Graph Covers and Generalised Snarks

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joint work with

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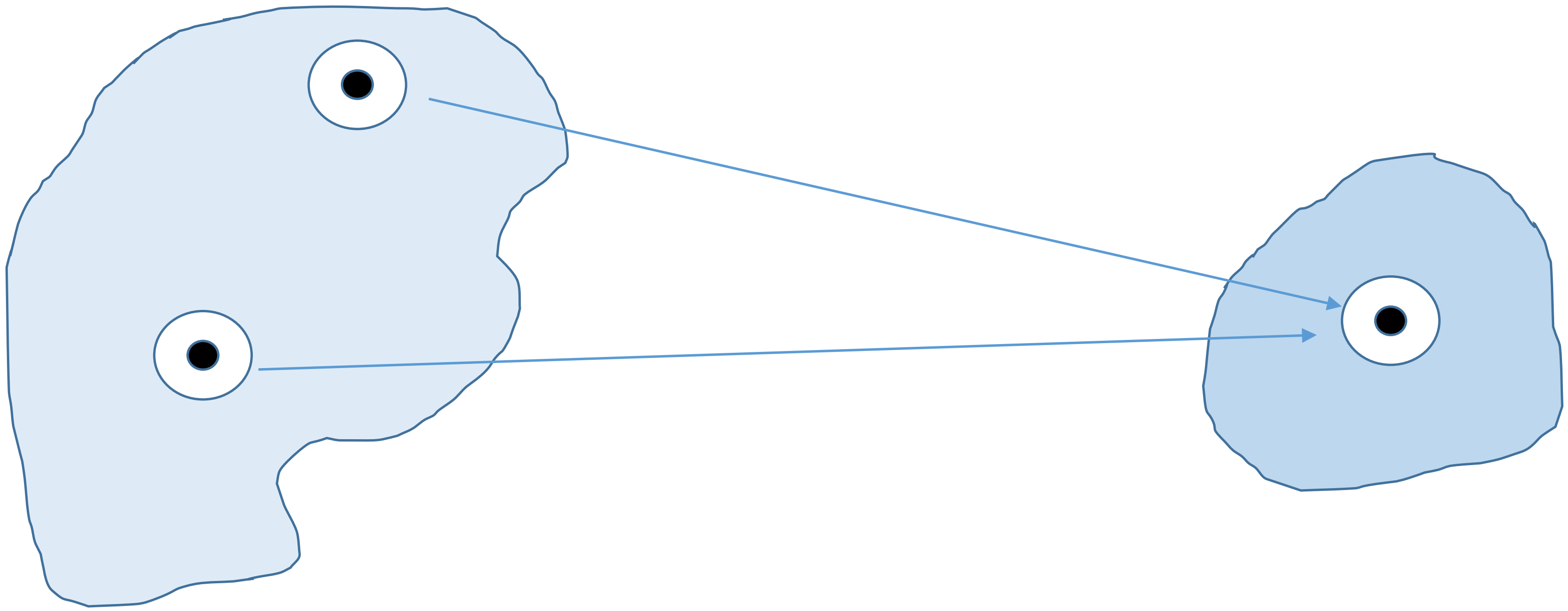
81

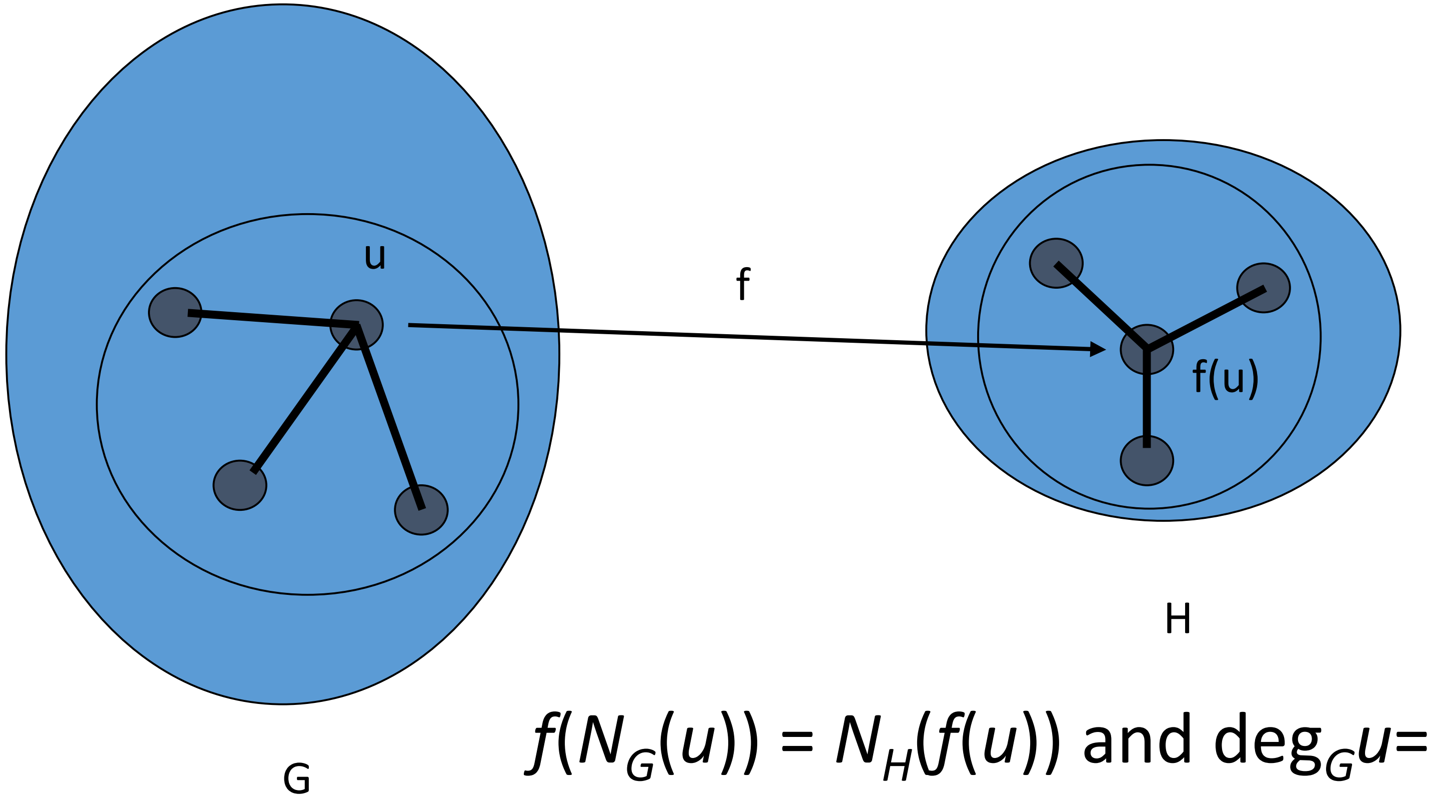
Covering spaces in topology

Euclidean and projective planes – the Euclidean plane is a double cover of the projective one

