Computational Complexity of Graph Covers – The Role of Cycles, Colours, and Lists

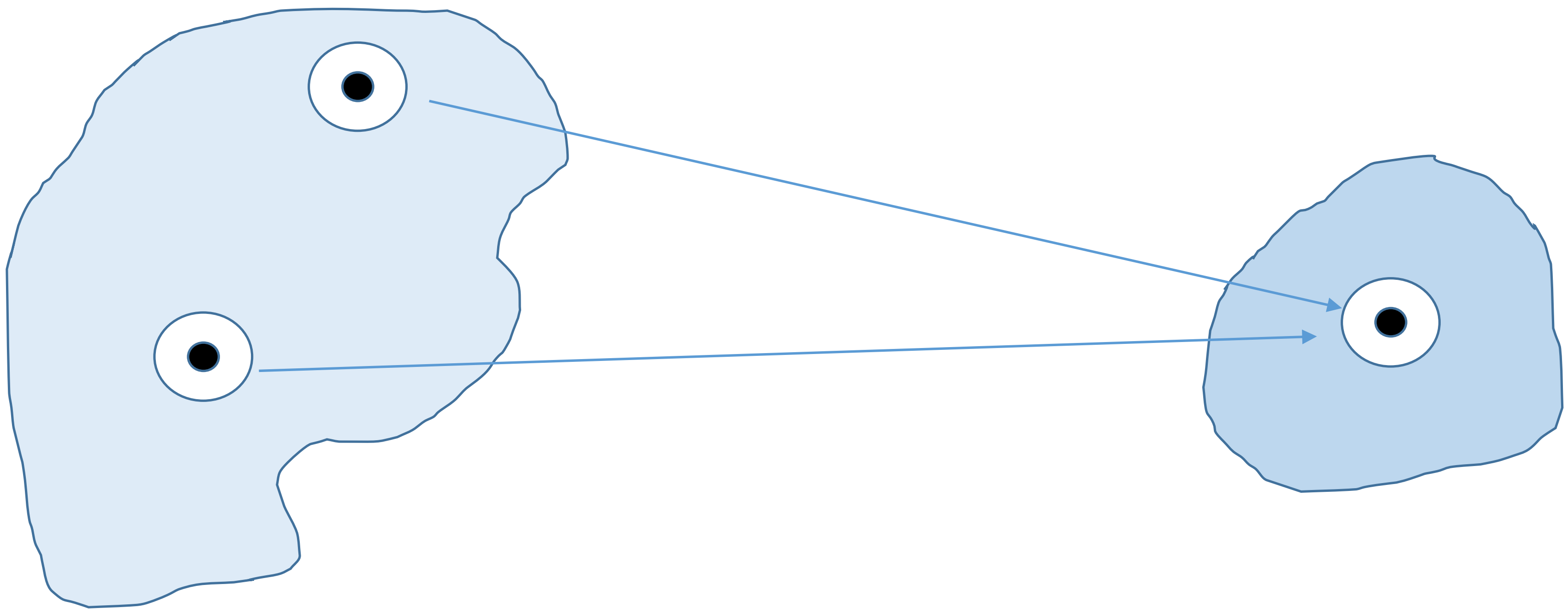
Jan Kratochvíl, Charles University, Prague

(based on joint work with J. Bok, J. Fiala, P. Hliněný, N. Jedličková, P. Rzazewski, and M. Seifrtová)

