Complexity of the List Version of Graph Covers

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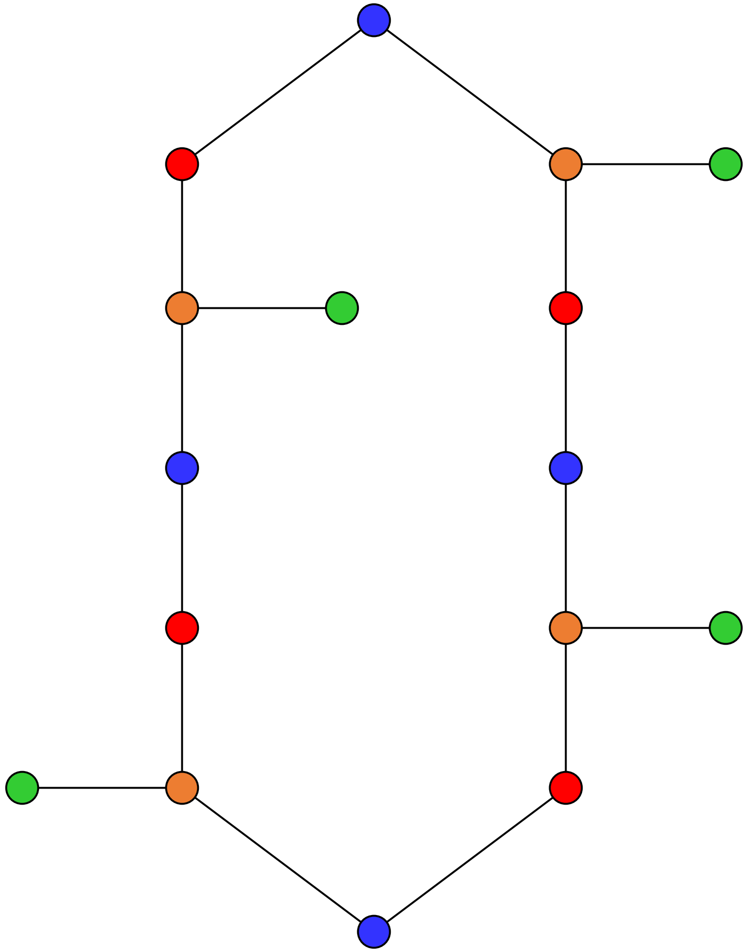
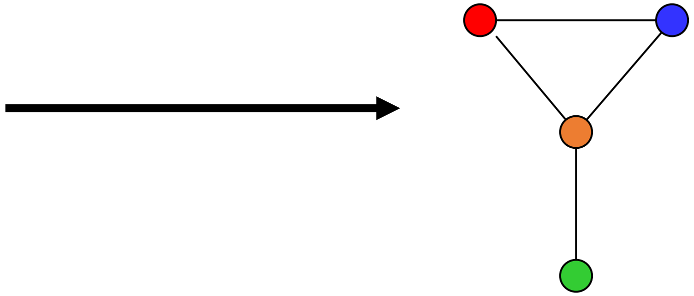
Computational complexity of graph covers

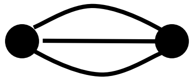


H-COVER
Input: A graph *G*
Question: Does *G* cover *H*?

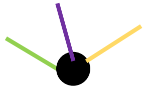


Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

= 3-edge-colorability of bipartite graphs

 -COVER is NP-complete

= 3-edge-colorability

Complexity of covering multigraphs

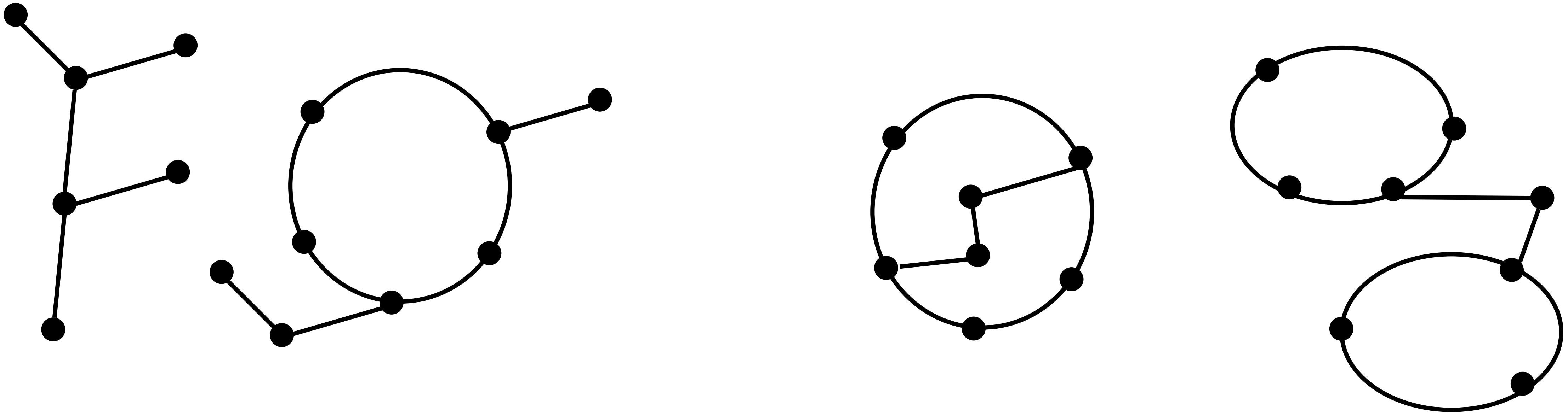
- Abello, Fellows, Stilwell 1991: Initiated the study of computational complexity of the *H*-COVER problem for fixed *H*.
- Kratochvil, Proskurowski, Telle 1997: Complete characterization of the computational complexity of *H*-COVER for colored mixed 2-vertex multigraphs *H (no semi-edges at that time).*
- Kratochvil, Telle, Tesař 2016: Complete characterization of the computational complexity of *H*-COVER for 3-vertex multigraphs *H.*
- Bok, Fiala, Hliněný, Kratochvíl MFCS 2021: First results on the computational complexity of *H*-COVER for (multi)graphs *with semi-edges*. Full classification for 1-vertex and 2-vertex graphs *H*.
- Semi-edges have been introduced in topological graph theory and are also widely used in mathematical physics. From now on *graph = multigraph with loops, multiple edges and semi-edges allowed.*

List covering problems

List-H-COVER
Input: A graph G, lists $L(u) \subseteq V(H)$ for $u \in V(G)$, $L(e) \subseteq V(H)$ for $e \in E(G)$.
Question: Does G allow a covering projection $f{:}G \to H$ such that $f(u) \in L(u)$ for every $u \in V(G)$ and $f(e) \in L(e)$ for every $e \in E(G)$?

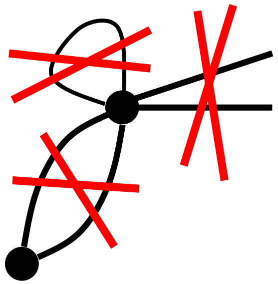
List covering problems

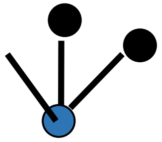
Partial cover (*locally injective homomorphism*) is a harder problem than graph cover, but a dichotomy has been proved for List-*H*-PartialCOVER [Fiala, Kratochvil WG 2006]



List covering problems

Theorem: If *H* is a *k*-regular graph, *k*≥3, with at least one *semi-simple vertex*, then List-*H*-COVER is NP-complete for simple input graphs.





semi-simple

Proof: Revisit the reduction for *k*-edge-colorable *k*-regular graphs from Kratochvil, Proskurowski, Telle [JCTB 1997].

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Covering Regular Graphs

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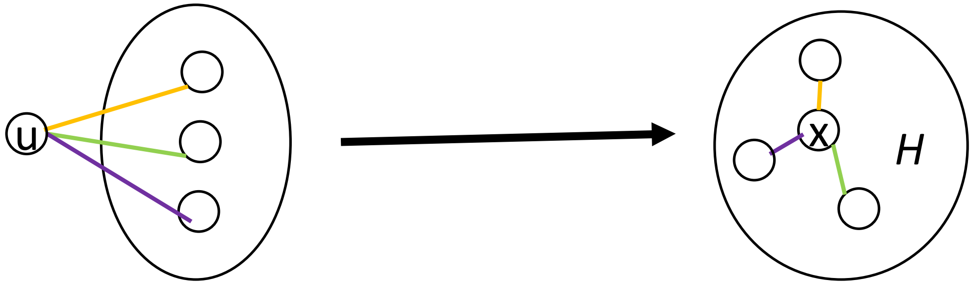
University of Bergen, Bergen, Norway

Received January 18, 1996

A covering projection from a graph G onto a graph H is a "local isomorphism": a mapping from the vertex set of G onto the vertex set of H such that, for every $v \in V(G)$, the neighborhood of v is mapped bijectively onto the neighborhood (in H) of the image of v. We investigate two concepts that concern graph covers of regular graphs. The first one is called "multicovers": we show that for any regular graph H there exists a graph G that allows many different covering projections onto H. Secondly, we consider *partial covers*, which require only that G be a subgraph of a cover of H. As an application of our results we show that there are infinitely many rigid regular graphs H for which the H-cover problem—deciding if a given graph G covers H—is NP-complete. This resolves an open problem related to the characterization of graphs H for which H-COVER is tractable.

List covering problems

A graph G is a *multicover* of H if it covers H in many ways, in the sense that G has a vertex u such that for every vertex x of H and for every bijective mapping of the edges of G incident with u to the edges of H incident with x, there is a covering projection $G \to H$ that extends this mapping.