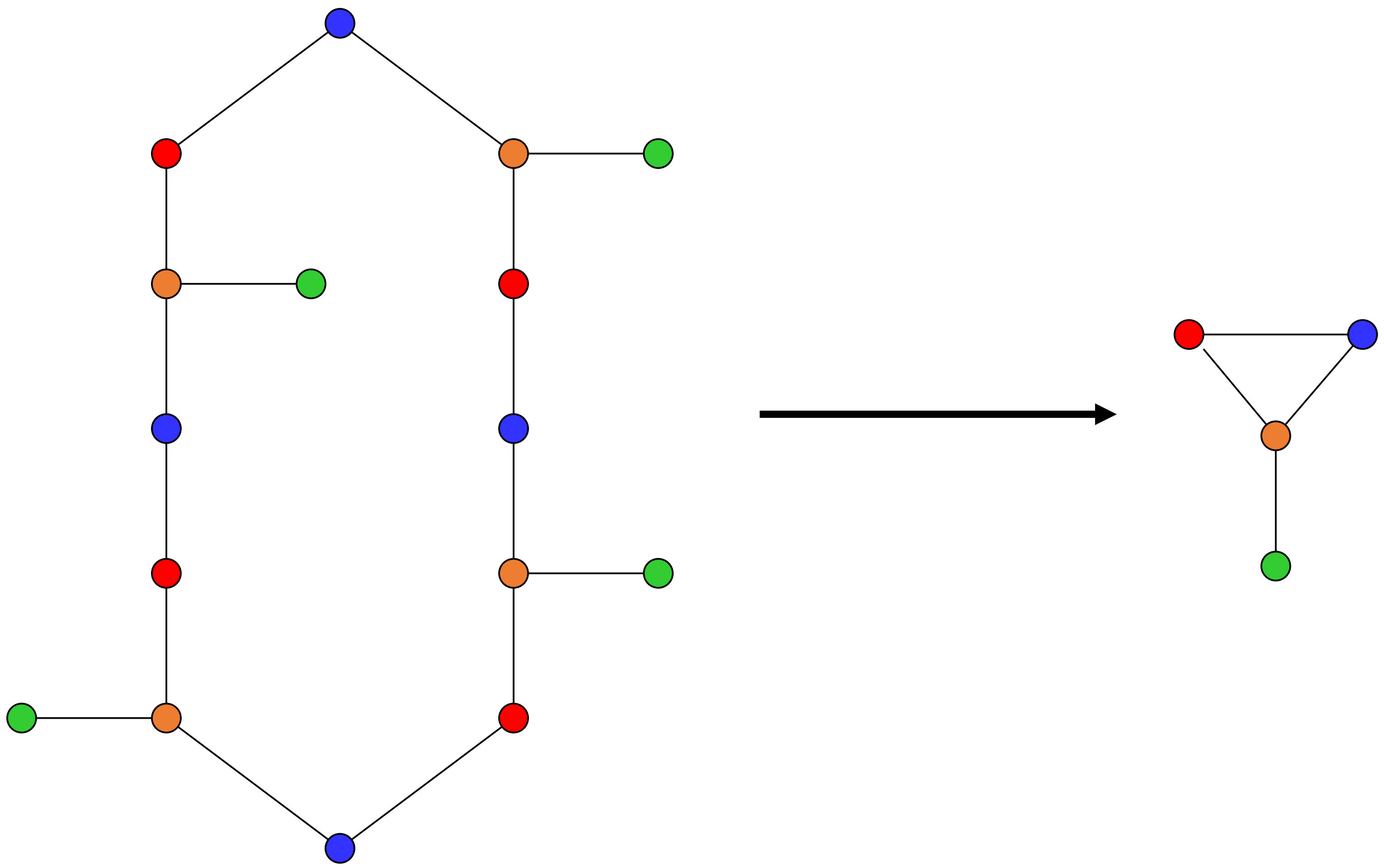


Definition of graph covering
(for connected simple graphs)

Definition: Mapping f: $V(G) \to V(H)$ is a *graph covering projection* if for every $u \in V(G)$, $f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$

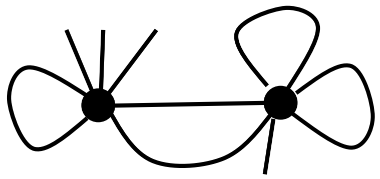


$f(N_G(u)) = N_H(f(u))$ and $\deg_G u = \deg_H f(u)$



General graphs

(with multiple edges, loops and semi-edges allowed)