Cycles & Colourings 2022

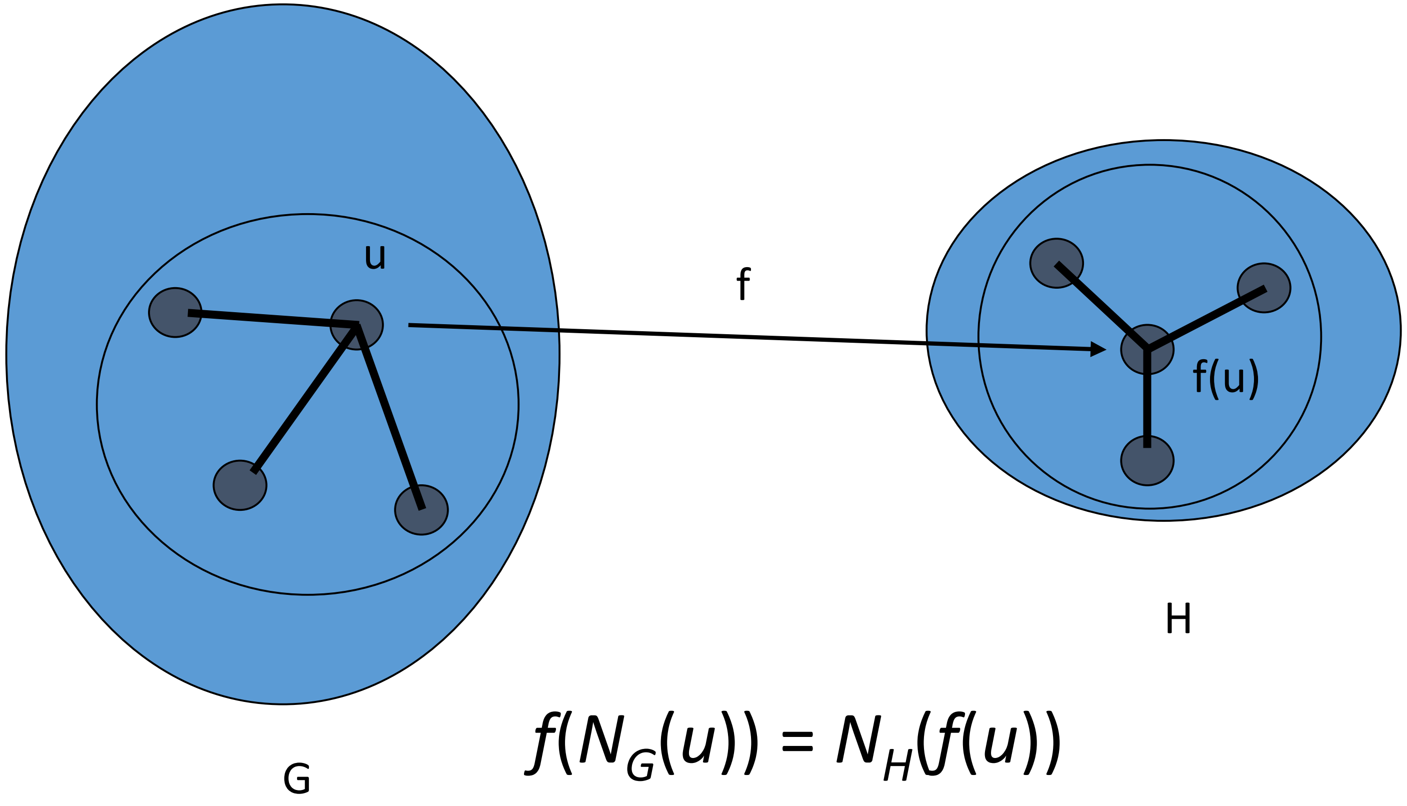
Novy Smokovec, September 05, 2022

Motivation from topology

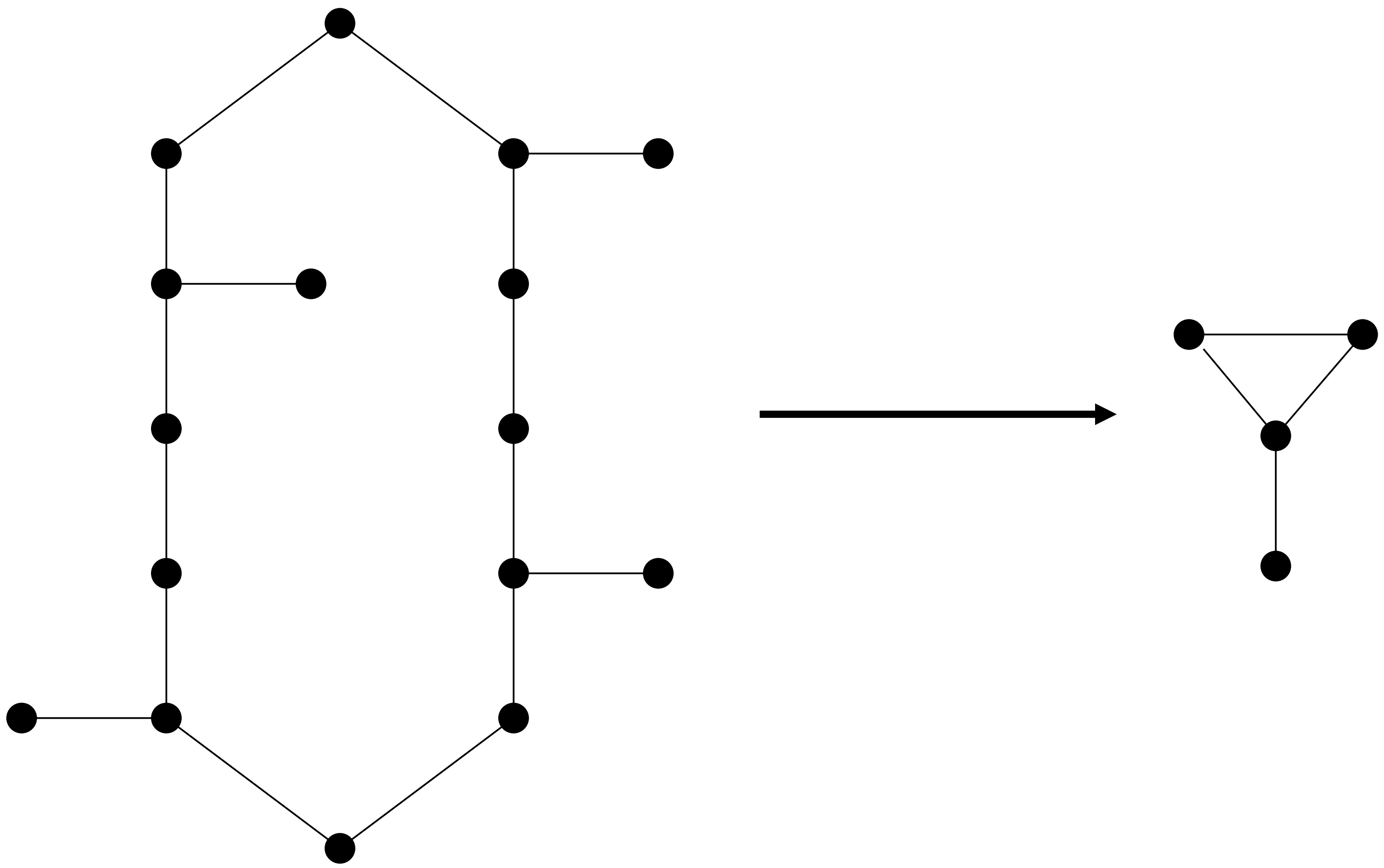


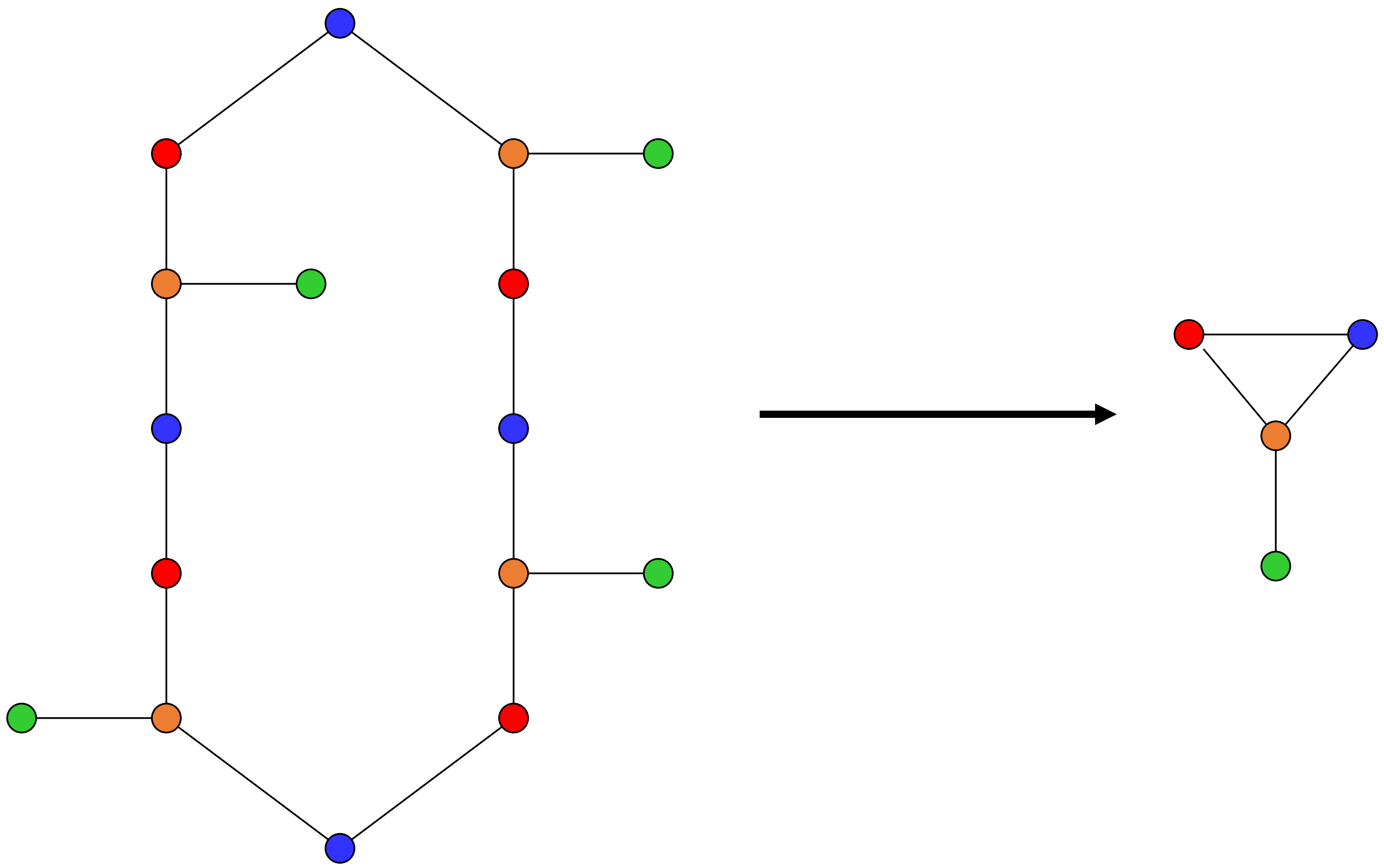
Definition of graph covering
(for connected simple graphs)

Definition: Mapping f: $V(G) \to V(H)$ is a *graph covering projection* if for every $u \in V(G)$, $f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$



$$f(N_G(u)) = N_H(f(u))$$





A bit of the history

- Topological graph theory, construction of highly symmetric graphs (Biggs 1974, Djokovic 1974, Gardiner 1974, Gross et al. 1977)
- Local computation (Angluin STOC 1980, Litovsky et al. 1992, Courcelle et al. 1994, Chalopin et al. 2006)
- Common covers (Angluin et al. 1981, Leighton 1982)
- Finite planar covers (Negami's conjecture 1988, Hliněný 1998, Archdeacon 2002, Hliněný et al. 2004)
- Regular covers and maps (Nedela et al. 1996, Malnic et al. 2000, …)

Computational complexity of graph covers

H-COVER
Input: A graph *G*
Question: Does *G* cover *H*?