List covering problems

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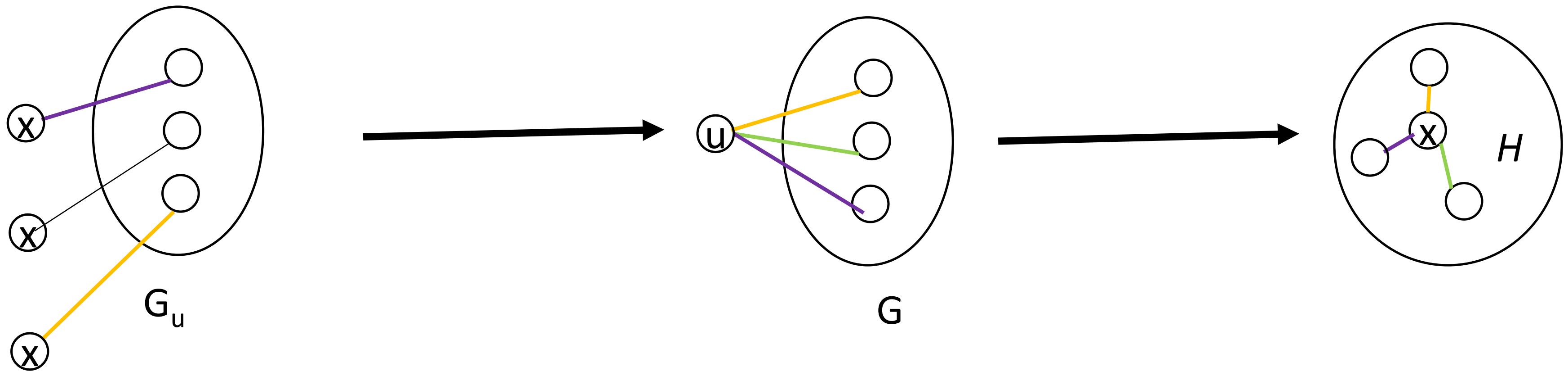
Lemma: Every H has a multicover.

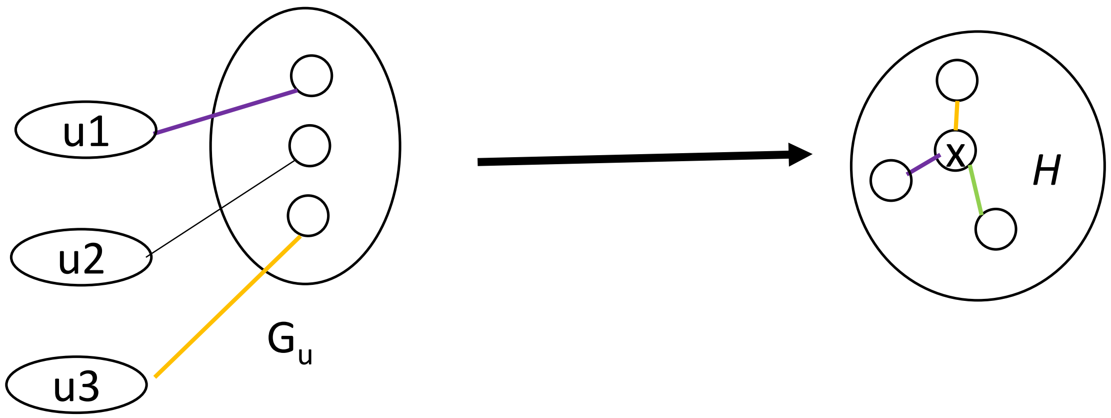


List covering problems

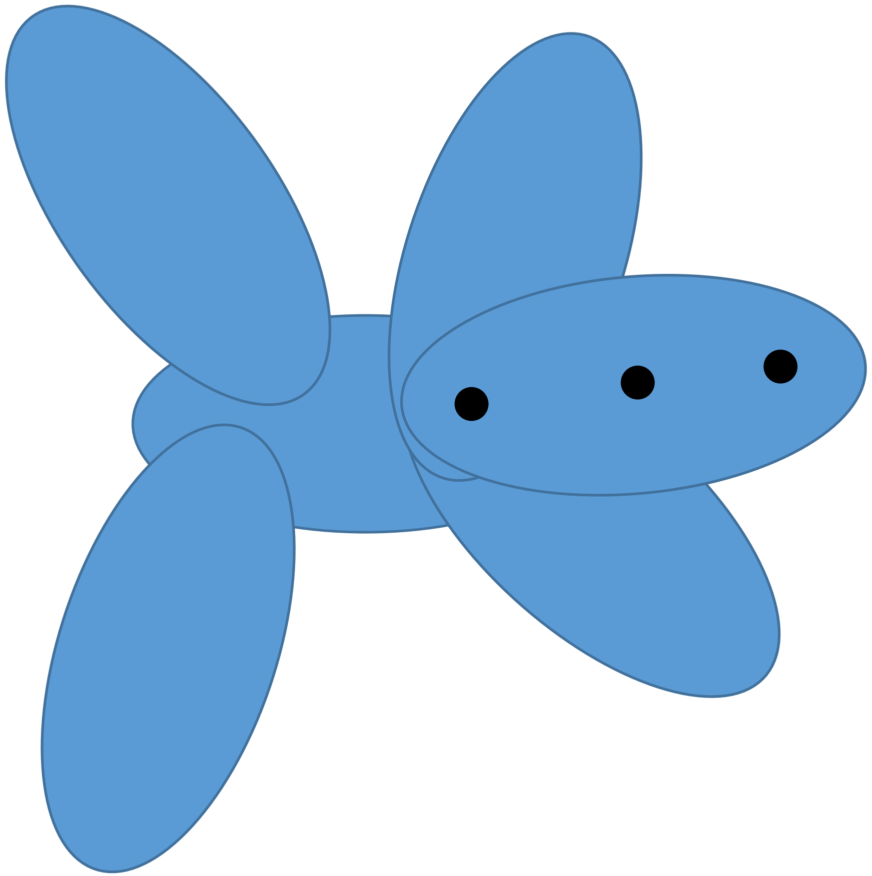
Lemma: Suppose a graph *G* covers *H*, and suppose the edges of *H* are properly colored by *k* colors by a coloring φ. Then for every partial covering projection $f{:}G_u \to H$, the following hold:

a) *f* is constant on the pendant vertices of G_u, i.e. $f(u1)=f(u2)=\ldots=f(uk)$,

b) if ψ is the coloring of edges of G_u obtained as $\psi(e)=\varphi(f(e))$, then the pendant edges of G_u are rainbow colored by ψ.



Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



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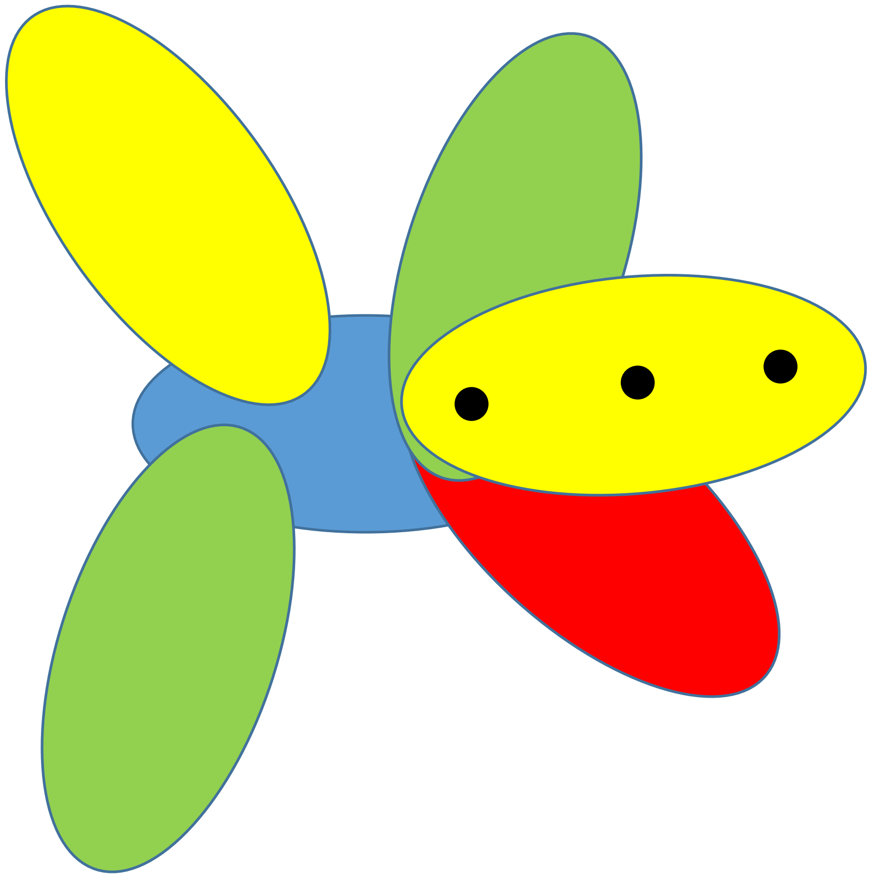
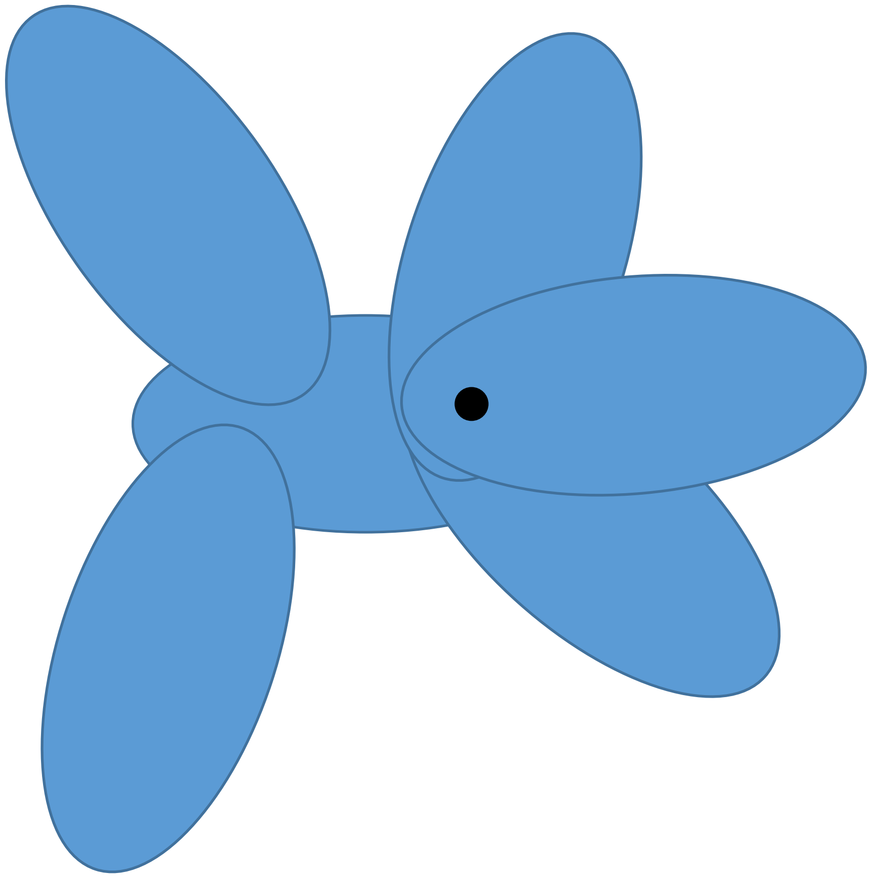
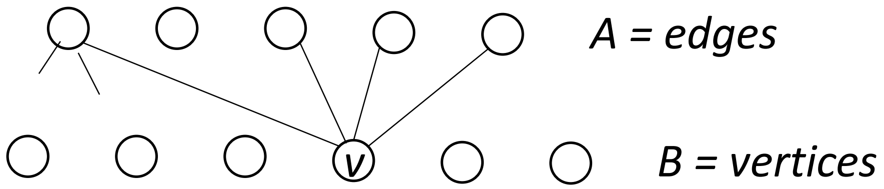


Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



Eq: Coloring vertices of one part of a (*k*,*k*-1)-regular bipartite graph by *k* colors so that every uncolored vertex has neighbors of all colors.



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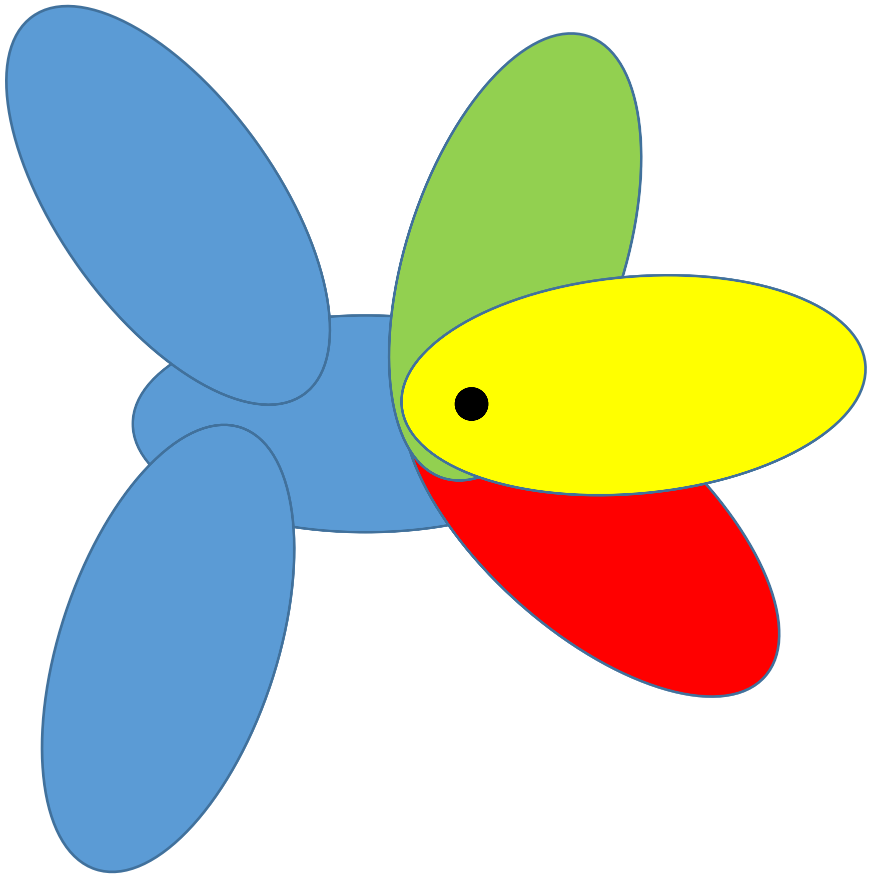

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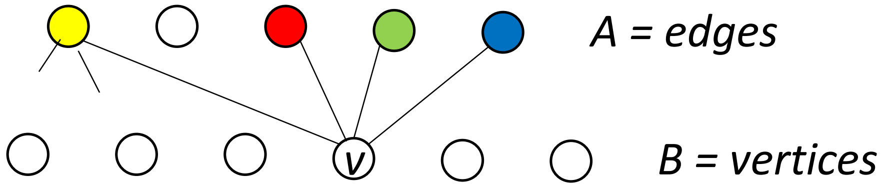
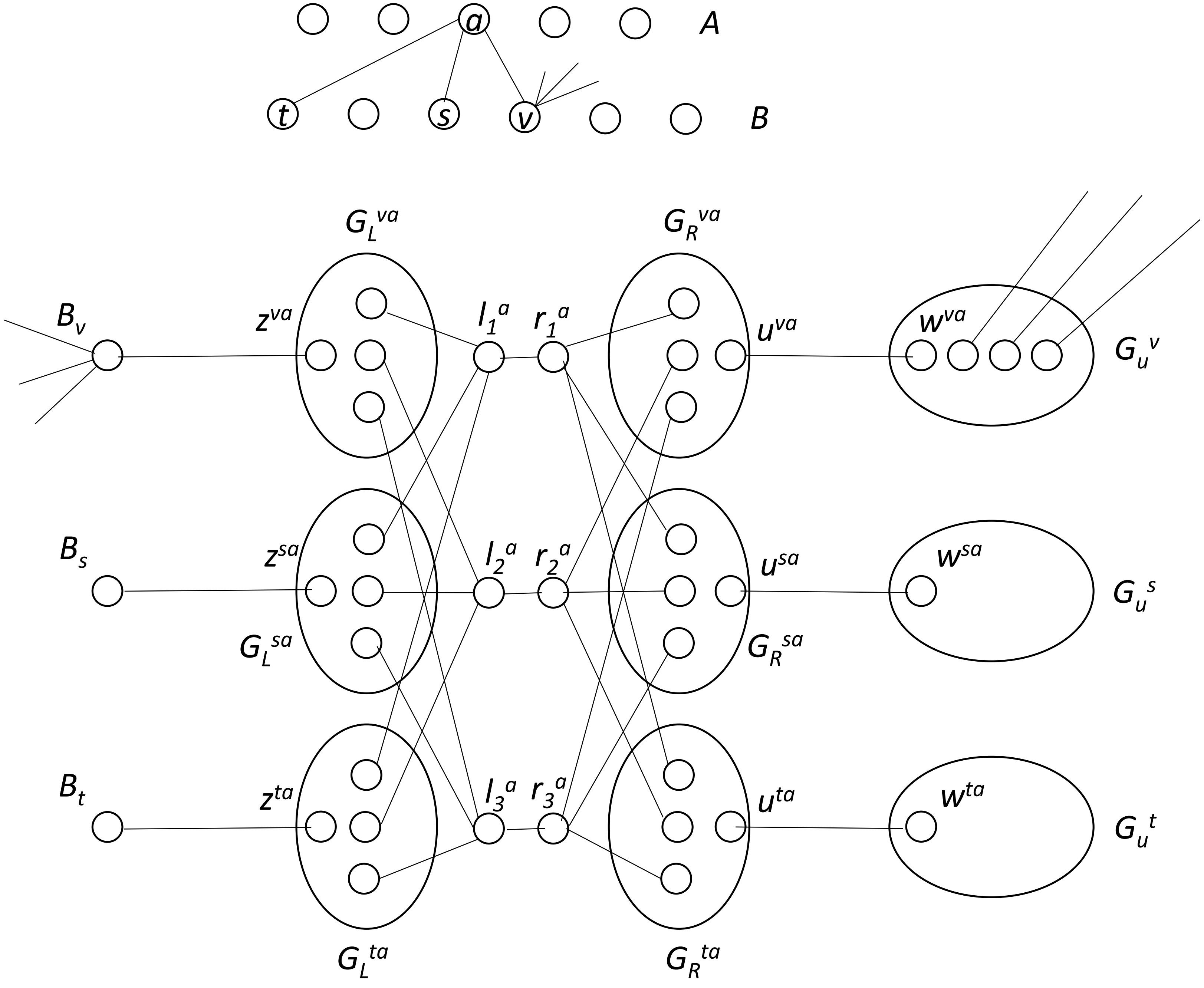
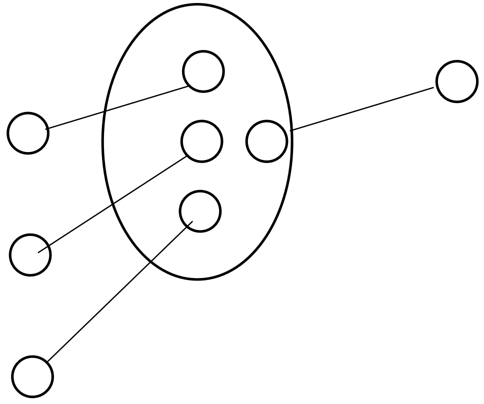
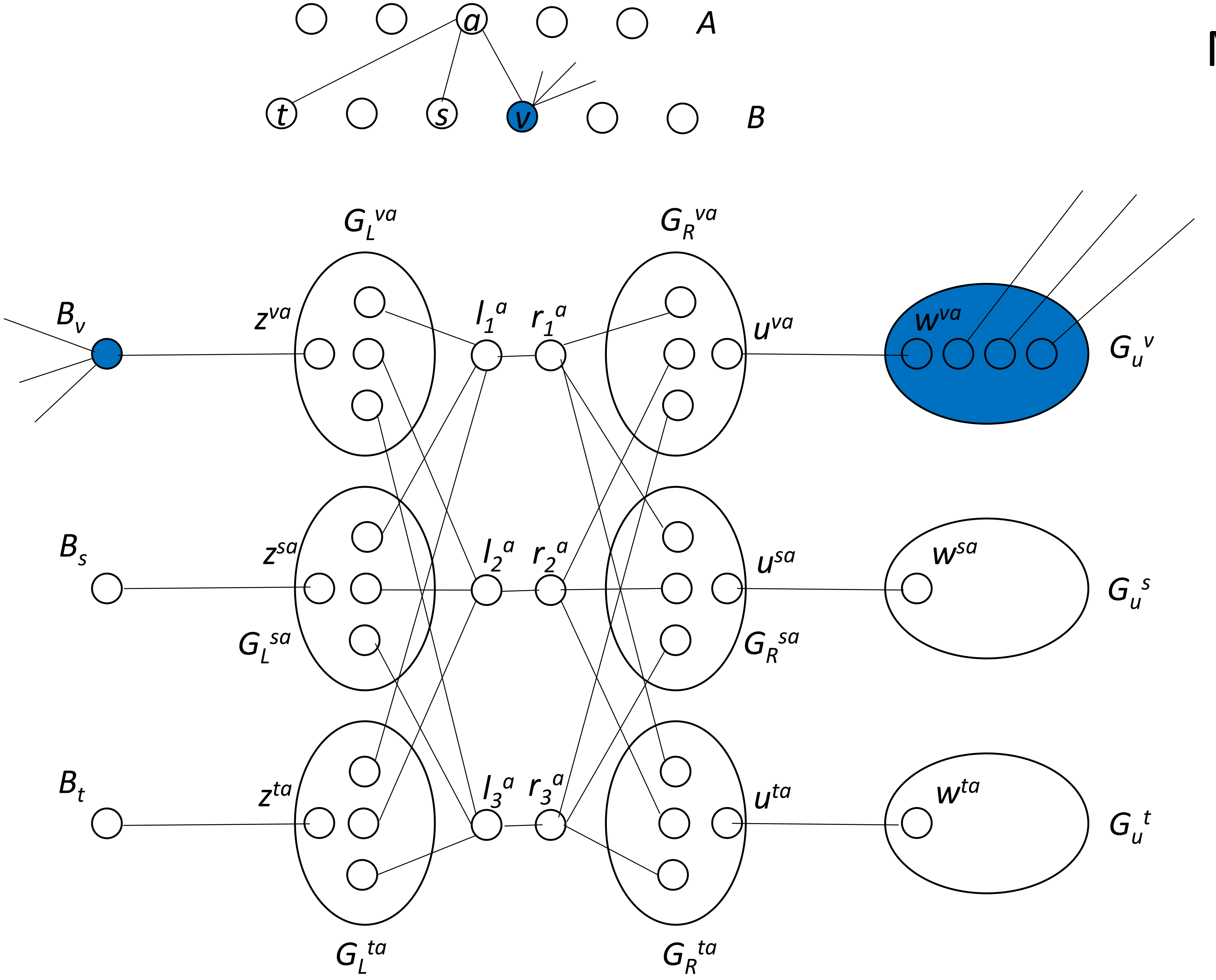