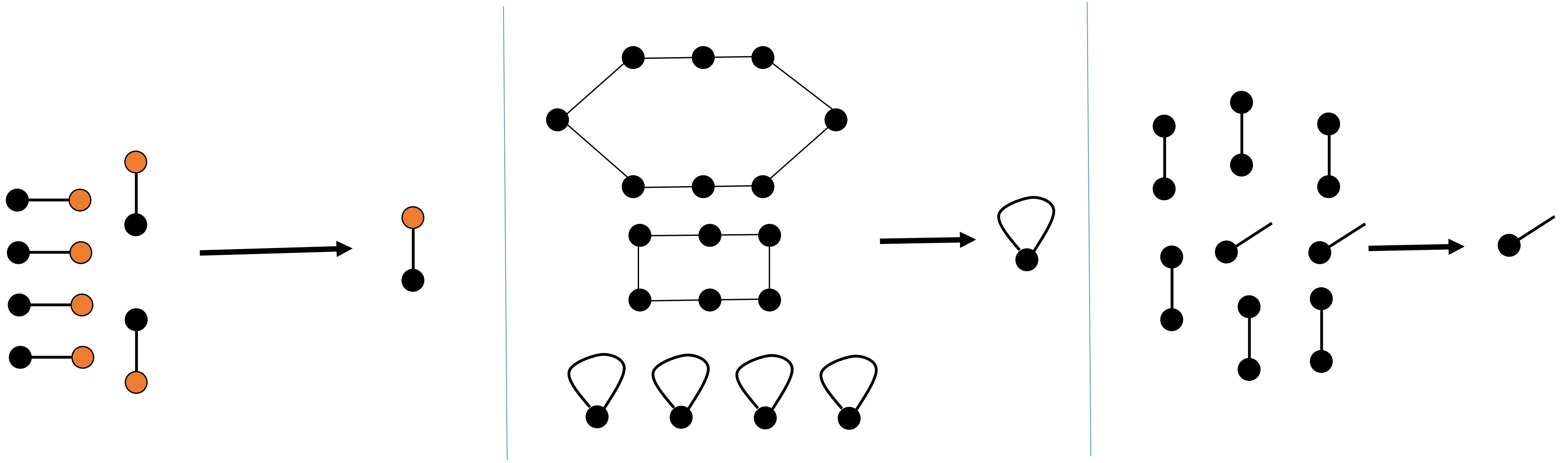


Covers of general graphs

(with multiple edges, loops and semi-edges)

Definition: A pair of mappings $f = (f_V, f_E)\colon G \to H$ is a graph covering projection if

- $f_V\colon V(G) \to V(H)$ is a homomorphism,
- $f_E\colon E(G) \to E(H)$ is compatible with f_V, and it is a bijection of {edges incident with u} onto {edges incident with $f_V(u)$} for every $u \in V(G)$



Computational complexity of graph covers

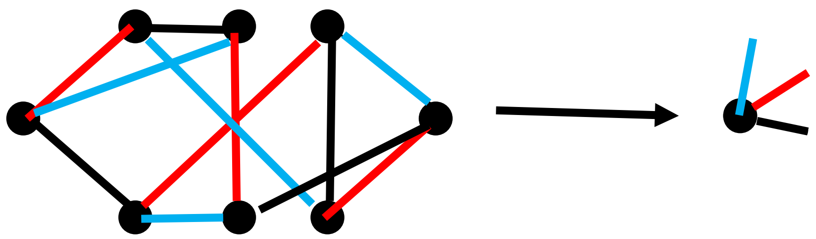
H-COVER
Input: A graph *G*
Question: Does *G* cover *H*?

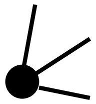
Complexity of covering multigraphs

- Kratochvil, Proskurowski, Telle 1997: Complete characterization of the computational complexity of *H*-COVER for colored mixed 2-vertex multigraphs (without semi-edges) *H.*
- Kratochvil, Telle, Tesař 2016: Complete characterization of the computational complexity of *H*-COVER for 3-vertex multigraphs *H* (monochromatic, undirected, without semi-edges).

- Bok, Fiala, Hliněný, Jdličková, Kratochvíl MFCS 2021: First results on the computational complexity of *H*-COVER for (multi)graphs with **semi-edges**. Full classification for 1-vertex and 2-vertex graphs *H*.
- Bok, Fiala, Jedličková, Kratochvíl, Rzazewski IWOCA 2022: If *H* is a *k*-regular (multi)graph, $k \geq 3$, with at least one semi-simple vertex, then List-*H*-COVER is NP-complete for simple input graphs.

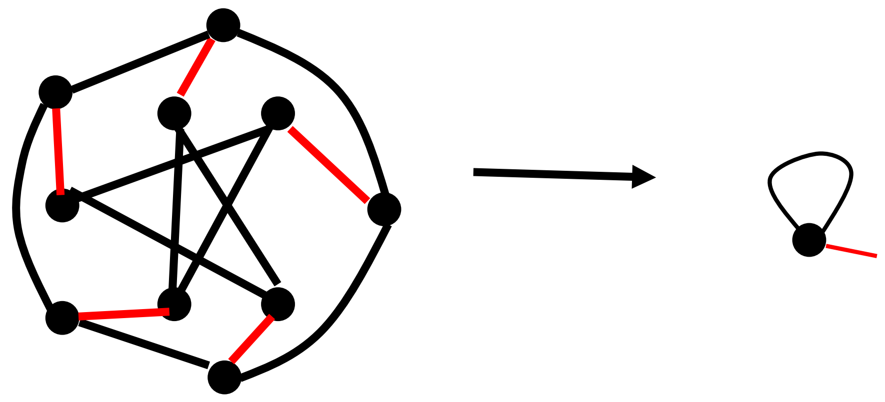
Some examples

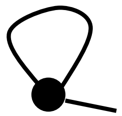


A graph covers  iff it is cubic and 3-edge-colorable.

NP-complete

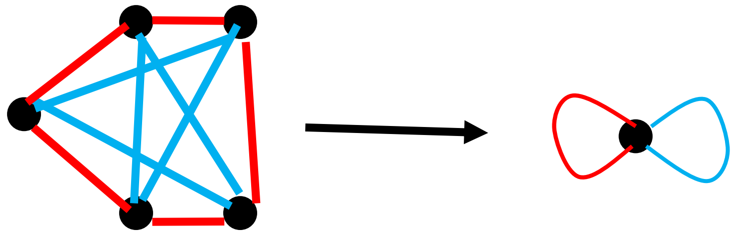
Some examples

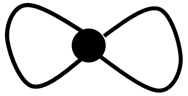


A graph covers  iff it is cubic and has a perfect matching.

Poly time

Some examples



A graph covers  iff it is 4-regular (Petersen/Konig-Hall thm).

