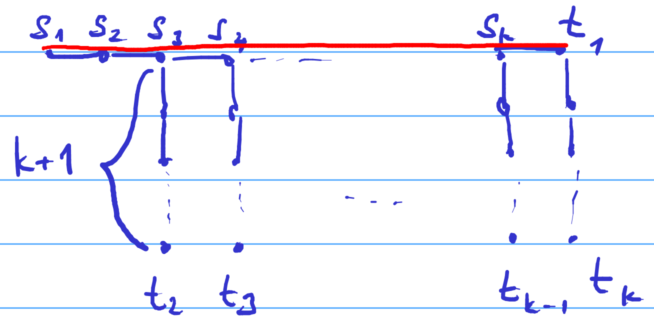
we say that P_j <u>blocks</u> P_i^* (or the pair s_i–t_i)

$\Rightarrow$ Every P_i, $i \in I$, blocks at most $\sqrt{m}$ paths from OPT.

$$|OPT_s \setminus I| \le \sqrt{m} \cdot |I|$$

$$|OPT| = |OPT_e| + |OPT_s| \le \sqrt{m} + |OPT_s \setminus I| + |I| \le \sqrt{m} + \sqrt{m}\,|I| + |I| \le (2\sqrt{m}+1)\,|I|$$ ∎

<u>Bad instance</u>



GREEDY ... 1

OPT $k-1$

$|E| \sim k^2$

<u>Notes</u>: for directed graphs ... almost the best possible

$\forall \varepsilon > 0$ $m^{1/2-\varepsilon}$-approx ... NP hard

for directed graphs: NP hard even for $k=2$

for undirected graphs: for any fixed k, in P

<u>Online algorithm?</u>

Alternative GREEDY: process the pairs in any order

if there exists an $s_i t_i$ path P_i, $|P_i| \le \sqrt{m}$,

use it.

slight difficulty: if $d(s_i, t_i) > \sqrt{m}$ $\forall i$

use randomization $\rightarrow$ alternative GREEDY works in ONLINE setting

PATHS WITH CONGESTION

Generalization of EDP: every edge has integer capacity $c \geq 1$; i.e. up to c paths can share any edge

$m := |E|$

GREEDY with capacities

1) $I = \emptyset$, $\beta = \lceil m^{\frac{1}{c+1}} \rceil$, $\forall e \in E \;\; d(e) = 1$
2) for each $i \in [k] \setminus I$, let P_i be the d-shortest $s_i - t_i$ path
   find $j \in [k] \setminus I$ s.t. $d(P_j) \leq d(P_i) \;\; \forall i \in [k] \setminus I$
   if no such j exists, or P_j violates capacities, go to (4)
3) $I := I \cup \{j\}$, $\forall e \in P_j$: $d(e) := d(e) \cdot \beta$, go to (2)
4) OUTPUT I, P_i, for each $i \in I$

THM: the GREEDY with capacities has approximation ratio $3c \cdot m^{\frac{1}{c+1}} + 1$.

Proof: consider an optimal solution OPT, P_i^* connects $s_i - t_i$ in OPT

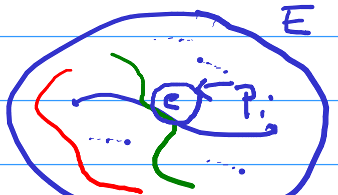
a path P selected by GREEDY is short if $d(P) = \sum_{e \in P} d(e) \leq m^{\frac{c}{c+1}}$

$d_1, d_2, d_3, \ldots d_\ell = \bar{d}$

let $\bar{d}$ be the length function at the end of the last iteration in which GREEDY used a short path [ℓ]

"1 $\bar{d}(E) \leq m(1 + 2|I|)$

proof: $\bar{d}(E) \leq m + \sum_{i \in I} m^{\frac{c}{c+1}} \cdot \beta \leq m + 2m|I|$ ∎

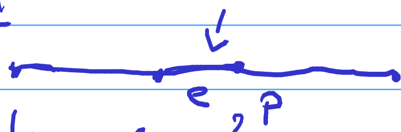


$\bar{d}(P_i) \leq m^{\frac{c}{c+1}} \cdot \beta$

"2 for each $P \in OPT \setminus I$, $\bar{d}(P) \geq m^{\frac{c}{c+1}}$

proof: Assume for contradiction that

$\bar{d}(P) < m^{\frac{c}{c+1}}$ for some $P \in OPT \setminus I$



How many paths from I use e in iteration $\ell+1$?

can it be $\geq c$? then $d(e) \geq \beta^c > m^{\frac{c}{c+1}}$ → NO!

$\leq c-1 \Rightarrow P$ is a short feasible path in iter. $\ell+1$ - contradiction ∎

"1 & 2 $\Rightarrow |OPT \setminus I| \leq \dfrac{c \cdot \bar{d}(E)}{m^{c/c+1}} \leq \dfrac{c \cdot m(1+2|I|)}{m^{c/c+1}} = c \cdot m^{\frac{1}{c+1}}(1 + 2|I|)$

$|OPT| \leq |OPT \setminus I| + |I| \leq (3c \cdot m^{\frac{1}{c+1}} + 1) I$ ∎

Note: $c = \lceil \log m \rceil$
$\Rightarrow$ ratio $O(\log m)$