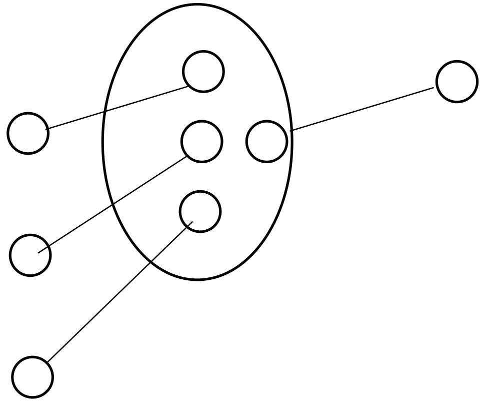


Multicover gadget:

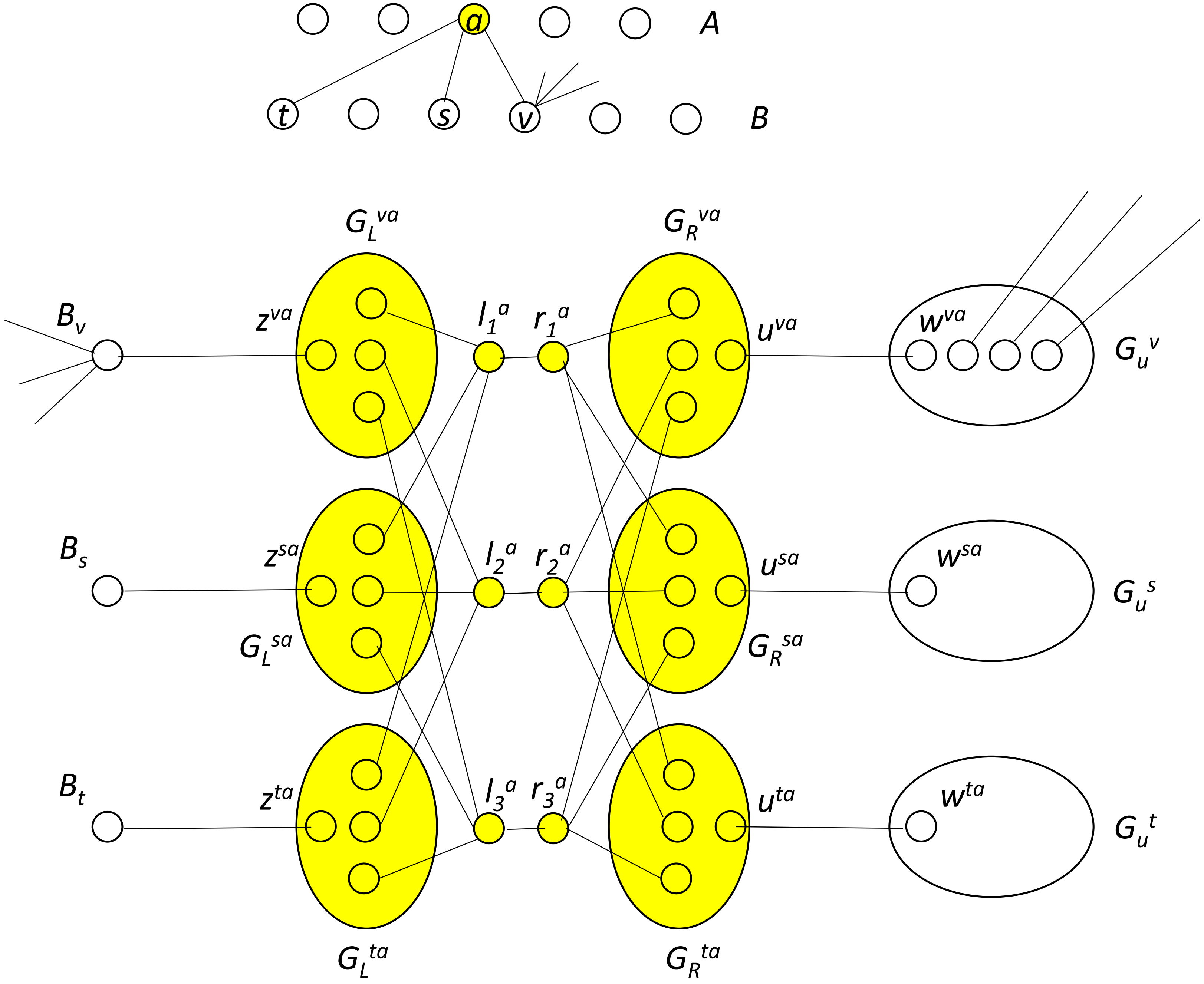


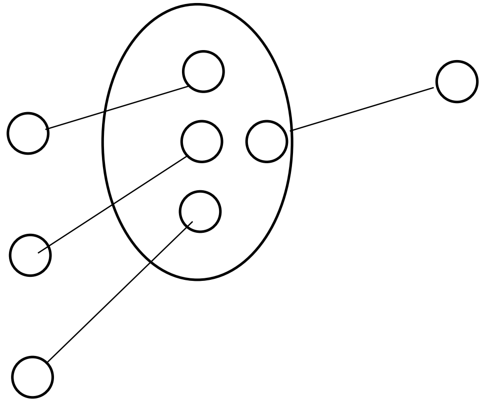
Multicover gadget:

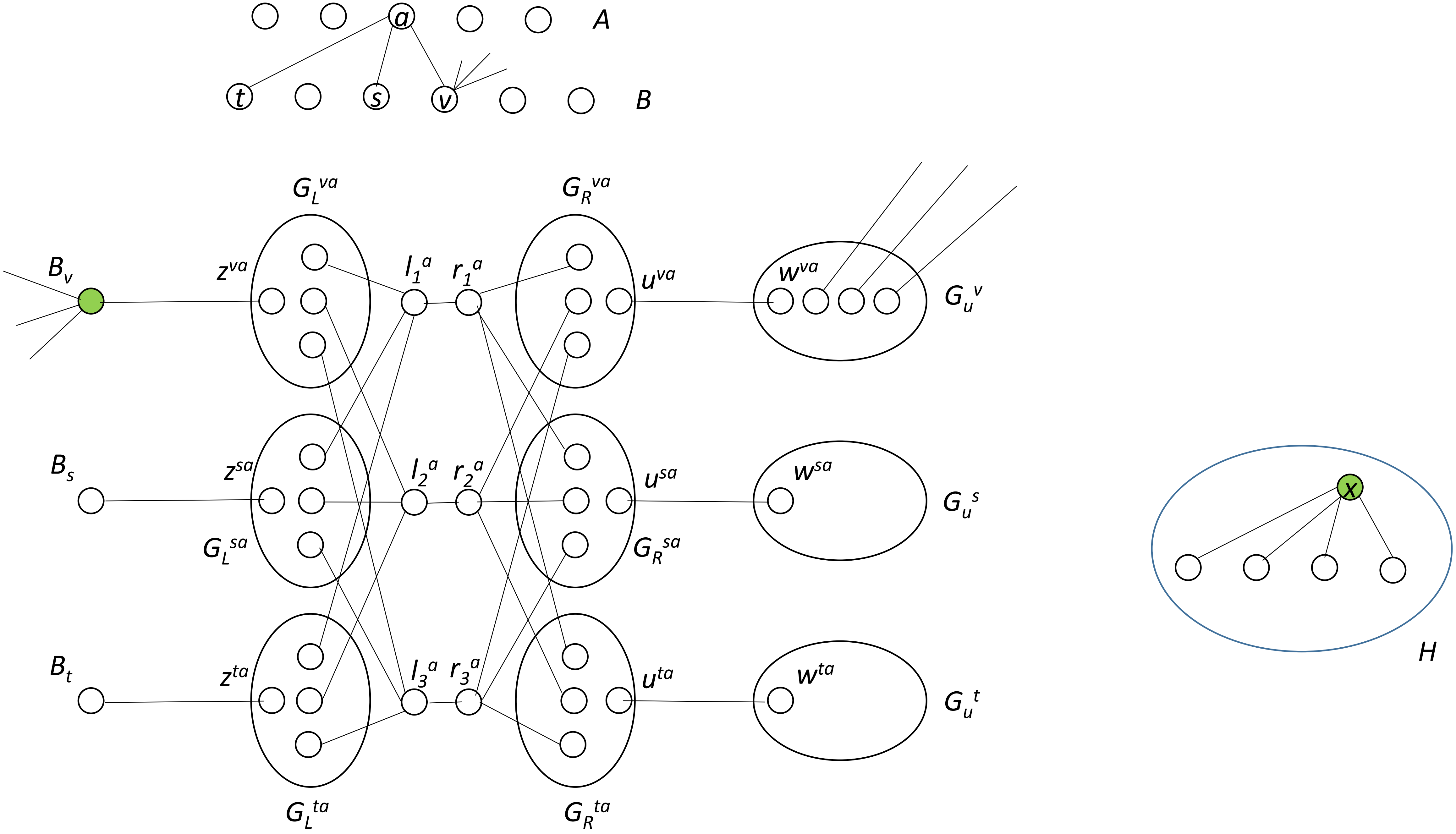




Multicover gadget:







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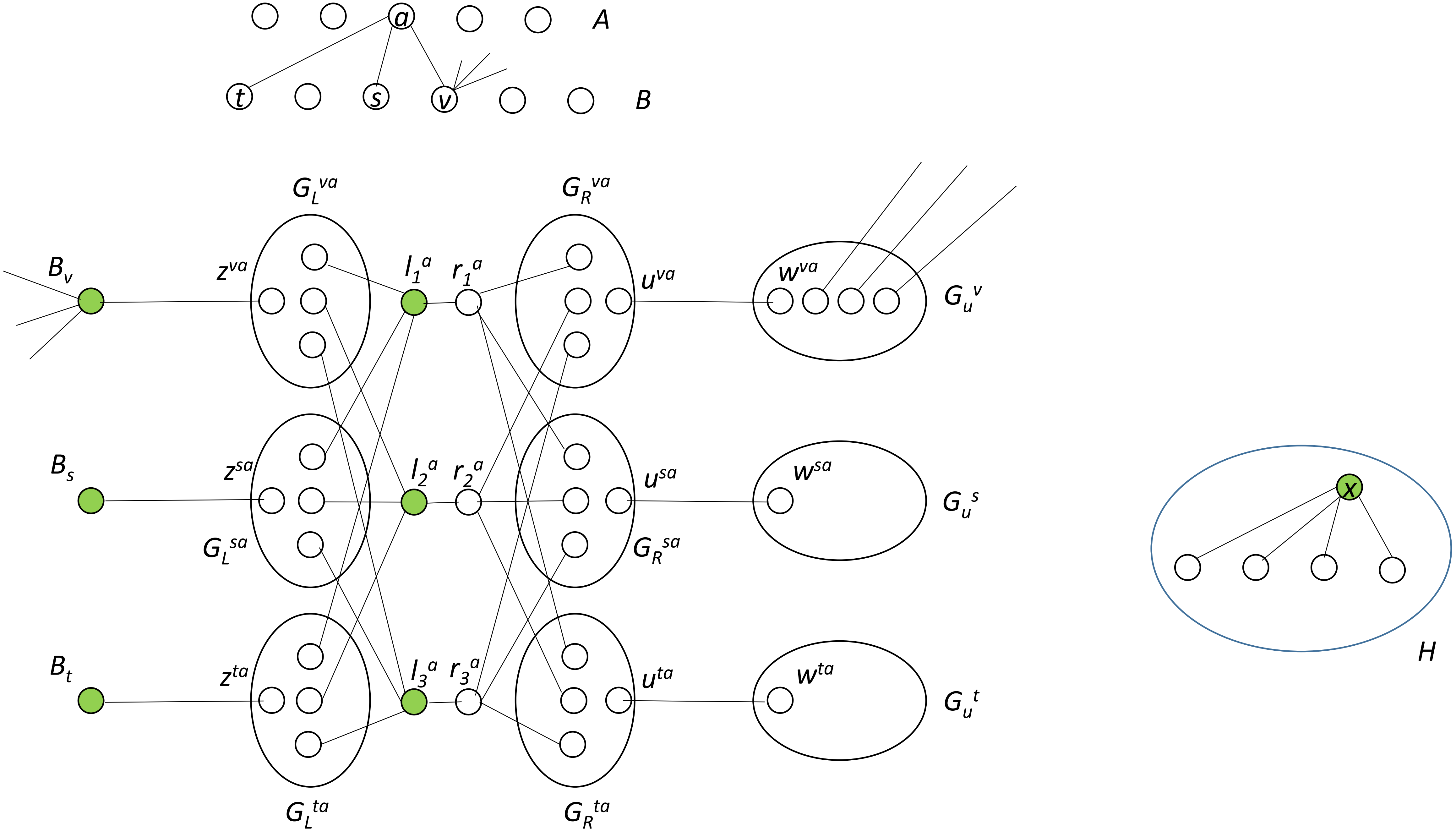