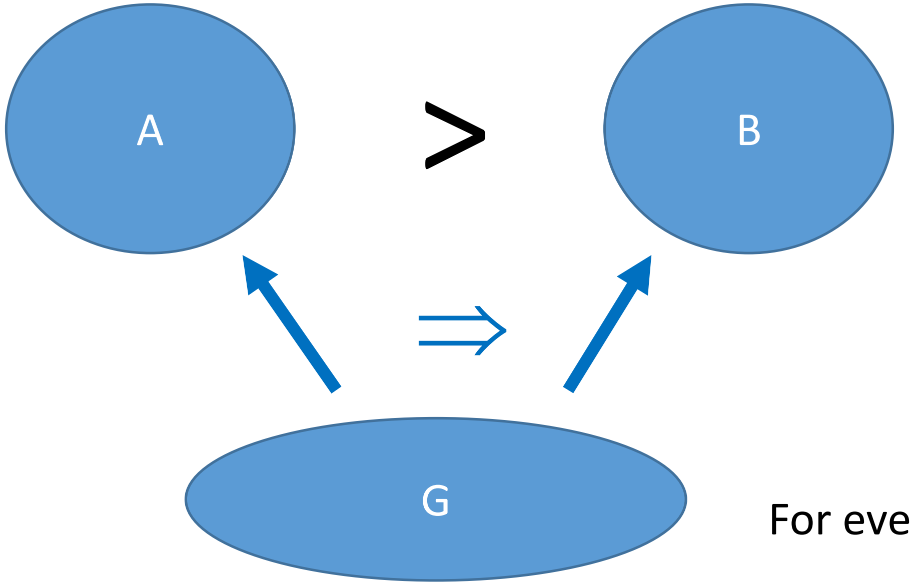
Poly time

Strong Dichotomy Conjecture

2021 Bok et al: For every fixed graph H, the H-COVER problem is either polynomial time solvable for arbitrary input graphs (loops, multiple edges, semi-edges allowed), or NP-complete for simple input graphs.

> relation on connected graphs

Definition: Given connected graphs A and B, we say that A > B if for every simple graph G, it is true that G covers B whenever G covers A.



For every simple connected G

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Example 1: If $A \to B$, then $A > B$.

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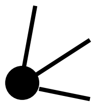
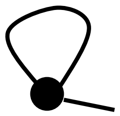
Example 1: If A $\rightarrow$ B, then A > B.

Example 2:  > 

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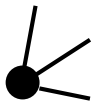
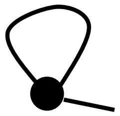
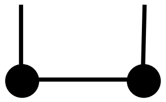
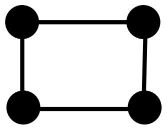
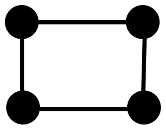
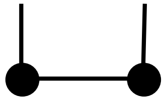
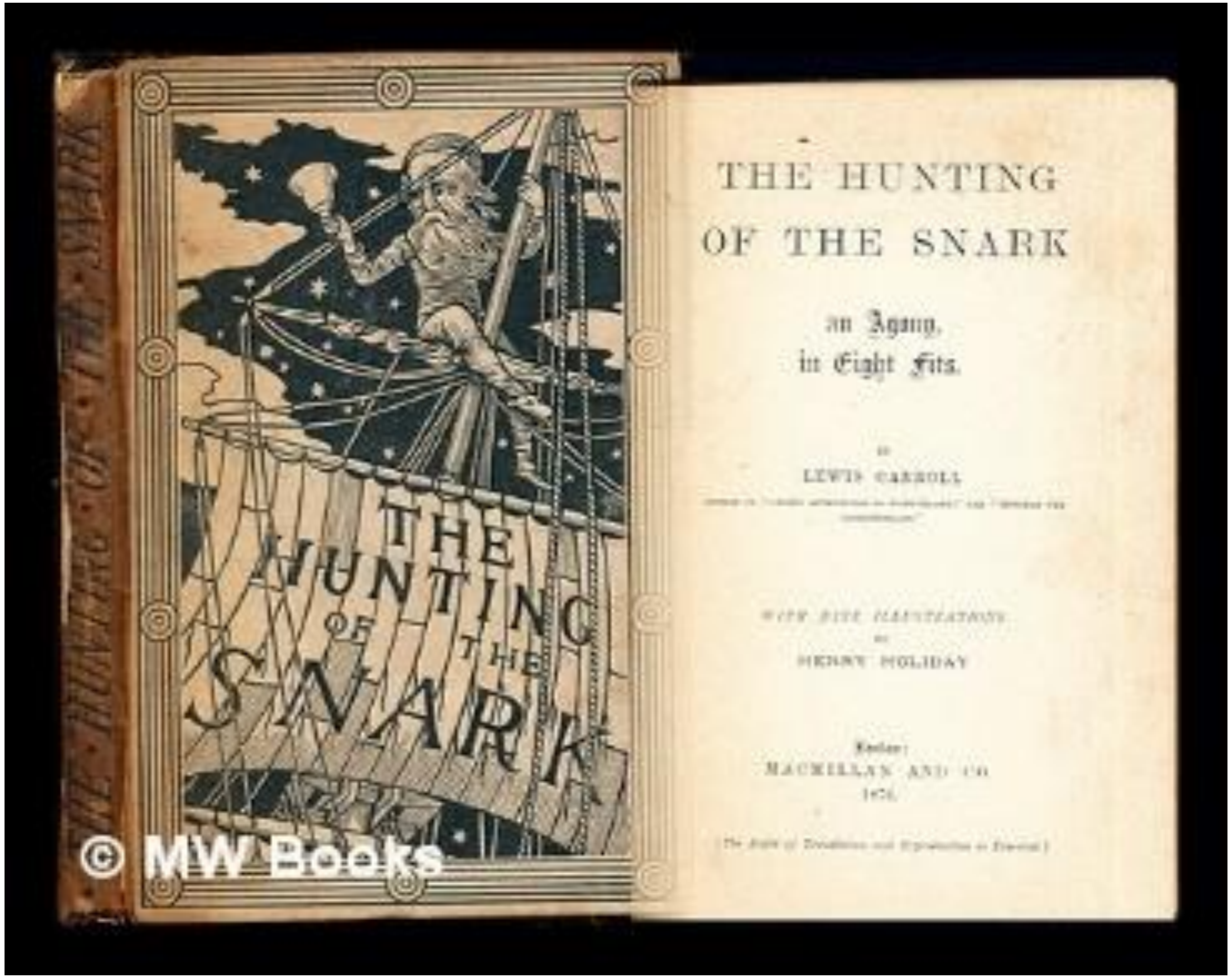
Example 2:  > 

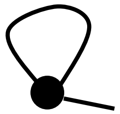
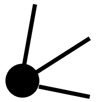
Example 3:  >  and  > 

Hunting for Snarks



> relation on connected graphs

Question: If $\neg$(A>B), then there is a witness G (a simple graph) such that G covers A but G does not cover B. How big would such a witness be? Can such a witness be constructed easily?