Computational complexity of graph covers

- Thm (Bodlaender 1989): *H*-COVER is NP-complete if *H* is also part of the input.
- Abello, Fellows, Stilwell 1991: Initiated the study of computational complexity of the *H*-COVER problem for fixed *H*.
- Thm (Kratochvil, Proskurowski, Telle 1994): *H*-COVER is polynomial time solvable for every simple graph with at most 2 vertices per equivalence class in its degree partition.
- Thm (Fiala, Kratochvil, Proskurowski, Telle 1998): *H*-COVER is NP-complete for every simple regular graph of degree at least 3.
- Fiala, Kratochvil 2008: Relation to CSP
- Bílka, Jirásek, Klavík, Tancer, Volec 2011: NP-hardness of covering small graphs by planar inputs.
- Bok, Fiala, Hlineny, Jedlickova, Kratochvil 2021: Covering multigraphs with semi-edges

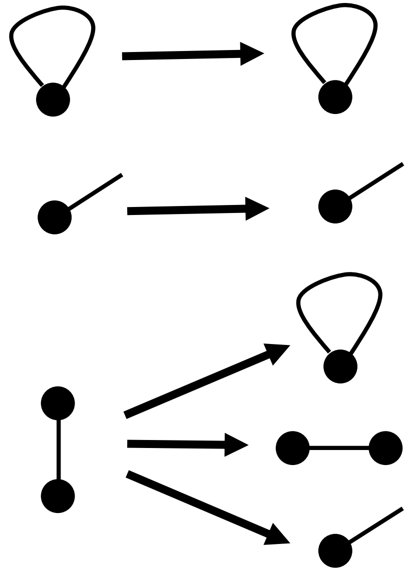
Outline of the presentation

- Multigraphs with semi-edges and the Strong Dichotomy Conjecture
- The Role of Cycles
- The Role Colours
- List Covering multigraphs with semi-simple vertices
- Complete characterization of List-H-Cover for cubic multigraphs H

1.1 Covers of multigraphs
(with multiple edges, loops and semi-edges)

Definition: A pair of mappings $f = (f_V, f_E)$: $G \to H$ is a graph covering projection if

- f_V:$V(G) \to V(H)$ is a homomorphism,
- f_E:$E(G) \to E(H)$ is compatible with f_V, and it is a bijection of {edges incident with u} onto {edges incident with $f_V(u)$} for every $u \in V(G)$

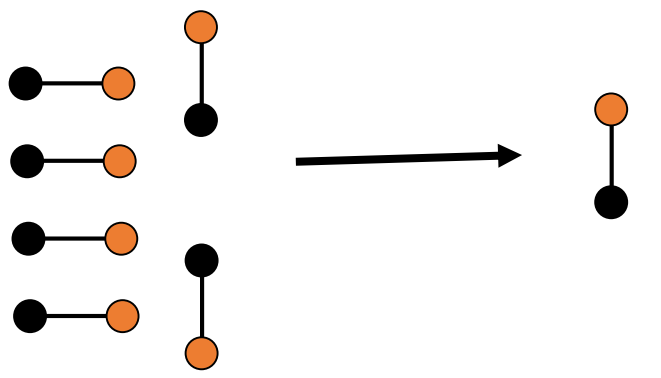


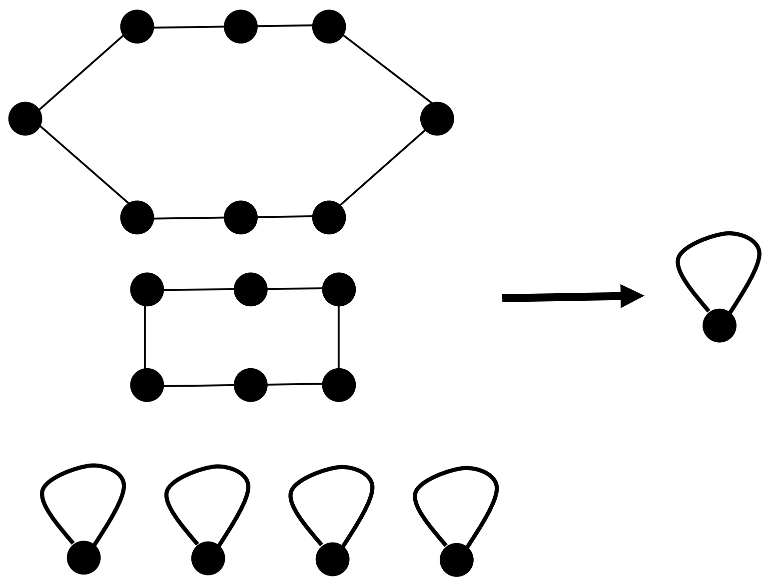
Covers of multigraphs

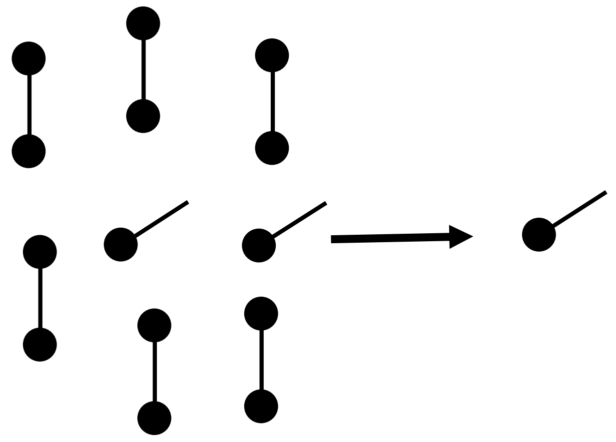
(with multiple edges, loops and semi-edges)

Definition: A pair of mappings $f = (f_V, f_E): G \to H$ is a graph covering projection if

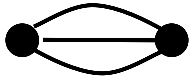
- $f_V: V(G) \to V(H)$ is a homomorphism,
- $f_E: E(G) \to E(H)$ is compatible with f_V, and it is a bijection of {edges incident with u} onto {edges incident with $f_V(u)$} for every $u \in V(G)$

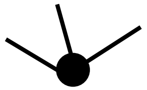






Covers of multigraphs
(few examples)

