with $f(n) = g(n) = 1$, $F(x) = G(x) = \lfloor x \rfloor$, and $y = \sqrt{x}$ gives

$$\begin{aligned}\sum_{n \le x} \tau(n) &= 2 \sum_{n \le x^{1/2}} \lfloor x/n \rfloor - \lfloor \sqrt{x} \rfloor^2 \\ &= 2x \cdot \sum_{n \le x^{1/2}} n^{-1} + O(\sqrt{x}) - x + O(\sqrt{x}) \,.\end{aligned}$$

Corollary C.1.16 implies that the last sum equals $\frac{1}{2}\log x + \gamma + O(1/\sqrt{x})$. The asymptotics follows. □

From $\sum_{n \le x} \log n = x \log x - x + O(\log x)$ (Corollary C.1.13) it follows that $\tau(n)$ has average order $\log n$ with relative error $O(1/\log x)$, and average order $2\gamma + \log n$ with relative error $O(1/\sqrt{x}\log x)$.

The next three asymptotic relations for $\sum_{n \le x} f(n)$ employ the sum

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6} = 1.64493\ldots$$

(Proposition C.4.4). By absolute convergence (Proposition C.2.4) and the identity in proof No. 10 of Euclid's theorem for $s = 2$, we have

$$\frac{6}{\pi^2} = \frac{1}{\zeta(2)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \,.$$

An integral estimate gives

$$\sum_{n > x} \frac{1}{n^2} < \frac{1}{x-1}, \quad x > 1 \,.$$

Proposition 1.4.2. *The function $\sigma(n)$ summing divisors of n satisfies*

$$\sum_{n \le x} \sigma(n) = (\pi^2/12)x^2 + O(x \log x), \quad x > 1 \,.$$

Proof. The sum equals

$$\begin{aligned}\sum_{mn \le x} m &= \sum_{n \le x} \sum_{m \le x/n} m = \sum_{n \le x} \lfloor x/n \rfloor (\lfloor x/n \rfloor + 1)/2 \\ &= \sum_{n \le x} (x^2/2n^2 + O(x/n)) = (x^2/2) \sum_{n \le x} n^{-2} + O(x \log x) \\ &= (x^2/2)\zeta(2) - O(x) + O(x \log x) \\ &= (\pi^2/12)x^2 + O(x \log x) \,,\end{aligned}$$

by the tail estimate of $\sum_{n \ge 1} n^{-2}$ and the formula for $\zeta(2)$. □

So $\sigma(n)$ has average order $(\pi^2/6)n$, with relative error $O((\log x)/x)$.

Proposition 1.4.3. *Euler's totient function $\varphi(n)$ satisfies*

$$\sum_{n \leq x} \varphi(n) = (3/\pi^2)x^2 + O(x \log x), \ \ x > 1 \, .$$

Proof. *First proof.* Since by Corollary 1.1.11

$$\varphi(n) = n \prod_{p \mid n} (1 - p^{-1}) = n \sum_{d \mid n} \mu(d)/d = \sum_{de=n} \mu(d)e \, ,$$

we calculate, using the above results on $\zeta(2)$, that the sum equals

$$\begin{aligned}
\sum_{de \leq x} \mu(d)e &= \sum_{d \leq x} \mu(d) \sum_{e \leq x/d} e = \sum_{d \leq x} \mu(d)(x^2/2d^2 + O(x/d)) \\
&= (x^2/2) \sum_{d \leq x} \mu(d)d^{-2} + O(x \log x) \\
&= x^2/2\zeta(2) - O(x) + O(x \log x) \\
&= (3/\pi^2)x^2 + O(x \log x) \, .
\end{aligned}$$

Second proof. Let $f(x)$ be the number of pairs $(a, b) \in \mathbb{N}^2$ with $a, b \leq x$, and $g(x)$ be the number of those with a and b coprime. Since $f(x) = \sum_{d \leq x} g(x/d)$ (pigeonholing the pairs by the greatest common divisor), Corollary 1.1.14 gives that $g(x) = \sum_{d \leq x} \mu(d)f(x/d)$. But $f(x) = \lfloor x \rfloor^2$, so $f(x/d) = x^2/d^2 + O(x/d)$ and $g(x) = x^2 \sum_{d \leq x} \mu(d)/d^2 + O(x \log x) = x^2/\zeta(2) + O(x \log x)$. But also $g(x) = 2\sum_{n \leq x} \varphi(n) - 1$ (pigeonholing the pairs by the maximum), which gives the result. □

So $\varphi(n)$ has average order $(6/\pi^2)n$, with relative error $O((\log x)/x)$. The second proof shows that the limit density of pairs of coprime positive integers equals

$$\lim_{n \to \infty} n^{-2} |\{(a, b) \in [n] \times [n] \mid (a, b) = 1\}| = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \, .$$

Next we count square-free natural numbers, whose number up to $x > 1$ we denote by $Q(x)$. In Proposition 5.1.6 we do the same for polynomials in $F_q[x]$; it is interesting to compare both results.

