Moreira's theorem: any finite coloring of $\{1, 2, \dots\}$ has a monochromatic triple $\{x, x + y, xy\}$

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The classical theorem of Hindman [2] says that for any finite coloring $\mathbb{N} = \{1, 2, \dots\} = C_1 \cup C_2 \cup \dots \cup C_r$ of natural numbers there is a color $b \in [r] = \{1, 2, \dots, r\}$ and an infinite sequence $1 \le n_1 < n_2 < \dots$ in $\mathbb{N}$ such that

$$n_{i_1} + n_{i_2} + \dots + n_{i_k} \in C_b$$

for every k-tuple $1 \le i_1 < i_2 < \dots < i_k$. The mapping $a \mapsto 2^a$ transforms sums into products, and we see that the Hindman theorem also holds for the product $n_{i_1} n_{i_2} \dots n_{i_k}$ in place of the sum. But can one combine together sums and products? Until now it was not known if every finite coloring of $\mathbb{N}$ has to contain a monochromatic pair $\{x + y, xy\}$. This problem has been recently resolved by Moreira [4]: every finite coloring of $\mathbb{N}$ has a monochromatic triple $\{x, x + y, xy\}$. We present his beautiful elementary proof ([4, Section 5] plus some details filled in by us) below. The proof is selfcontained except for the proof of van der Waerden's theorem, which is easily found in the literature or on the Internet. We intend to include Moreira's theorem in [3] in the chapter on arithmetic progressions.

Two comments before we plunge in the proof. By a general argument, existence of one monochromatic triple $\{x, x+y, xy\}$ in any finite coloring of $\mathbb{N}$ forces existence of infinitely many such triples; if a finite coloring existed with only finitely many such triples, we could recolor points in them and obtain a finite coloring with no such triple. This tacitly uses that always $|\{x, x + y, xy\}| \ge 2$; for $\mathbb{N}_0 = \{0, 1, \dots\}$ in place of $\mathbb{N}$ we would have to exclude the trivial triple for $x = y = 0$. Also, the theorem is equivalent to the finite version that for every $r \in \mathbb{N}$ there is an $n = n(r) \in \mathbb{N}$ such that for any r-coloring $f : [n] \to [r]$ there exist $x, y \in [n]$ such that $x + y, xy \in [n]$ and $f(x) = f(x + y) = f(xy)$. Equivalence of both versions of the theorem follows by the usual compactness argument; if for some r there were no such $n(r)$, we could paste the bad r-colorings of $[n]$, $n = 1, 2, \dots$, into a bad r-coloring of $\mathbb{N}$. We prove the infinite version.

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Theorem (Moreira [4], 2016). *For any finite coloring*

$$\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$$

there exist a $b \in [r]$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, x+y, xy\} \subset C_b$.

Proof. A set $X \subset \mathbb{N}$ is *piecewise syndetic*, or briefly a *ps set*, if there is a $c \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there is a k-tuple $n_1 < n_2 < \cdots < n_k$ of elements in X such that $n_{i+1} - n_i \le c$ for every $i = 1, 2, \ldots, k-1$. We call such tuple a *c-run in X with length k*. $\mathbb{N}$ is a ps set. It is clear that if $X, Y \subset \mathbb{N}$, X is a ps set, and $Y = f(X)$ where $f\colon \mathbb{R} \to \mathbb{R}$ is an affine (nonconstant) mapping, then Y is a ps set. The same if $Y \supset X$. We begin with proving that ps sets have Ramsey property.

- If $\mathbb{N} \supset X = C_1 \cup C_2 \cup \cdots \cup C_r$ and X is a ps set then some C_i is a ps set.

It suffices to prove it for $r = 2$. Suppose that X is red-blue colored and t_k, $k = 1, 2, \ldots$, are c-runs in X with length k. Let $b(t_k)$ be the number of blue numbers in t_k. If $b(t_k)$ are bounded, say $b(t_k) \le d \in \mathbb{N}$ for every k, then by omitting all blue numbers from all t_k we get $c(d+1)$-runs of red numbers with length going to infinity. So we assume that $b(t_k)$ are unbounded and passing to a subsequence we may assume that $b(t_k) \to \infty$. Let $g(t_k)$ be the maximum distance between two consecutive blue numbers in t_k. If it is unbounded then the red numbers between two distant consecutive blue numbers in t_k form c-runs of red numbers with length going to infinity. So we may assume that $g(t_k) \le d$ for a $d \in \mathbb{N}$ for every k. But then the blue numbers in t_k form d-runs of blue numbers with length going to infinity.

The following is the classical van der Waerden theorem.

- For every $k, r \in \mathbb{N}$ there is an $n = n(r, k) \in \mathbb{N}$ such that any r-coloring $f\colon [n] \to [r]$ contains a monochromatic k-term arithmetic progression,

  $$f(a) = f(a+d) = f(a+2d) = \cdots = f(a+(k-1)d)$$

  for some $a, d \in [n]$ with $a + (k-1)d \in [n]$.