We know that $\neg$ ( > ). 2-connected witnesses are **snarks**.

> relation on connected graphs

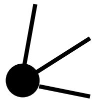
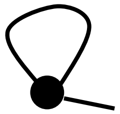
Open problem: Describe all pairs of connected graphs A and B such that A > B and A does not cover B.

Conjecture (Bok et al. 2022): If A has no semi-edges, then
A > B if and only if A covers B.

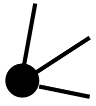
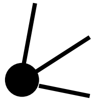
> relation on connected graphs

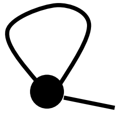
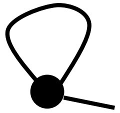
Open problem: Describe all pairs of connected graphs A and B such that A > B and A does not cover B.

Conjecture (Bok et al. 2022): If A has no semi-edges, then
A > B if and only if A covers B.

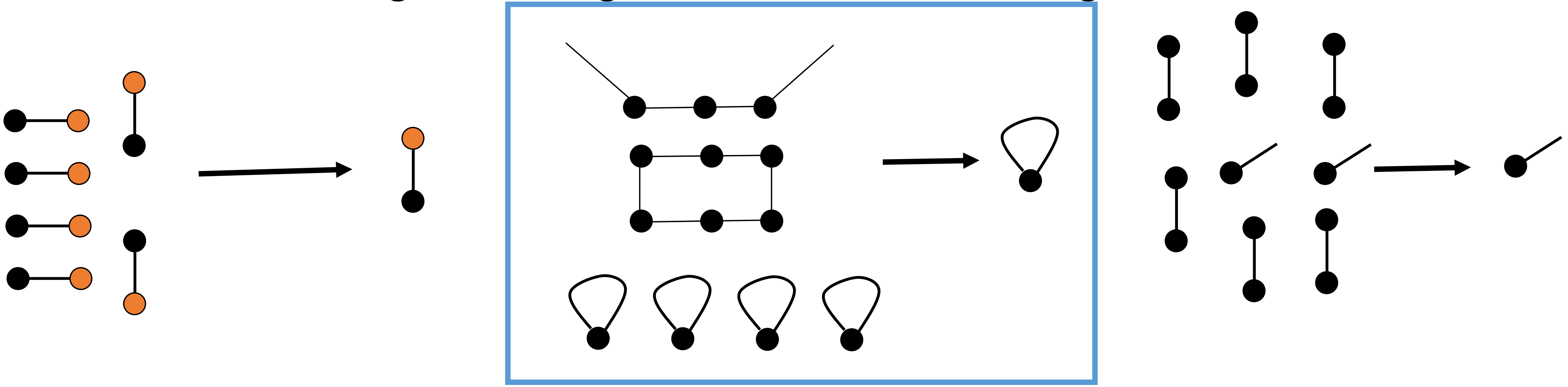
JK, Nedela (EUROCOMB 2023): True for B =  and B = 
with arbitrary A.

> relation on connected graphs

Thm 1 (JK,RN): For any graph A, A >  iff A $\rightarrow$  .

Thm 2 (JK,RN): For any graph A, A >  iff A *semi-covers*  .

Definition: Preimages of edges in a semi-covering



Covering directed graphs

Thm (JK, Proskurowski, Telle + Fiala 1997): If H is simple undirected k-regular graph, $k>2$, then H-COVER is NP-complete.

Thm (Bok, Fiala, Hlineny, Jedlickova, JK 2021): If H is semi-simple undirected k-regular graph, $k>2$, then H-COVER is NP-complete.

Conjecture: If H is simple connected directed k-in-k-out-regular graph with $k>2$, then H-COVER is NP-complete.

Covering directed graphs

Observation: If H is connected undirected 2-regular graph, then H-COVER is polynomial time solvable.

Question: What about connected directed 2-in-2-out-regular graphs?

Covering directed graphs

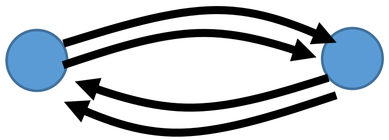
Observation: If H is connected undirected 2-regular graph, then H-COVER is polynomial time solvable.

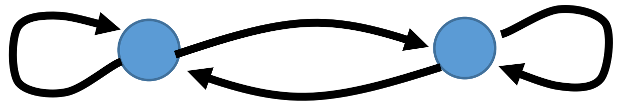
Question: What about connected directed 2-in-2-out-regular graphs?

Answer: A complete jungle.

Covering directed 2-in-2-out regular graphs

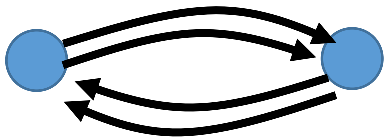
2-vertex graphs



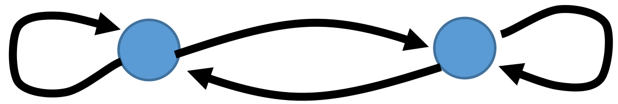


Covering directed 2-in-2-out regular graphs

2-vertex graphs



Polynomial time



Covering directed 2-in-2-out regular graphs

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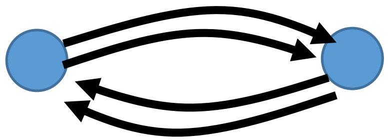