Proposition 1.4.4. *We have*

$$Q(x) = \sum_{n \leq x} \mu(n)^2 = \sum_{n \leq x, \ \mu(n) \neq 0} 1 = (6/\pi^2)x + O(\sqrt{x}), \ \ x > 1 \, .$$

Therefore $\mu(n)^2$ has constant average order $6/\pi^2 = 0.60792\ldots$, with relative error $O(1/\sqrt{x})$.

Proof. *Proof by inclusion and exclusion.* Let $x > 1$. The principle of inclusion and exclusion (Proposition C.1.1) shows that $Q(x)$ equals, by the above results on $\zeta(2)$,

$$\begin{aligned}
& \lfloor x\rfloor - \sum_{p\le\sqrt{x}} \left\lfloor \frac{x}{p^2}\right\rfloor + \sum_{p<q,\ pq\le\sqrt{x}} \left\lfloor \frac{x}{(pq)^2}\right\rfloor - \sum_{p<q<r,\ pqr\le\sqrt{x}} \left\lfloor \frac{x}{(pqr)^2}\right\rfloor + \dots \\
= {} & x\sum_{n\le\sqrt{x}} \frac{\mu(n)}{n^2} + \Delta,\ |\Delta| \le Q(\sqrt{x}), \\
= {} & x\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - x\sum_{n>\sqrt{x}} \frac{\mu(n)}{n^2} + O(\sqrt{x}) = \frac{x}{\zeta(2)} - O(\sqrt{x}) + O(\sqrt{x}) \\
= {} & (6/\pi^2)x + O(\sqrt{x})\ .
\end{aligned}$$

Proof by the Dirichlet convolution. We define $\nu(n)$ as $\nu(n) = 0$ if n is not a square, and $\nu(n) = \mu(m)$ if $n = m^2$. Then $\mu(n)^2 = \sum_{ab=n} \nu(a)$:

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \prod_p (1+p^{-s}) = \prod_p (1-p^{-2s}) \prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} \sum_{n=1}^{\infty} \frac{1}{n^s}\ .$$

Thus

$$\sum_{n\le x} \mu(n)^2 = \sum_{ab\le x} \nu(a) = \sum_{a\le x} \nu(a)\lfloor x/a\rfloor = x\sum_{a\le x} \frac{\nu(a)}{a} + \Delta,\ |\Delta| \le \sum_{a\le x} |\nu(a)|$$

— the same sum and the same error term as in the first proof. □

So $1/\zeta(2) = 0.60792\dots$ is also the limit density of square-free numbers, the limit probability that a random natural number is square-free:

$$\lim_{n\to\infty} n^{-1}|\{a \in [n] \mid \mu(a) \ne 0\}| = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}\ .$$

For polynomials, this probablity (that a random monic degree n polynonomial in $F_q[x]$ is square-free) equals to the constant value $1-1/q$, see Proposition 5.1.6. The two proofs exploit the same idea in two ways, and the Dirichlet convolution is the more flexible way. We demonstrate it by the next result, needed later.

Proposition 1.4.5. *For $x > 1$ we have*

$$\sum_{n\le x} \frac{\mu(n)^2}{n} = \frac{\log x}{\zeta(2)} + O(1)\ .$$

Proof. As $\mu(n)^2 = \sum_{ab=n} \nu(a)$ (by the previous proof), the sum equals

$$\begin{aligned}
\sum_{ab\le x} \frac{\nu(a)}{a}\cdot\frac{1}{b} &= \sum_{b\le x} \frac{1}{b} \sum_{a\le x/b} \frac{\nu(a)}{a} = \sum_{b\le x} \frac{1}{b}(\zeta(2)^{-1} + O(\sqrt{b/x})) \\
&= \frac{1}{\zeta(2)} \sum_{b\le x} \frac{1}{b} + \frac{O(1)}{\sqrt{x}} \sum_{b\le x} \frac{1}{\sqrt{b}} = \frac{\log x}{\zeta(2)} + O(1) + \frac{O(1)}{\sqrt{x}} O(\sqrt{x}) \\
&= \frac{\log x}{\zeta(2)} + O(1)\ .
\end{aligned}$$