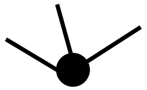
 -COVER is polynomial time solvable

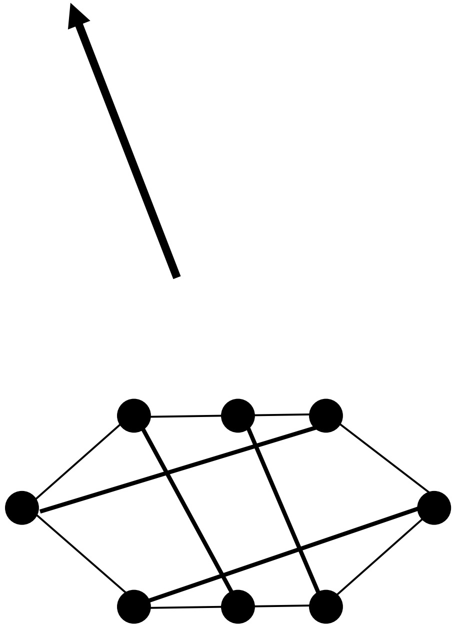
 -COVER is NP-complete

Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

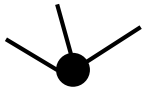
 -COVER is NP-complete

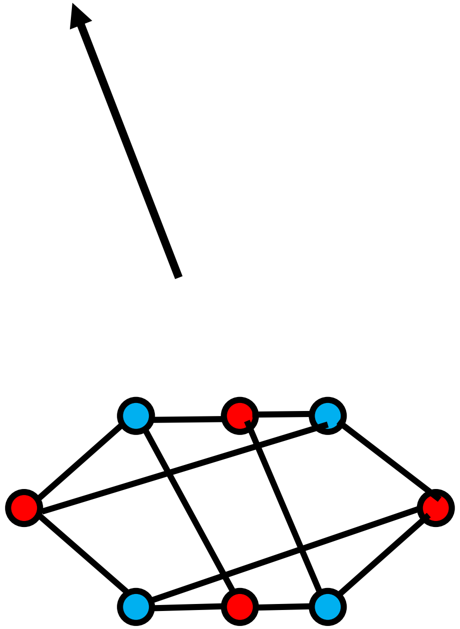


Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

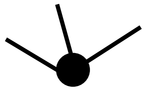
 -COVER is NP-complete

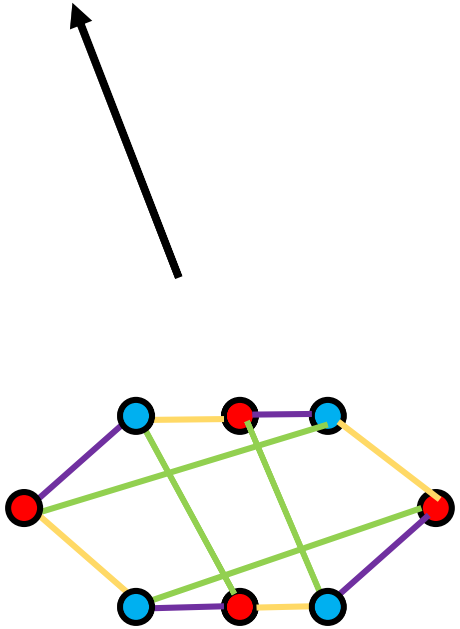


Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

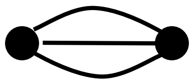
 -COVER is NP-complete

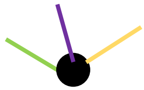


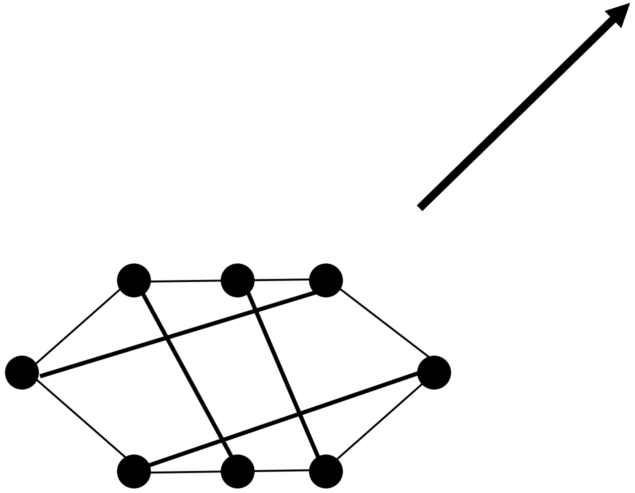
Konig-Hall

Covers of multigraphs

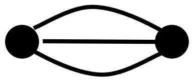
(few examples)

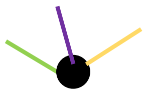
 -COVER is polynomial time solvable

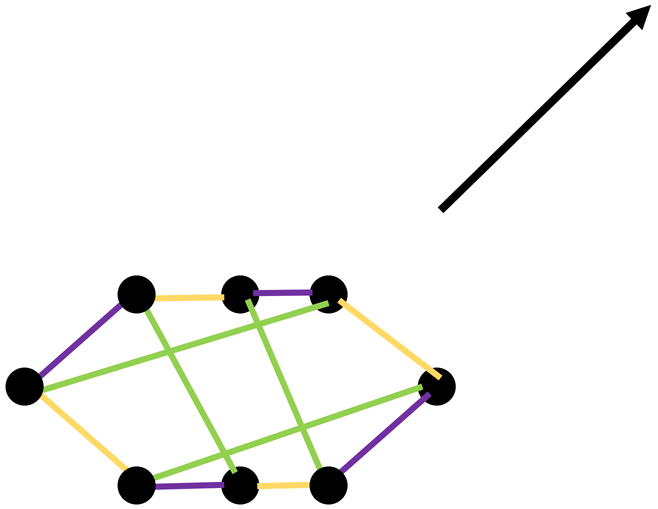
 -COVER is NP-complete



Covers of multigraphs
(few examples)

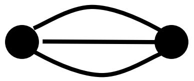
 -COVER is polynomial time solvable

 -COVER is NP-complete

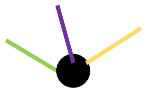


Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

= 3-edge-colorability of bipartite graphs

 -COVER is NP-complete

= 3-edge-colorability

Complexity of covering multigraphs

- Kratochvil, Proskurowski, Telle 1997: Complete characterization of the computational complexity of *H*-COVER for colored mixed 2-vertex multigraphs *H (no semi-edges at that time).*
- Kratochvil, Telle, Tesař 2016: Complete characterization of the computational complexity of *H*-COVER for 3-vertex multigraphs *H.*
- Bok, Fiala, Hliněný, Jedličková, Kratochvíl MFCS 2021: First results on the computational complexity of *H*-COVER for (multi)graphs *with semi-edges*. Full classification for 1-vertex and 2-vertex graphs *H* (report at CSGT 2020).
- Bok, Fiala, Jedličková, Kratochvíl, Seifrtová FCT 2021: Covers of disconnected multigraphs (also at CSGT 2021)
- Bok, Fiala, Jedličková, Kratochvíl, Rzazewski IWOCA 2022: List Covering version

1.2 Hoping for a stronger dichotomy

Strong dichotomy conjecture: For all connected graphs H, the H-COVER problem is either polynomial time solvable for general input graphs, or NP-complete for simple input graphs (i.e., no loops, no multiple edges, no semi-edges are allowed).

Or does there exist a connected graph H (loops, multiple edges and semi-edges allowed) such that the H -COVER problem is NP-complete for general inputs, but polynomial time solvable for simple graphs on the input?

2. The Role of Cycles (and Semi-edges)

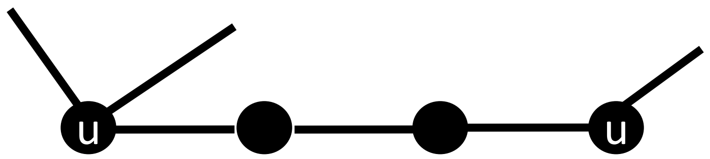
2. The Role of Cycles (and Semi-edges)

Observation: If *G* and *H* are connected acyclic simple graphs (i.e., trees), then *G* covers *H* iff *G* is isomorphic to *H*.

2. The Role of Cycles (and Semi-edges)

Observation: If *G* and *H* are connected acyclic simple graphs (i.e., trees), then *G* covers *H* iff *G* is isomorphic to *H*.

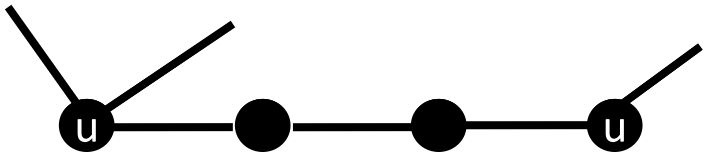
Proof: Every covering projection from *G* to *H* is 1-fold. If not, consider two vertices of *G* that map on the same vertex (say u) of *H* and whose distance is shortest possible.