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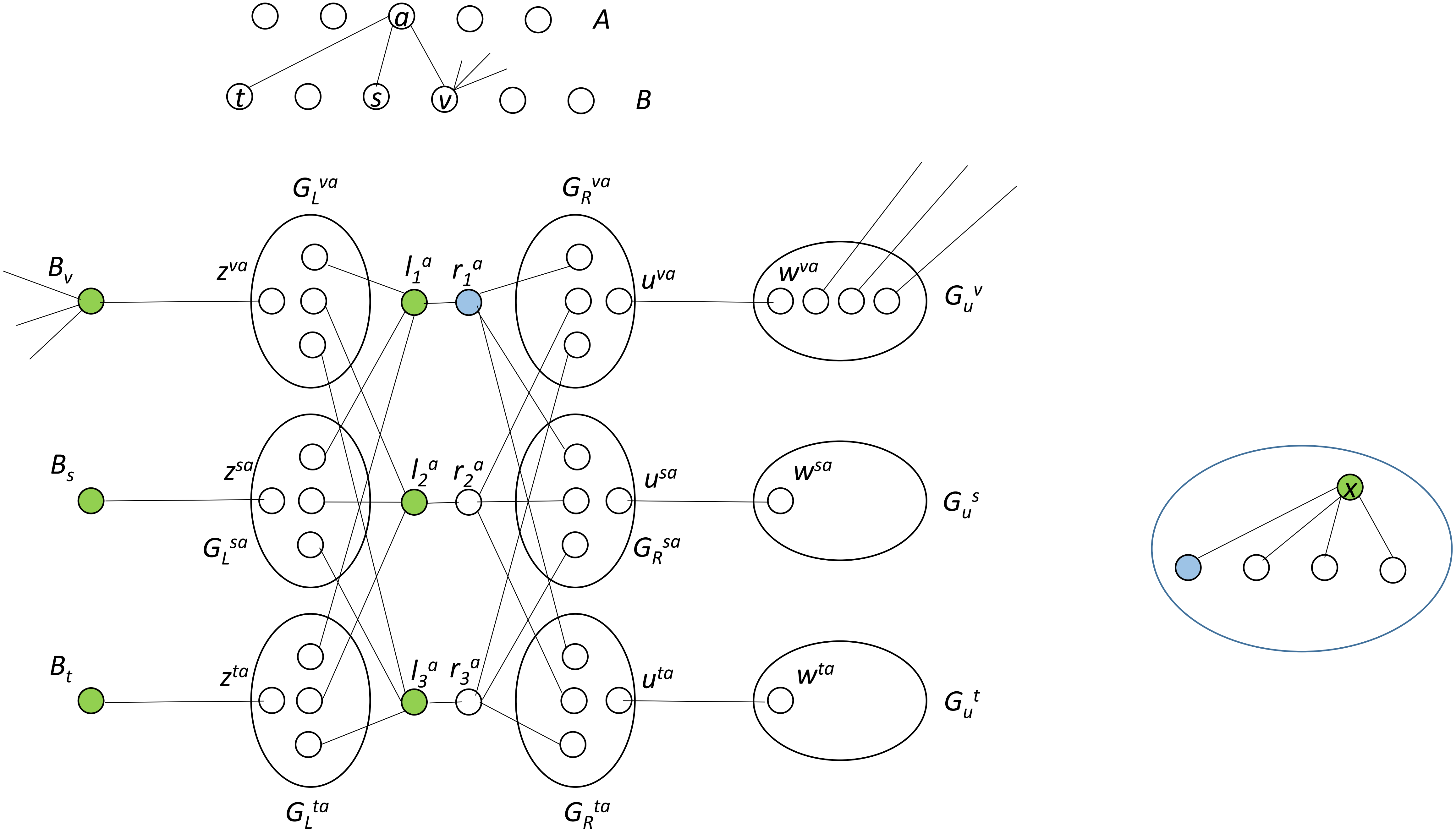
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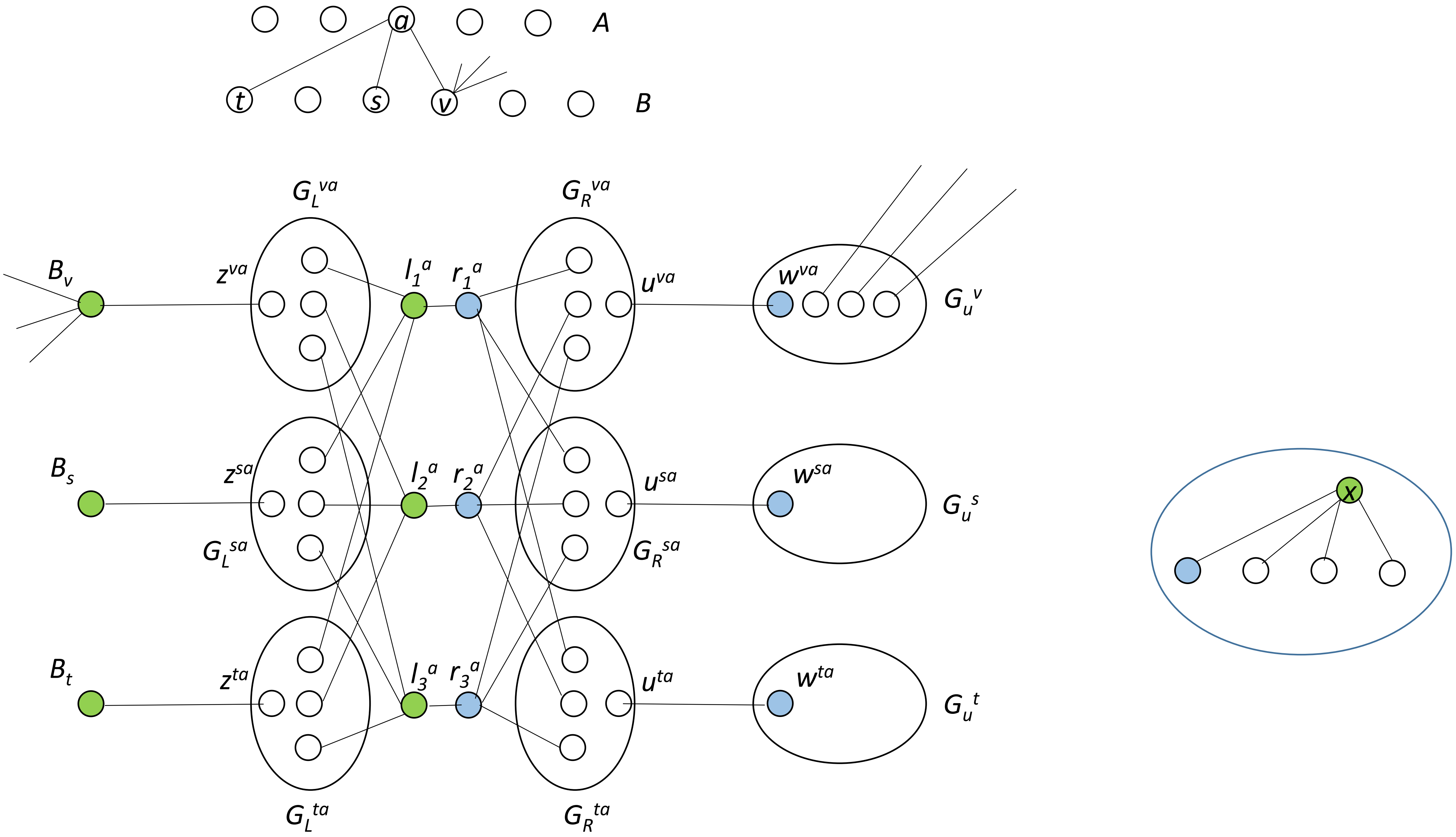
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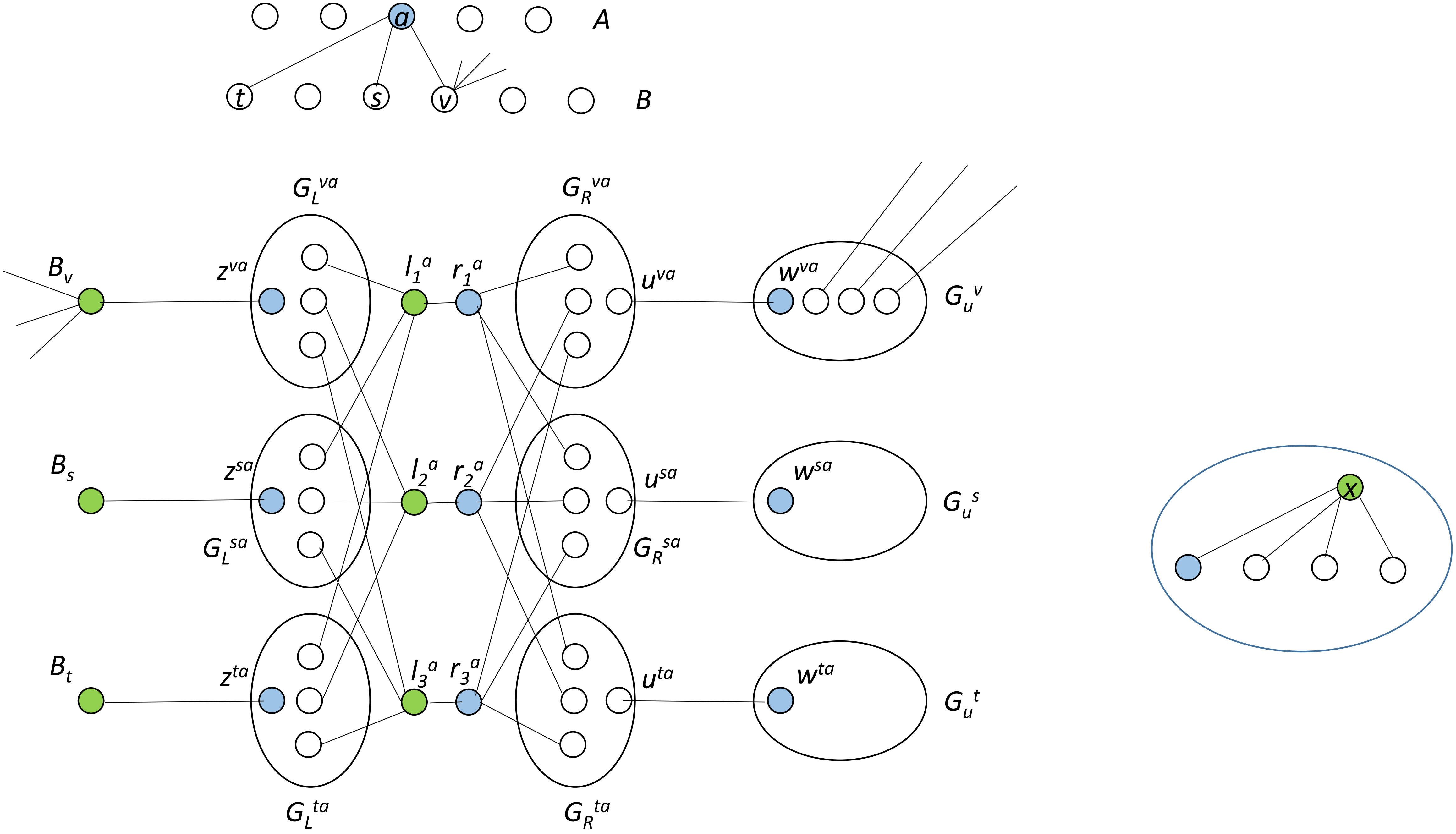

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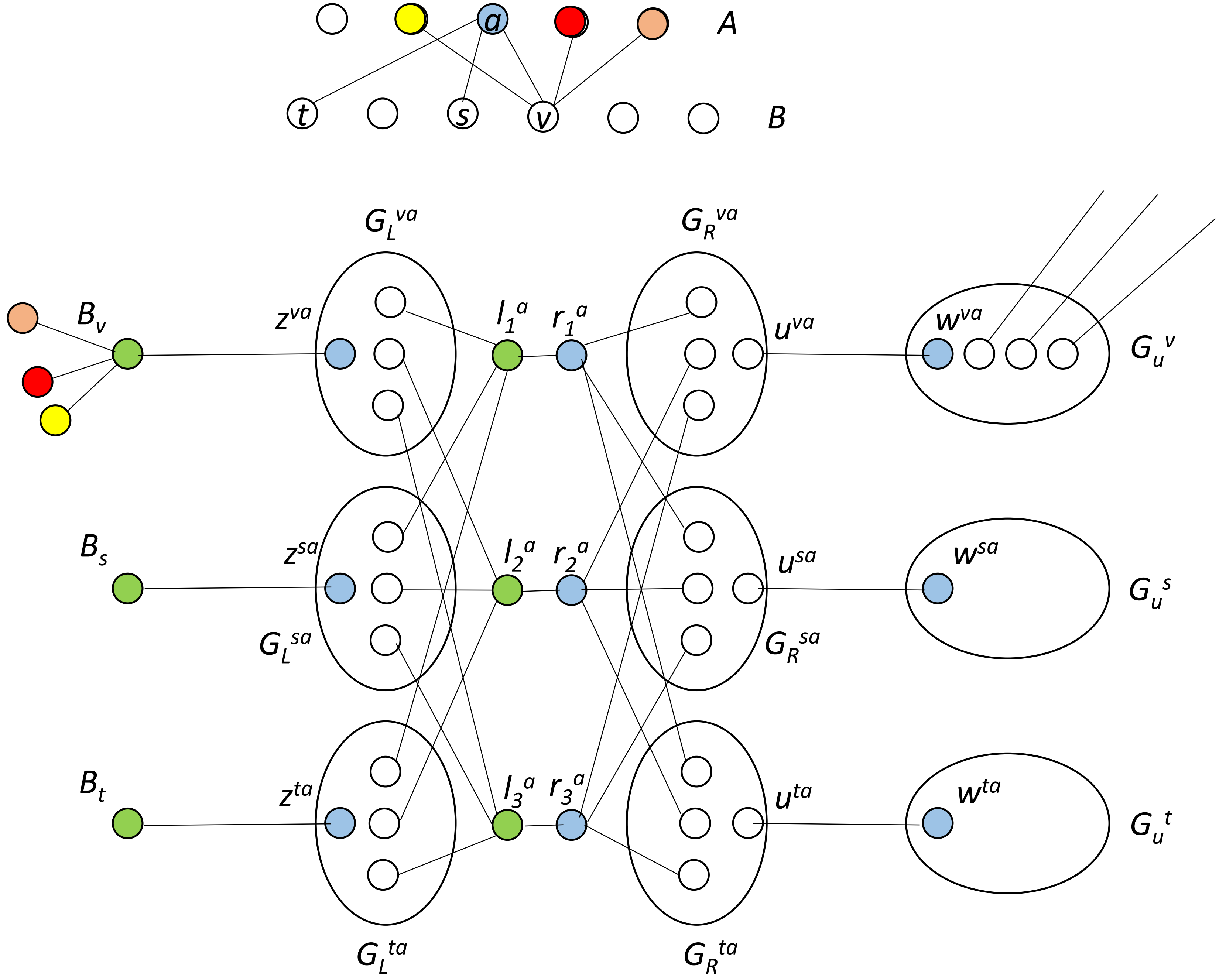
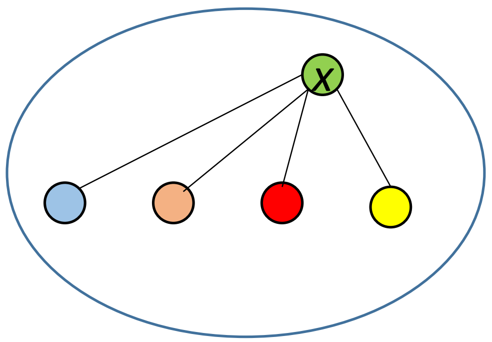
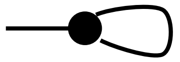
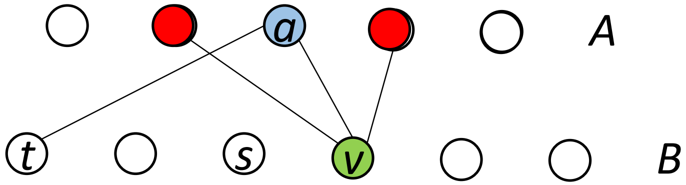
Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