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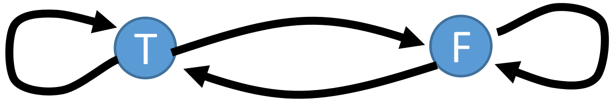
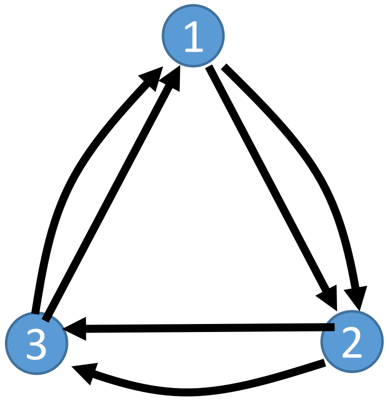


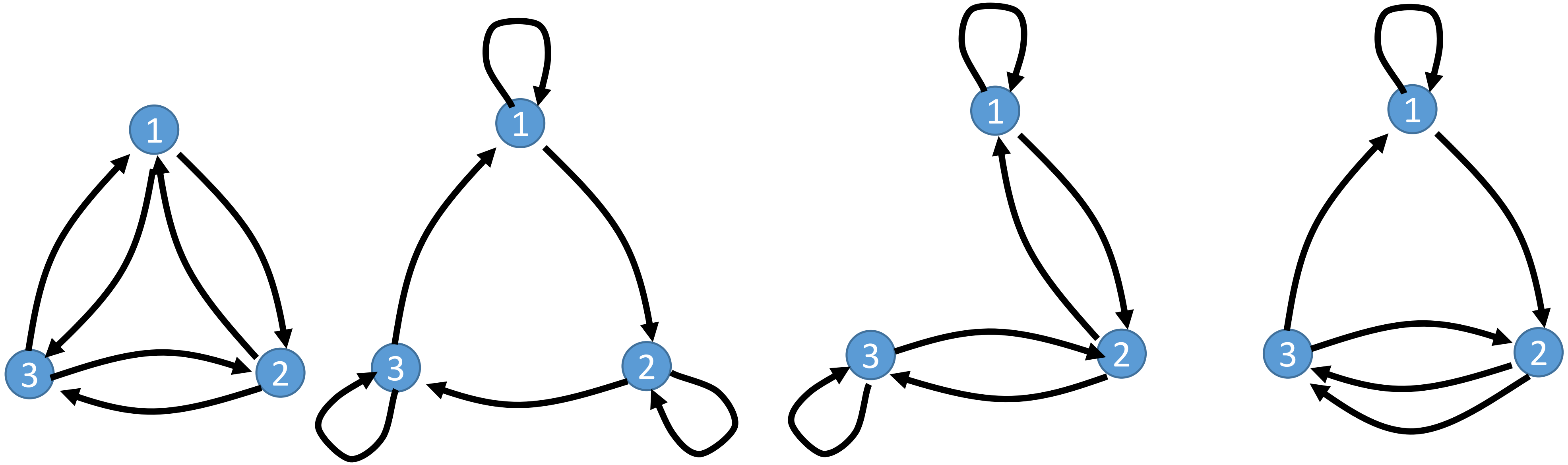
Polynomial time
via 2-SAT

Covering directed 2-in-2-out regular graphs

3-vertex graphs

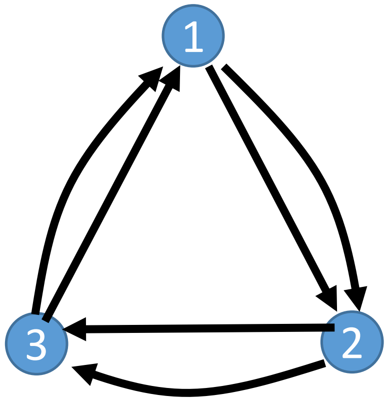


Polynomial time

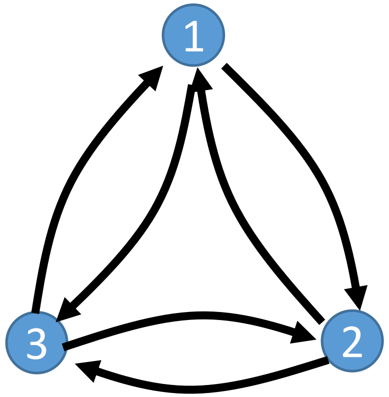


Covering directed 2-in-2-out regular graphs

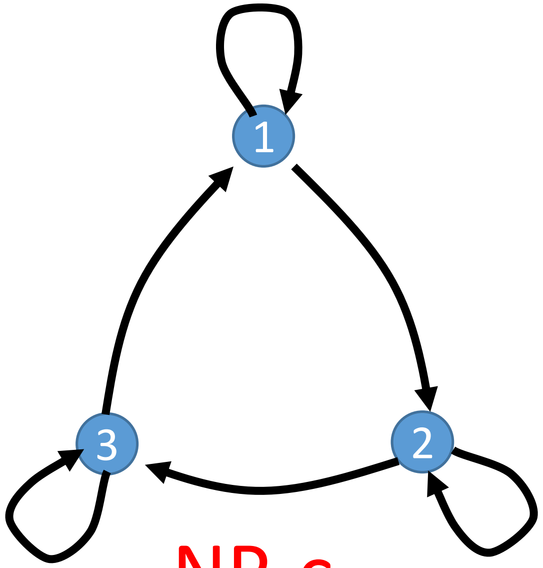
3-vertex graphs



Polynomial time



NP-c



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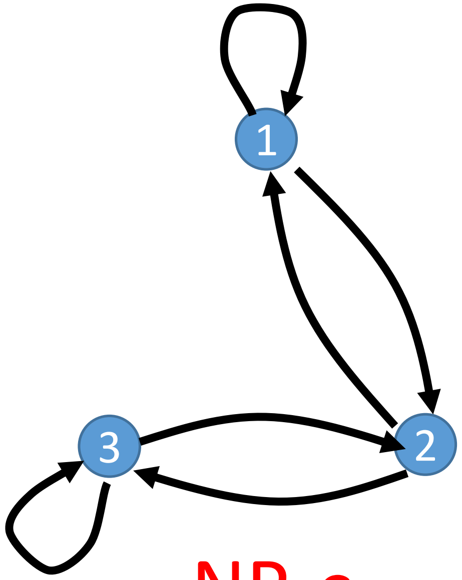