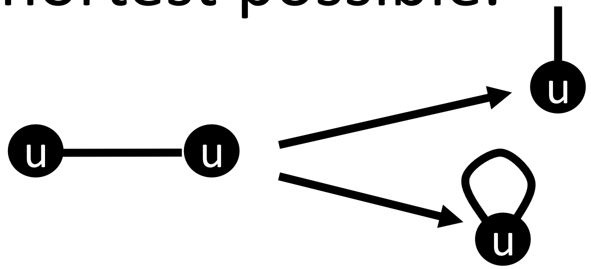
2. The Role of Cycles (and Semi-edges)

Observation: If *G* and *H* are connected acyclic simple graphs (i.e., trees), then *G* covers *H* iff *G* is isomorphic to *H*.

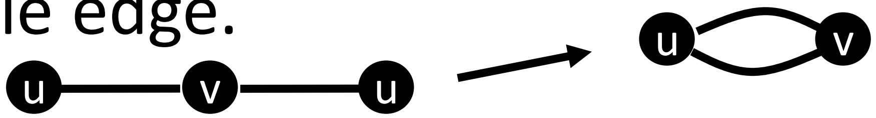
Proof: Every covering projection from *G* to *H* is 1-fold. If not, consider two vertices of *G* that map on the same vertex (say u) of *H* and whose distance is shortest possible.



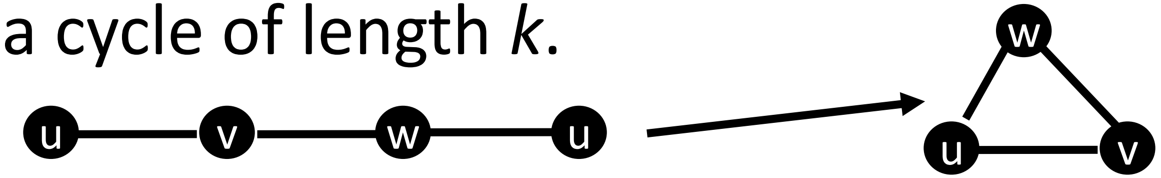
If the distance is 1, then *H* has a loop or a semi-edge.



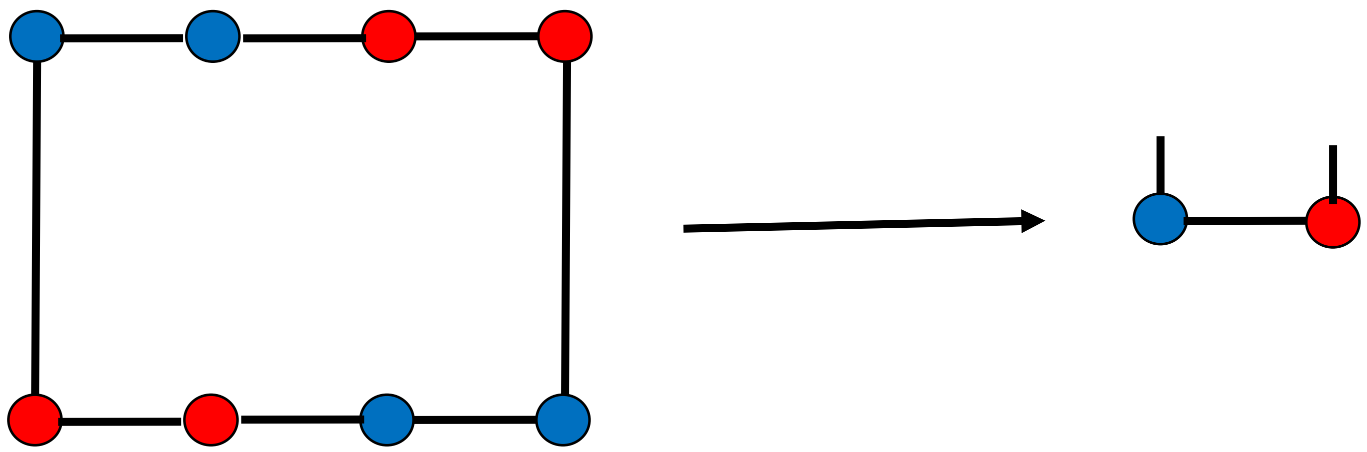
If the distance is 2, then *H* has a double edge.



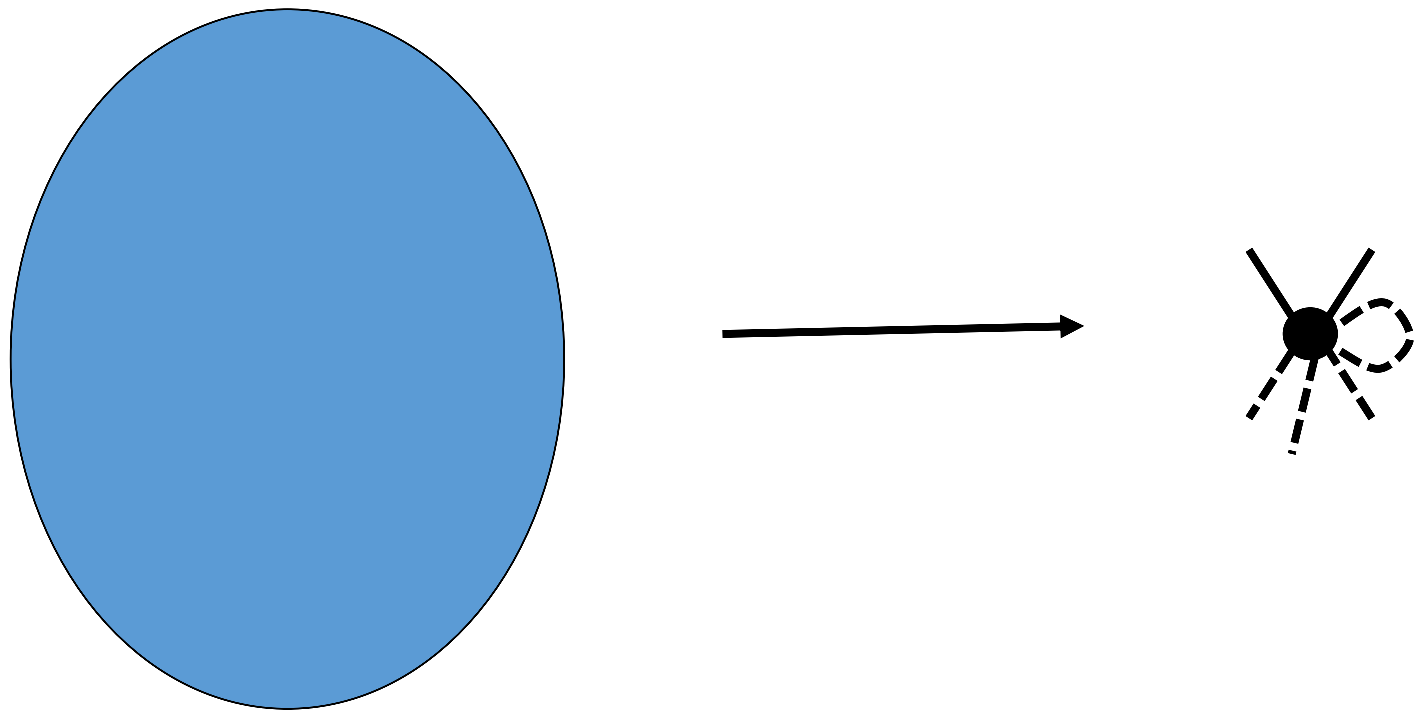
If the distance is *k*>2, then *H* has a cycle of length *k*.