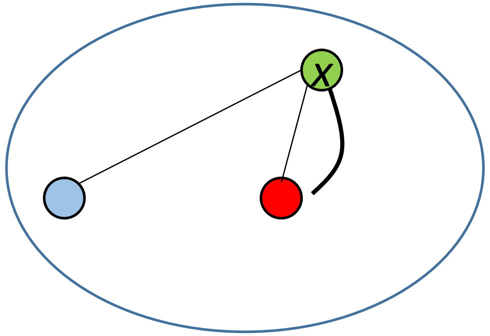


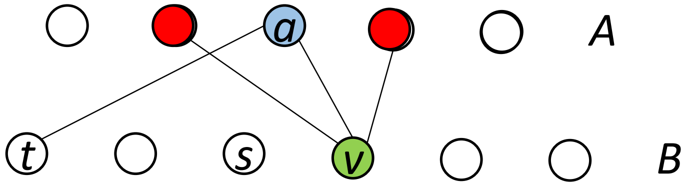
Multigraphs – what can go wrong?





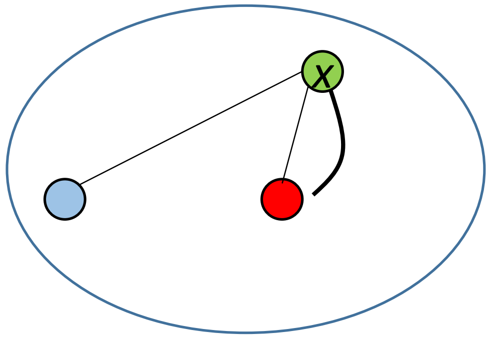
Reduction from 3-edge colorability of 3-regular 2-uniform hypergraphs.





Reduction from 3-edge colorability of 3-regular 2-uniform hypergraphs.

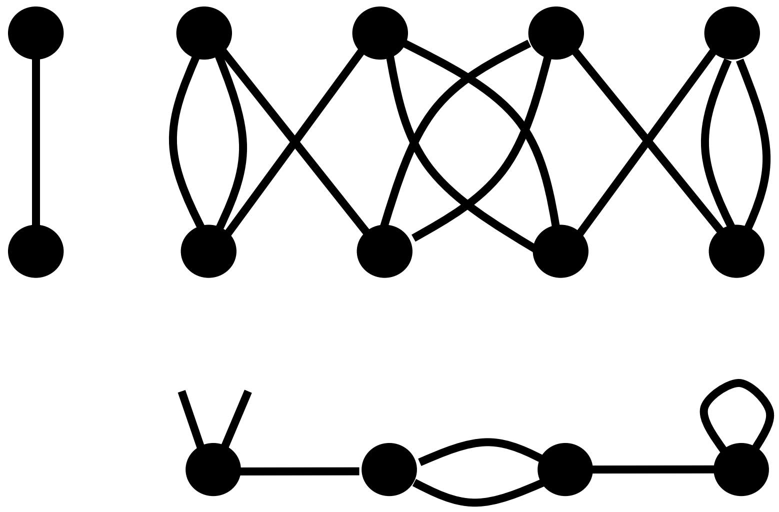
Perfect matching



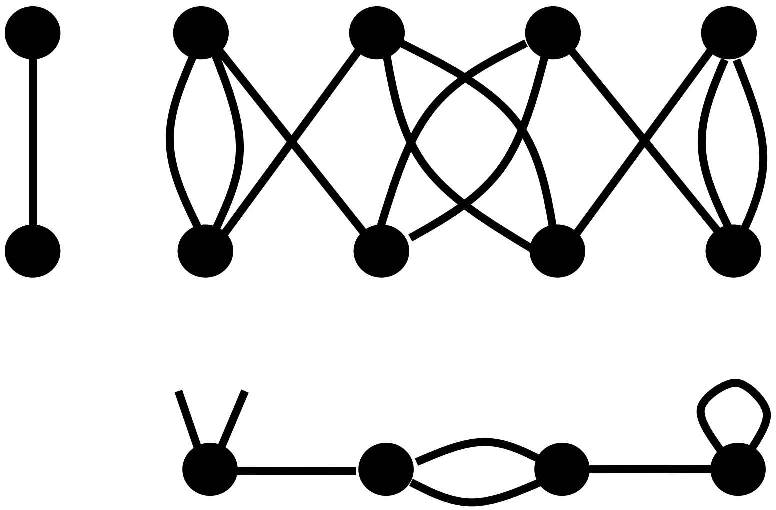
Multigraphs – what can be fixed?

Fiala trick: If H is not bipartite, then G covers HxK_2 iff G is bipartite and covers H.

Note: HxK_2 is bipartite and hence k-edge-colorable. And has no loops nor semi-edges, but may have multiple edges.

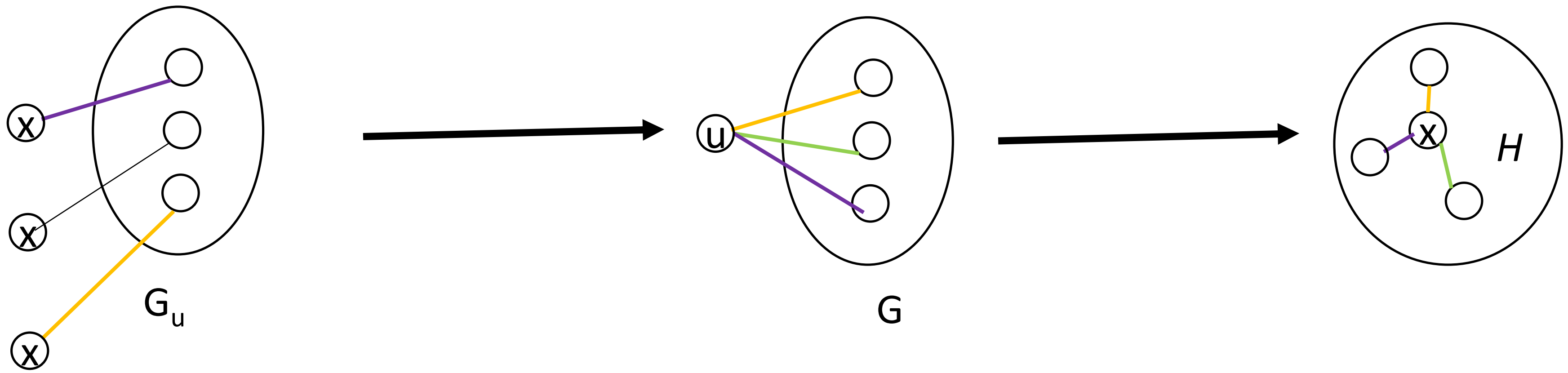


Observation: A semi-simple vertex in H becomes simple in $H\text{x}K_2$.