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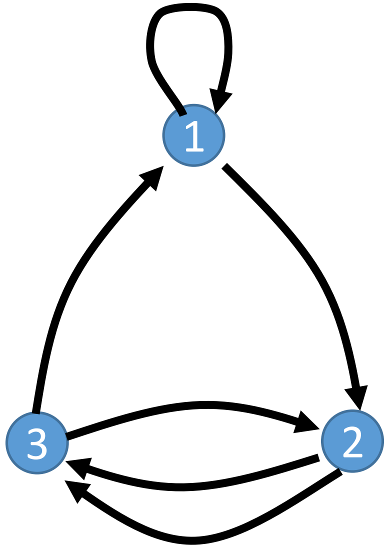
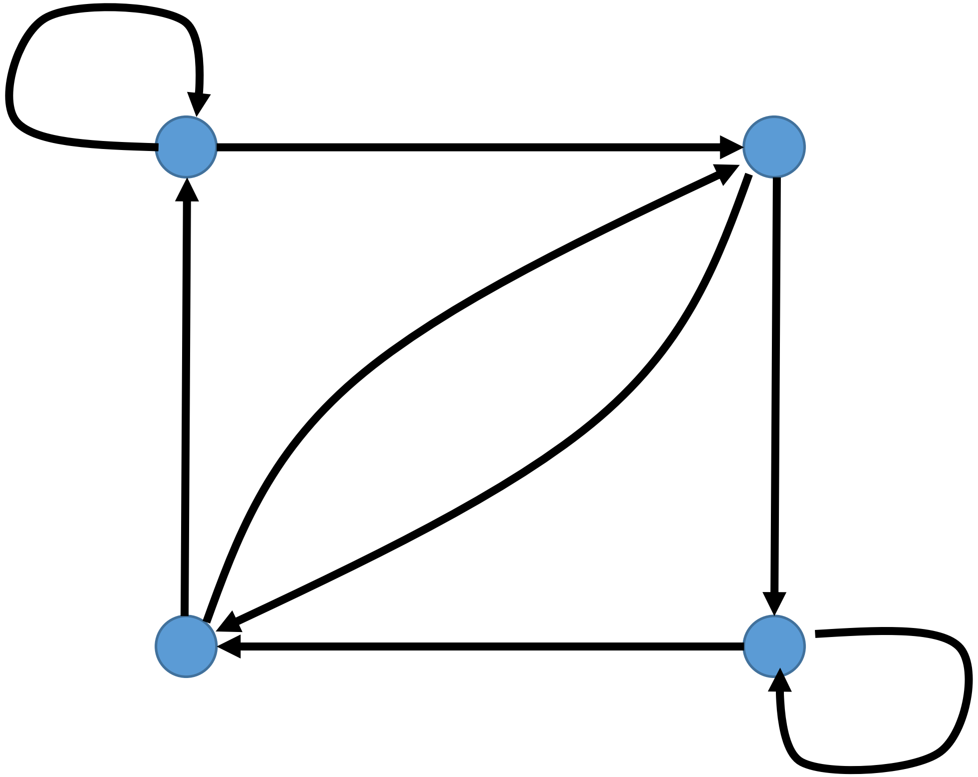


NP-c

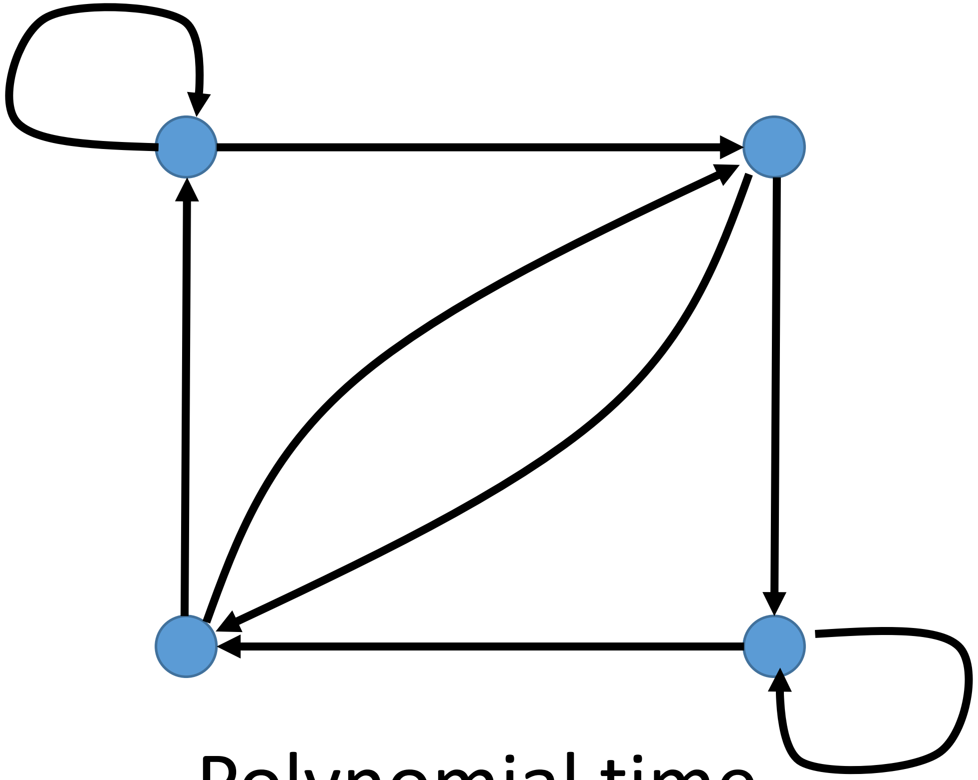
Covering directed 2-in-2-out regular graphs

