First homework assignment

The numbers in boxes indicate the maximum number of points available for a given exercise.

[4] 1. Show that an unoriented graph G has an orientation in which each vertex has at most k outgoing edges if and only if each subgraph H of G satisfies $|E(H)| \leq k|V(H)|$.

[4] 2. For what values of k is it true that every filling of the first k rows of a 9×9 matrix according to the rules of sudoku can be extended into a filling of $k+1$ rows without violating the rules?

[4] 3. Let M be a matrix whose entries have values 0 or 1. Show that the largest independent set of M has the same cardinality as the smallest covering set of lines of M. An *independent set* of M is a set of entries which are all equal to 1 and no two of them belong to the same row or column. A *covering set of lines* is a set of rows and columns that together cover all the positive entries of M.

4. Let M be an $n \times n$ matrix of numbers $1, 2, \ldots, n$, where each number appears in M exactly n times.

   [4] (a) Show that M has a row or column with at least $\sqrt{n}$ different numbers.

   [2] (b) Show that if $\sqrt{n}$ is an integer, then the estimate in part (a) cannot be improved.

[2] 5. Find a connected 3-regular graph with 100 vertices that has no perfect matching.

6. We say that a matching M in a graph G *covers* a vertex $v \in V(G)$, if v belongs to an edge of M. Prove the following generalizations of Tutte's theorem. (Your proof may use the standard Tutte's theorem that you know from the lecture.)

   [4] (a) Let $k \geq 0$ be an integer. A graph $G = (V, E)$ has a matching that covers at least $|V| - k$ vertices if and only if for each set $A \subseteq V$, the graph $G \setminus A$ has at most $|A| + k$ odd components.

   [4] (b) Let $G = (V, E)$ be a graph, let $T \subseteq V$ be a set of its vertices. G has a matching that covers all the vertices from T if and only if for each set $A \subseteq V$, at most $|A|$ odd components of the graph $G \setminus A$ have the property that their vertices all belong to T.

[4] 7. Deduce Hall's theorem from Tutte's theorem.

[4] 8. Deduce Hall's theorem from Gallai-Milgram's theorem.

[5] 9. Deduce Tutte's theorem from Hall's theorem. For example, you may follow these steps:

   - Assume there is a graph which satisfies Tutte's condition but has no perfect matching. Let $G = (V, E)$ be such a graph with the smallest possible number of vertices. Let $X \subseteq V$ be a maximal set of vertices that satisfies Tutte's condition with equality, i.e., $G \setminus X$ has exactly $|X|$ odd components. Show that such a set X exists and is nonempty.
   - Show that $G \setminus X$ has no even components.
   - Show that if C is a component of $G \setminus X$ and v any vertex of C, then $C \setminus \{v\}$ has a perfect matching.
   - Use Hall's theorem to show that G has a perfect matching.

[4] 10. For each $k \geq 2$, show that there is a $(2k-1)$-edge-connected graph that does not have k disjoint spanning trees. (You may begin by showing that for any q, a q-regular graph with at most $2q$ vertices is q-edge-connected.)

11. We claim that any sequence $x_1, x_2, \ldots, x_{rs+1}$ of $rs+1$ real numbers contains a nondecreasing subsequence of length $r+1$ or a decreasing subsequence of length $s+1$.

   [3] (a) Show that the claim follows from Dilworth's theorem.

   [3] (b) Prove the claim directly, without using Dilworth's or Gallai-Milgram's theorem. Hint: for each element x_i of the sequence, consider the length $l(x_i)$ of the longest nondecreasing subsequence ending in x_i and the length $l'(x_i)$ of the longest decreasing subsequence ending in x_i.

3+2 12. Let $G = (V, E)$ be a directed graph, let $\chi(G)$ be its chromatic number (i.e., $\chi(G)$ is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices have the same color). Show that G contains a directed path with at least $\chi(G)$ vertices. If you can prove this for any directed graph G, you get 5 points, if you can only prove it for a graph G that has no directed cycles, you get 3 points.

13. For each of the following decision problems, show that the problem is NP-hard or find a polynomial algorithm. A *formula* always means a formula in conjunctive normal form. A k-*formula* is a formula whose every clause has k literals. A *positive formula* is a formula that does not contain any negated literal.

3 (a) (2-SAT) Input: a 2-formula F. Question: does F have a satisfying assignment?

3 (b) (NAE-SAT) Input: a formula F. Question: does F have an assignment such that in each clause at least one literal is satisfied and at least one literal is not satisfied? (Note: a constant like "TRUE" or "FALSE" cannot be used as a literal in a formula. Each literal must be a variable or a negated variable.)

3 (c) (positive NAE-SAT) like NAE-SAT, but the input is a positive formula.

1 (d) (positive SAT) like SAT, but the input is a positive formula.

2 (e) (2-NAE-SAT) like NAE-SAT, but the input is a 2-formula.

4 (f) (ODD-SAT) Input: a formula F. Question: is there an assignment that satisfies an odd number of literals in each clause of F?

3 (g) (Hypergraph bicoloring) Input: a hypergraph H. Question: does H have a bicoloring? (See the summary of the recitation of 23. 10. for the necessary definitions.)

2 14. Show that for each k, there is an unsatisfiable $(k, 2^k)$-formula. (Recall that a $(k, 2^k)$-formula is a formula in conjunctive normal form whose clauses have k different literals and each variable appears in at most 2^k clauses.)