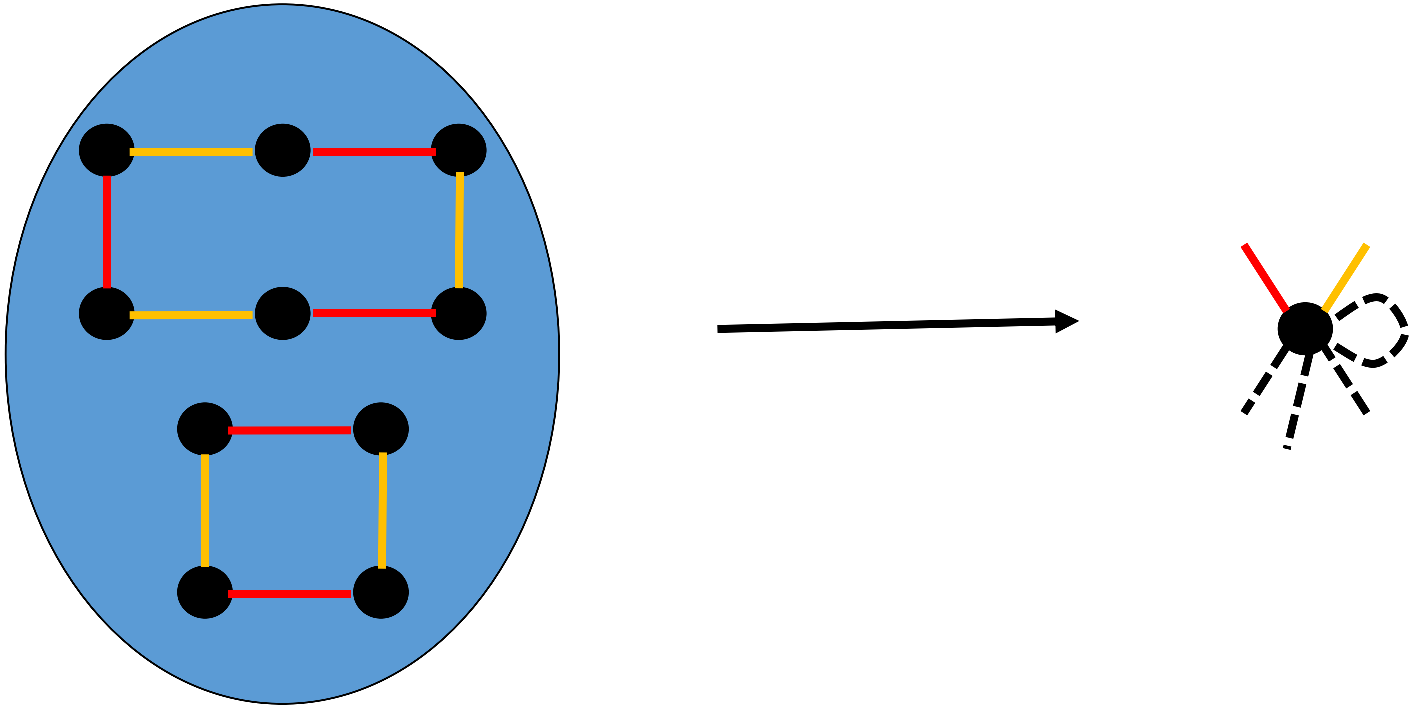
2. The Role of Cycles (and Semi-edges)



2. The Role of Cycles (and Semi-edges)



2. The Role of Cycles (and Semi-edges)



Observation: The preimage of two semi-edges pending on the same vertex is an even 2-factor.

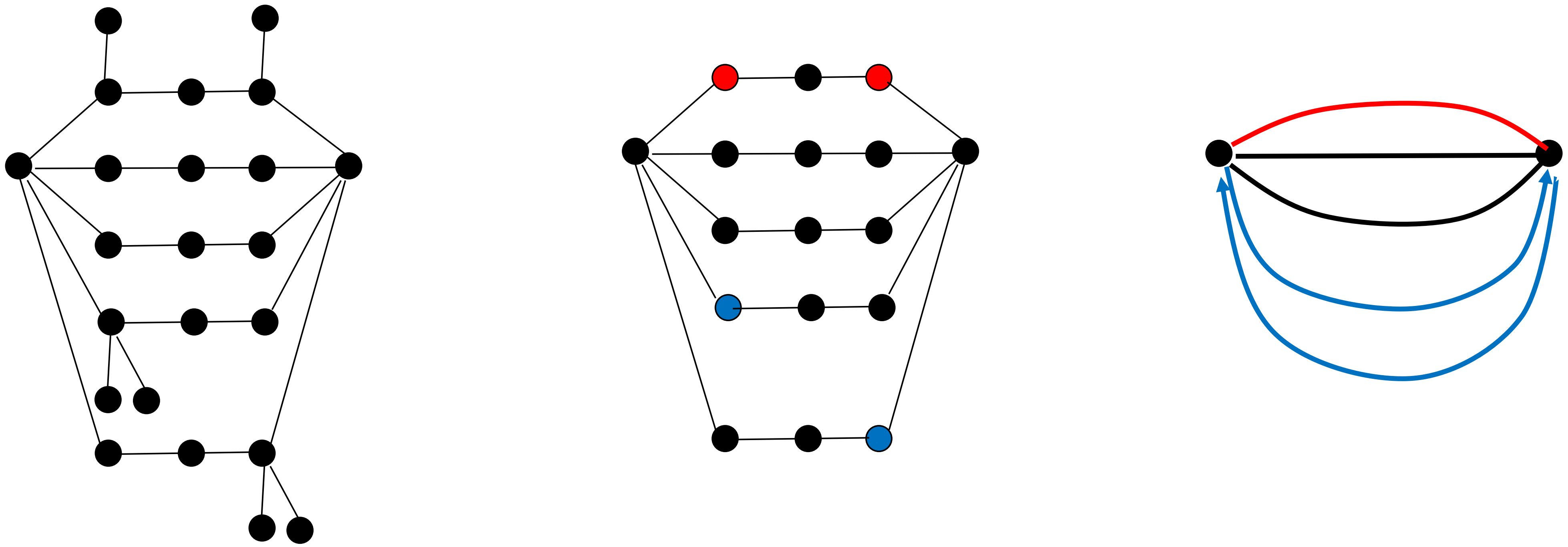
2. The Role of Cycles (and Semi-edges)

Theorem (Bok, Fiala, Hlineny, Jedlickova, Kratochvil MFCS 2021): For every k>2, it is NP-complete to decide if a k-regular simple graph contains an even 2-factor.

3. The Role of Colours

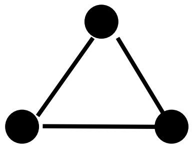
Theorem (Kratochvil, Proskurowski, Telle WG 1997): To fully understand the complexity of *H*-COVER for simple undirected graphs, it is necessary and sufficient to have a complete characterization for coloured mixed multigraphs of minimum degree greater than 2.

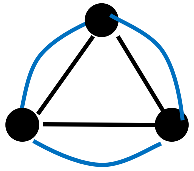
Reduction to coloured graphs



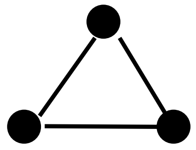
Apply the same reductions to *G* and *H*. Every covering projection must respect the colors.

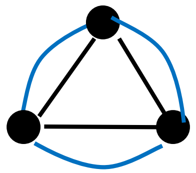
3. The Role of Colours

Example:  -COVER is polynomial time solvable, while

 -COVER is NP-complete.

3. The Role of Colours

Example:  -COVER is polynomial time solvable, while

 -COVER is NP-complete.