4-vertex graphs



Covering directed 2-in-2-out regular graphs

4-vertex graphs



Polynomial time

Covering directed 2-in-2-out regular graphs

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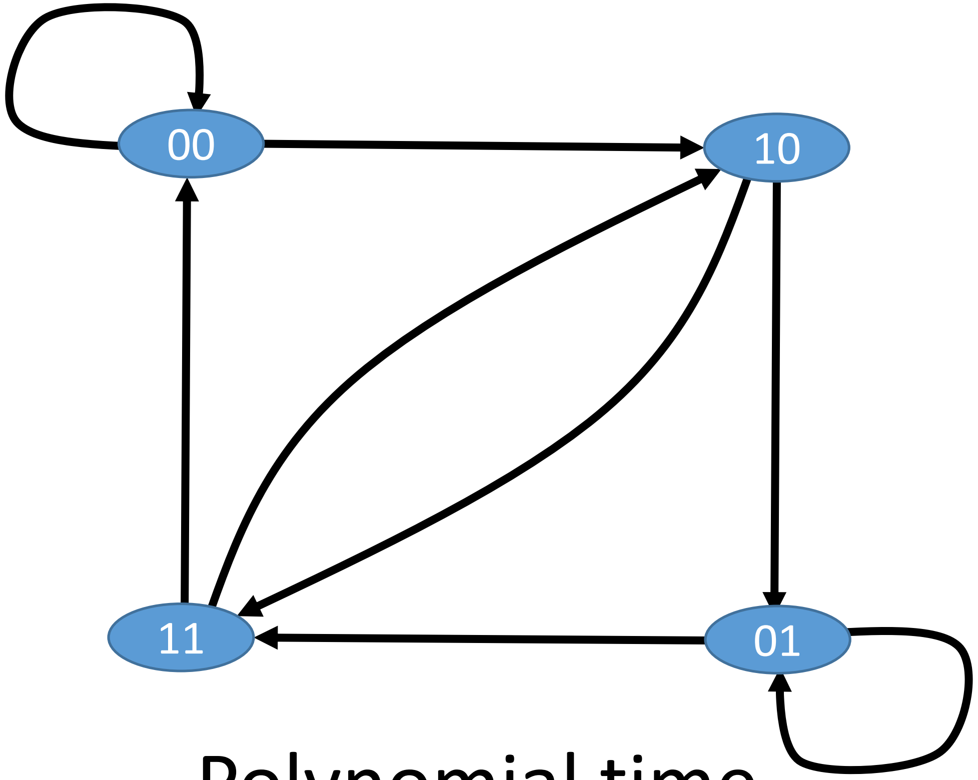
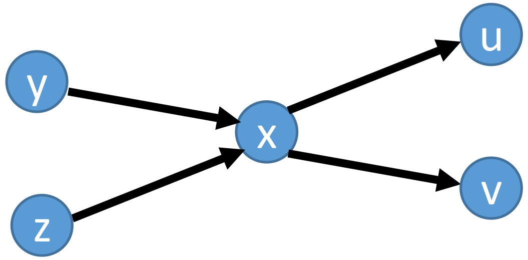


Polynomial time



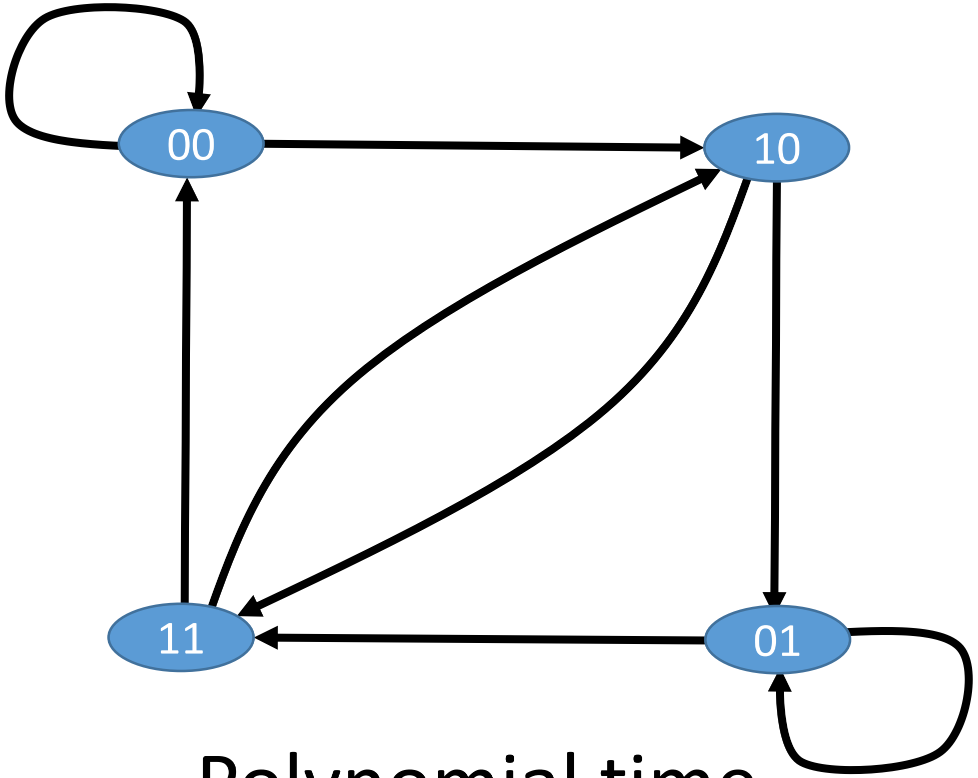
$u(2)+v(2)=0$

$u(1)+v(1)=1$

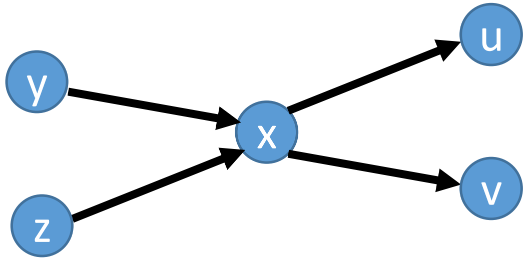
: $(x(1),x(2)) \in GF(2)^2$

Covering directed 2-in-2-out regular graphs

4-vertex graphs



Polynomial time



x : $(x(1),x(2)) \in GF(2)^2$

$u(2)+v(2) = 0$

$u(1)+v(1) = 1$

$x(1)+x(2)+u(2)=0$

$y(2)+z(2) = 1$

$y(1)+z(1) = 1$

$y(1)+y(2)+x(2) = 0$

Thank you