This was proven by van der Waerden [5]. For a short (less than 1 page) combinatorial proof see Graham, Spencer and Rothschild [1].

The next property of ps sets is crucial.

- If $F \subset \mathbb{N}$ is finite and $X \subset \mathbb{N}$ is a ps set then there is an $n \in \mathbb{N}$ such that

  $$Y = X \cap \bigcap_{m \in F} (X - mn)$$

  is a ps set.

To prove it, we take c-runs t_k, $k \in \mathbb{N}$, in X with length k and, using the vdW theorem, set $m = n(c, 1+f)$ where $f = \max F$. We partition each interval of integers $I_k = [\min t_k, \max t_k]$ into (disjoint and consecutive) intervals $J_{k,i}$,

$i = 1, 2, \ldots, \lfloor |I_k|/m \rfloor$, each with length m, and the possibly shorter residual interval. We color each $J_{k,i}$ with colors $\{0, 1, \ldots, c-1\}$, the color of $a \in J_{k,i}$ is the distance to the preceding closest element of X (so a has color 0 iff $a \in X$). By the vdW theorem, every $J_{k,i}$ contains a monochromatic $(1+f)$-term AP. Shifting it down by its color, we get a $(1+f)$-term AP $A_{k,i} \subset J'_{k,i} \cap X$, where $J'_{k,i} = [\min J_{k,i} - c, \max J_{k,i}]$. Let $a_{k,i}$ and $d_{k,i}$ be, respectively, the first term and the common difference of $A_{k,i}$. Clearly, $a_{k,i} \in Y$ if $n = d_{k,i}$ (where Y is as stated). Let Y' be the set of all $a_{k,i}$. We have $Y' \subset X$ and from $a_{k,i} \in J'_{k,i}$ and $a_{k,i+1} - a_{k,i} < 2m + c$ it follows that Y' is a ps set. We give to each $a_{k,i}$ the color $d_{k,i}$. As $d_{k,i} < m$, this is a finite coloring of Y'. By the Ramsey property of ps sets (proved above), there is a monochromatic ps set $Y'' \subset Y'$. For n equal to the color of Y'' one has $Y \supset Y''$ and so Y is a ps set.

We start the proper proof of the theorem. Let $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, $r \geq 2$, be a finite coloring of natural numbers. We define inductively numbers $n_1, \ldots, n_r \in \mathbb{N}$, ps sets $X_0, \ldots, X_r, Y_1, \ldots, Y_r \subset \mathbb{N}$, and colors $r_0, \ldots, r_r \in [r]$ so that $X_i \subset C_{r_i}$ for every i. First we select $r_0 \in [r]$ so that $X_0 := C_{r_0}$ is a ps set (using the Ramsey property of ps sets). If $0 < i \leq r$ (and the quantities n_-, X_-, Y_-, and r_- with indices smaller than i have been already defined), we apply the above crucial property of ps sets and take $n_i \in \mathbb{N}$ such that

$$Y_i := X_{i-1} \cap \bigcap_{j=1}^{i} (X_{i-1} - n_j^2 \ldots n_{i-1}^2 n_i)$$

is a ps set (for $i = j$ the empty product $n_j^2 \ldots n_{i-1}^2 = 1$). Since $n_i Y_i$ is a ps set, there is an $r_i \in [r]$ such that $X_i := n_i Y_i \cap C_{r_i}$ is a ps set. This finishes the inductive definition.

From $X_i \subset n_i Y_i \subset n_i X_{i-1}$ we get by iteration, for any $0 \leq j < i \leq r$, the inclusion $X_i \subset n_{j+1} \ldots n_i X_j$. By the pigeonhole principle, there exist $0 \leq j < i \leq r$ with $r_i = r_j$. We take any $x' \in X_i$ and set

$$y := n_{j+1} \ldots n_i \quad \text{and} \quad x := x'/y$$

($x \in \mathbb{N}$ by the inclusion). We show that $\{x, x+y, xy\} \subset C_{r_i}$, which will finish the proof. Indeed, $xy = x' \in X_i \subset C_{r_i}$ and, by the inclusion, $xy \in X_i \subset yX_j$, giving $x \in X_j \subset C_{r_j} = C_{r_i}$. Finally,

$$\begin{array}{rcl} (x+y)y & = & x' + y^2 \in X_i + y^2 \subset n_i Y_i + y^2 \\ \text{(the definition of } Y_i\text{)} & \subset & n_i(X_{i-1} - n_{j+1}^2 \ldots n_{i-1}^2 n_i) + y^2 \\ \text{(the inclusion)} & \subset & n_i(n_{j+1} \ldots n_{i-1} X_j - n_{j+1}^2 \ldots n_{i-1}^2 n_i) + y^2 \\ \text{(the definition of } y\text{)} & = & yX_j - y^2 + y^2 = yX_j \end{array}$$

and $x + y \in X_j \subset C_{r_j} = C_{r_i}$. □

References

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