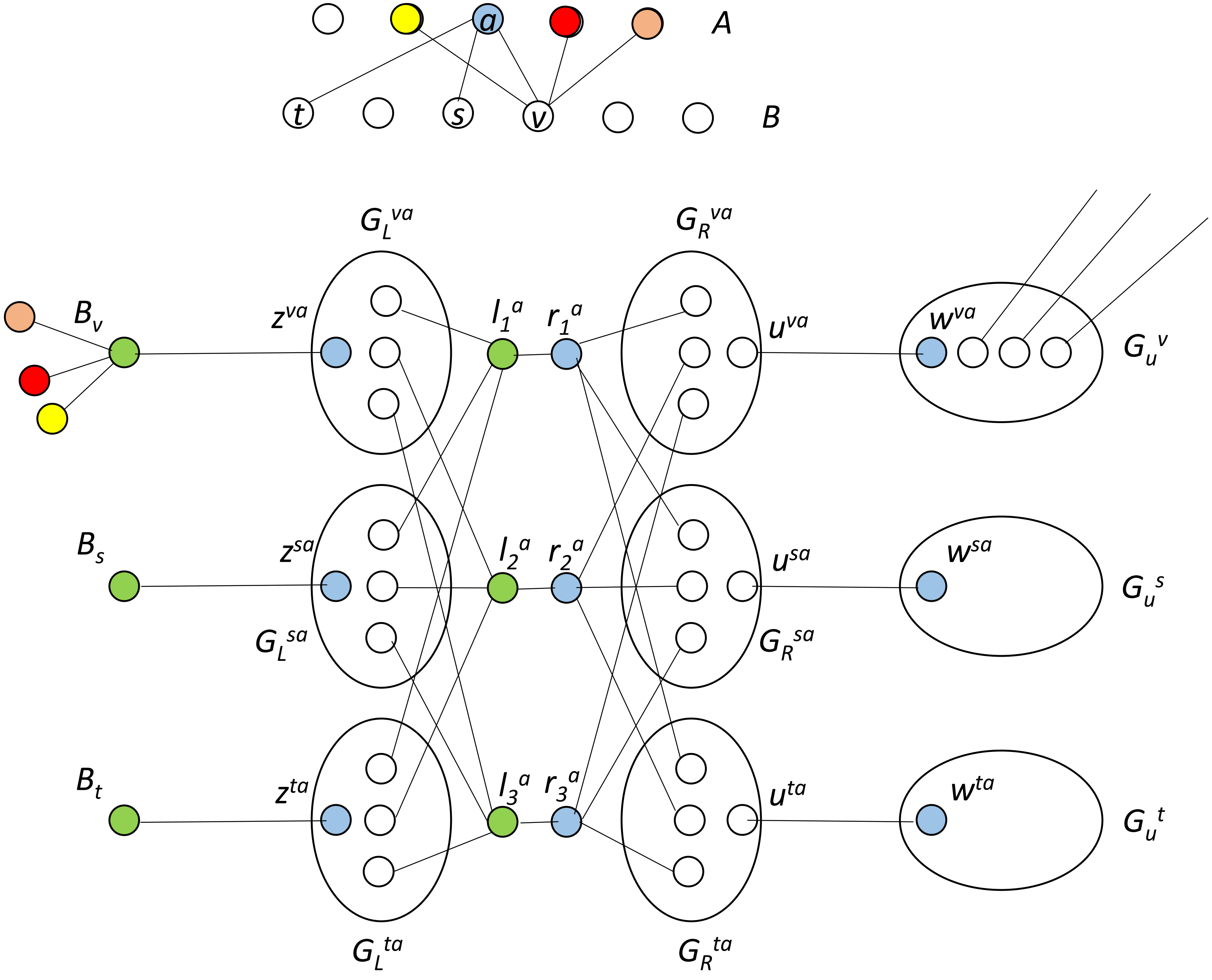


Multigraphs

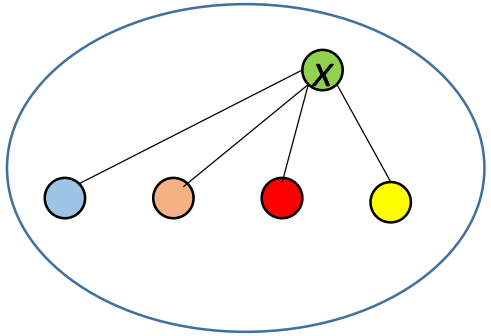
Lemma: Every multigraph *H* has a multicover which is a simple graph.



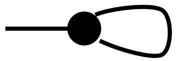
Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



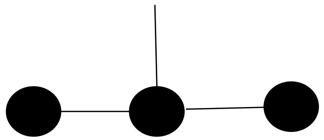
$L(B_v)=\{x\}$



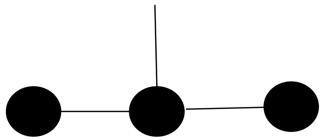
2. Strong Dichotomy for cubic graphs

Theorem (strong dichotomy for 3-regular target graphs): List-*H*-COVER is polynomial time solvable for  and NP-complete for all other target graphs *H*, even for simple inputs.

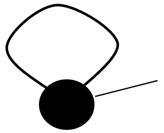
Strong Dichotomy for cubic graphs

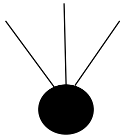
Case A: *H* has a vertex with 3 different neighbors  this is a semi-simple vertex and List-*H*-COVER is NP-complete for simple input graphs by the Theorem.

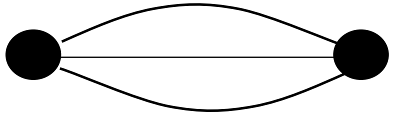
Strong Dichotomy for cubic graphs

Case A: *H* has a vertex with 3 different neighbors  this is a semi-simple vertex and List-*H*-COVER is NP-complete for simple input graphs by the Theorem.

Case B: H has a vertex whose all 3 neighbors are the same vertex

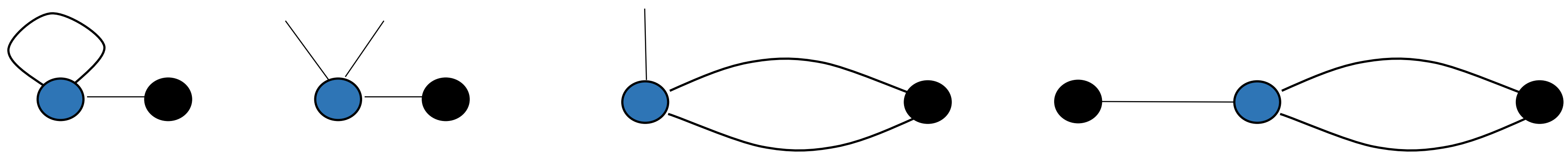
Case B1:  List-*H*-COVER is polynomial time solvable via perfect matching

Case B2:  *H*-COVER is NP-complete for simple inputs (3-edge-colorability)

Case B3:  List-*H*-COVER is NP-complete for simple inputs (via Precoloring extension for line graphs of cubic bipartite graphs, Fiala 1998)

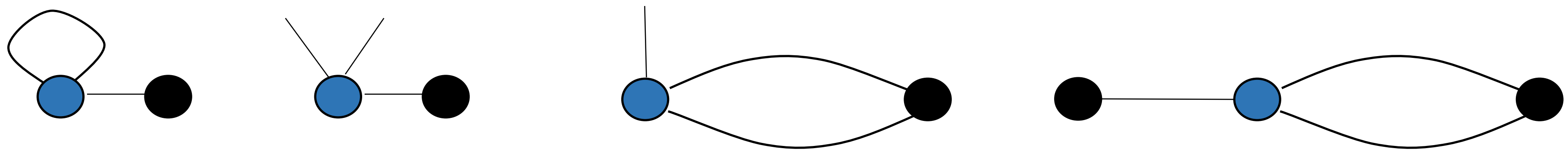
Strong Dichotomy for cubic graphs

Case C: Every vertex of *H* has exactly 2 neighbors, one adjacent via a double edge or via a loop or via 2 semi-edges, and the other one via a single edge or via a semi-edge.

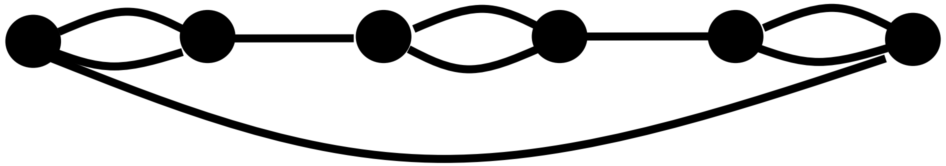


Strong Dichotomy for cubic graphs

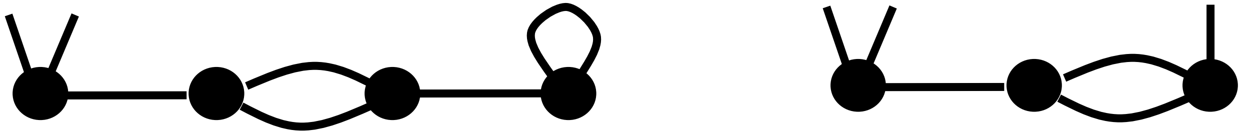
Case C: Every vertex of *H* has exactly 2 neighbors, one adjacent via a double edge or via a loop or via 2 semi-edges, and the other one via a single edge or via a semi-edge.



Case C1: *H* is a ring



Case C2: *H* is a sausage graph



Research questions

Problem 1: Full characterization and strong dichotomy for List-*H*-COVER for *k*-regular target graphs *H* for $k \geq 4$?

Problem 2: Can we do without lists?

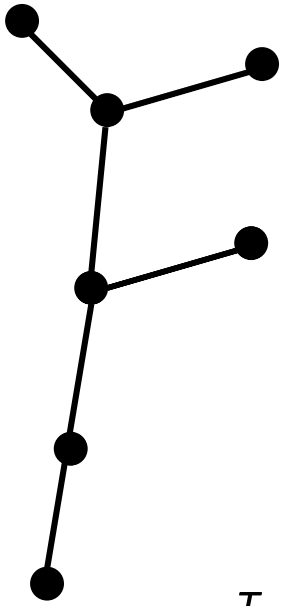
Problem 3: Can we do without semi-simple vertices?

Conjecture: Let *H* be a connected k-regular graph (loops, multiple edges and semi-edges allowed), with $k \geq 3$. Then both *H*-COVER and List-*H*-COVER are polynomial time solvable if *H* is a single-vertex graph with at most one semi-edge, *H*-COVER is solvable in polynomial time if H is a two-vertex graph with *k* parallel edges between its vertices, and both problems are NP-complete for simple input graphs otherwise.

One partial result

Sometimes we can do without lists.

Theorem (BFJK 2023+): Let *T* be a tree of max degree $d \geq 3$, and let *T*’ be the *d*-regular graph obtained from *T* by adding semi-edges. Then *T*’-COVER is NP-complete for simple input graphs.



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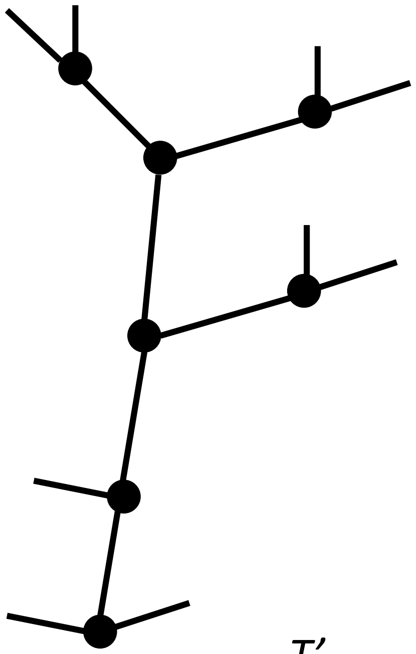

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