Theorem (Bok, Fiala, Jedlickova, Kratochvil, Seifrtova FCT 2021): For a colorured mixed multigraph (with semi-edges) *H*, the *H*-COVER problem is solvable in polynomial time if and only if it is P-time solvable for any of its monochromatic spanning subgraphs, and it is NP-complete even for simple input graphs otherwise (cf. the Strong Dichotomy Conjecture).

4. List covering problems

List-H-COVER

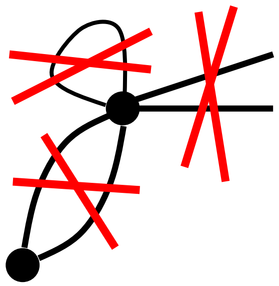
Input: A graph G, lists $L(u) \subseteq V(H)$ for $u \in V(G)$, $L(e) \subseteq V(H)$ for $e \in E(G)$.

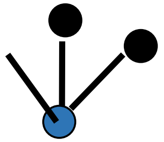
Question: Does G allow a covering projection $f{:}G \to H$ such that $f(u) \in L(u)$ for every $u \in V(G)$ and $f(e) \in L(e)$ for every $e \in E(G)$?

List covering problems

Theorem (Fiala, Kratochvil WG 2006): List-*H*-PartialCOVER is solvable in polynomial time if *H* has at most 1 cycle, and it is NP-complete otherwise (for simple undirected graphs *H*).

Theorem (Bok, Fiala, Jedlickova, Kratochvil, Rzazewski): If *H* is a *k*-regular graph, $k \geq 3$, with at least one *semi-simple vertex*, then List-*H*-COVER is NP-complete for simple input graphs.

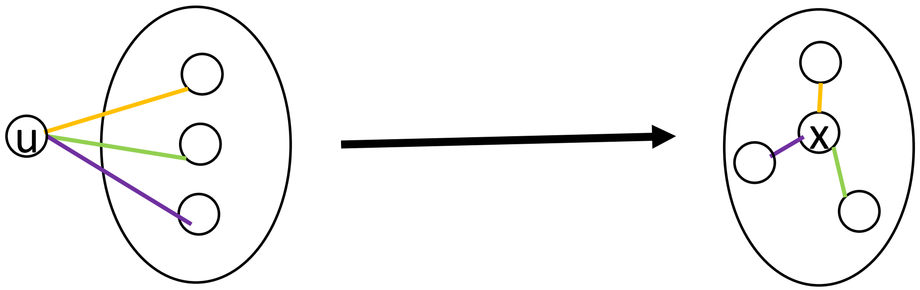




List covering problems

Sketch of proof: Revisit the reduction for *k*-edge-colorable *k*-regular graphs from Kratochvil, Proskurowski, Telle [JCTB 1997].

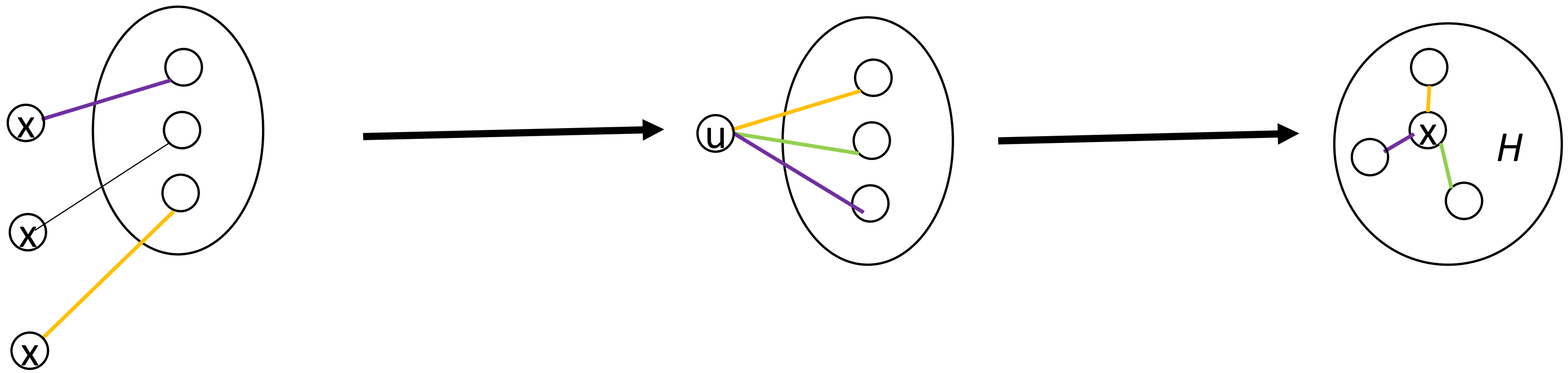
A graph *G* is a *multicover* of *H* if it covers *H* in many ways, in that sense that *G* has a vertex *u* such that for every vertex *x* of *H* and for every bijective mapping of the edges of *G* incident with *u* to the edges of *H* incident with *x*, there is a covering projection $G \to H$ that extends this mapping.



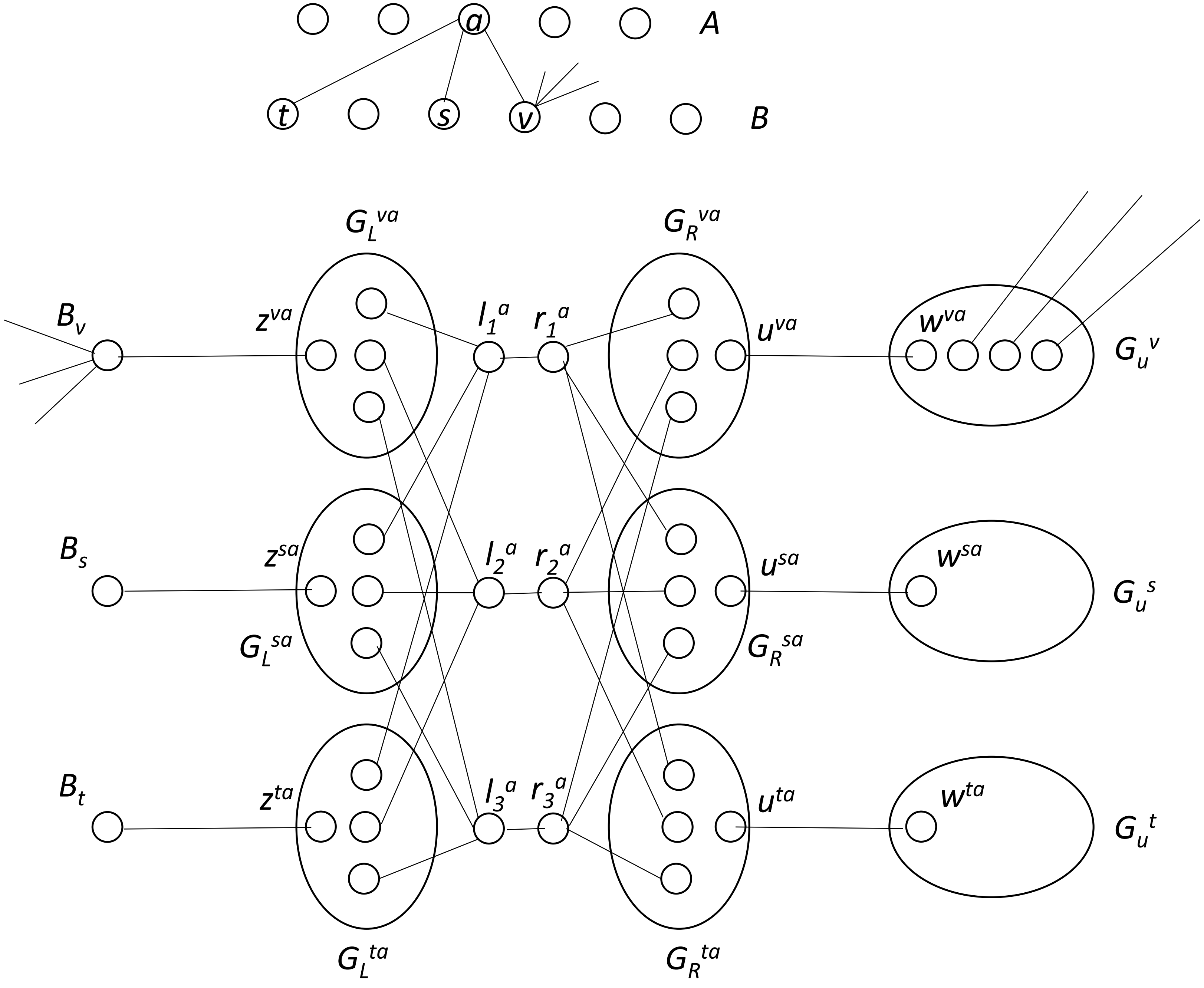
List covering problems

Sketch of proof: Revisit the reduction for k-edge-colorable k-regular graphs from Kratochvil, Proskurowski, Telle [JCTB 1997].

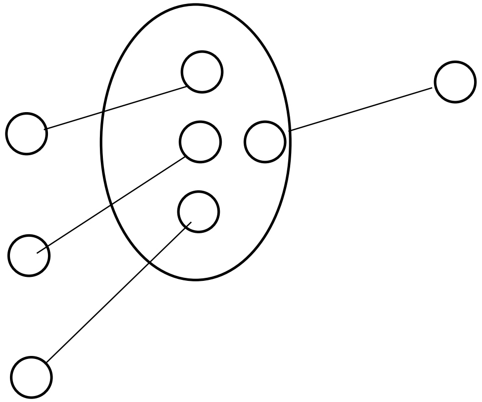
A graph G is a *multicover* of H if it covers H in many ways, in that sense that G has a vertex u such that for every vertex x of H and for every bijective mapping of the edges of G incident with u to the edges of H incident with x, there is a covering projection $G \rightarrow H$ that extends this mapping.



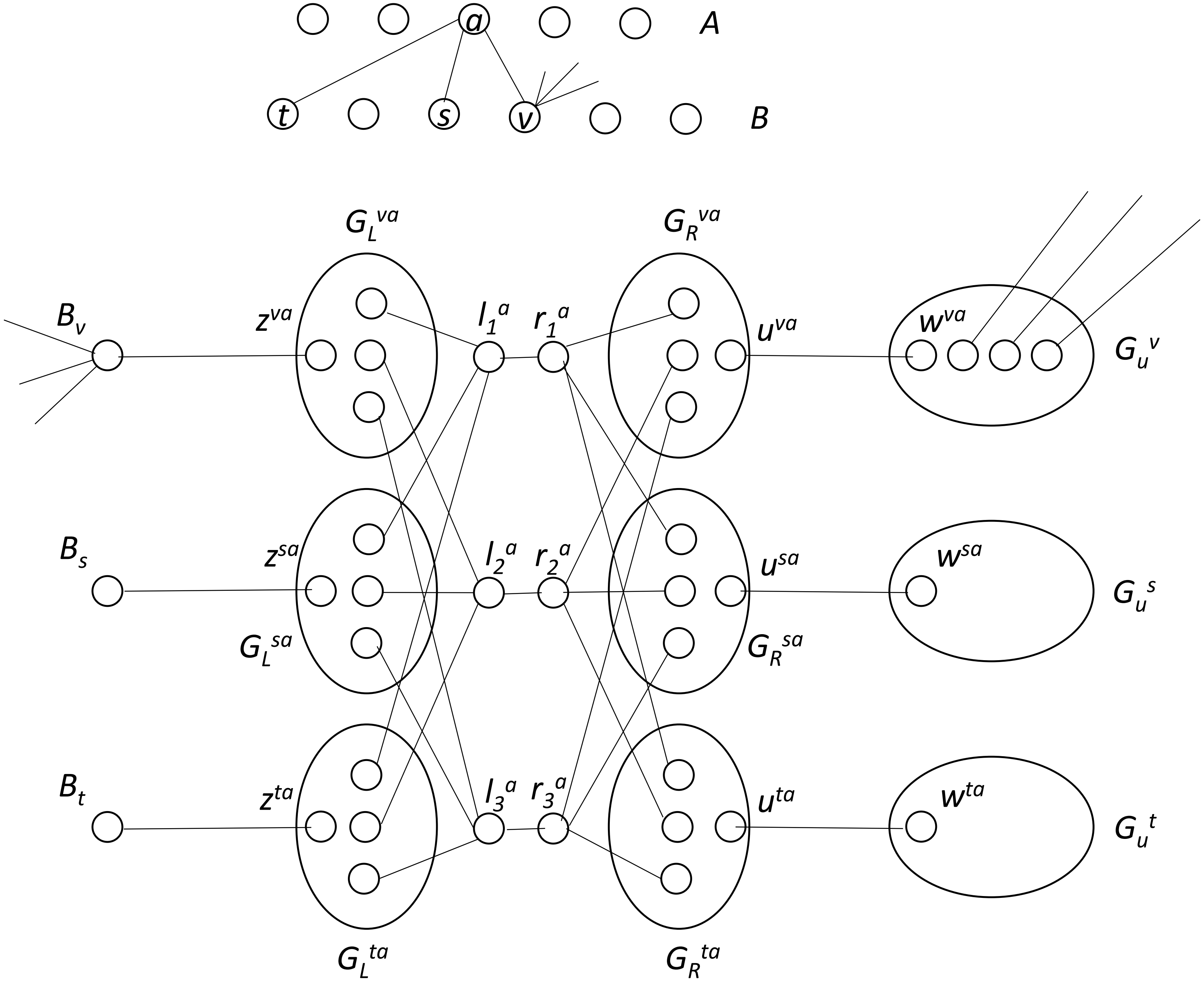
Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



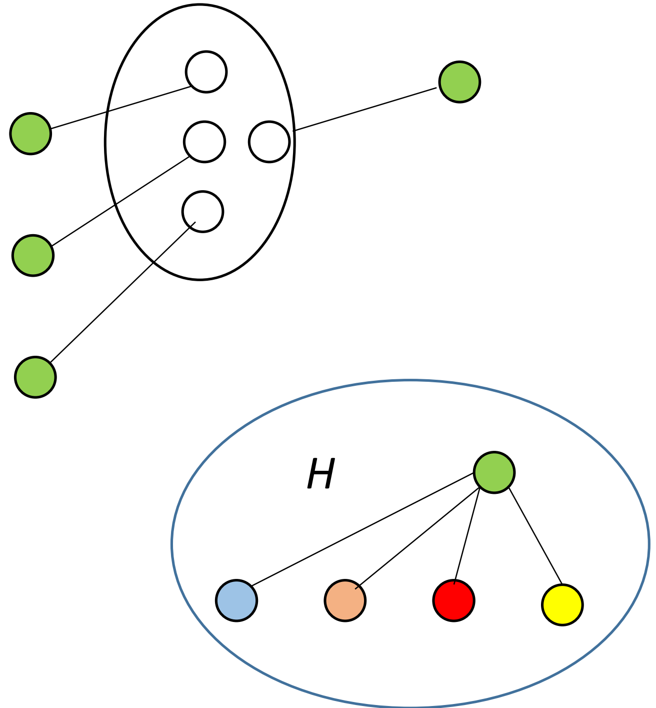
Multicover gadget:



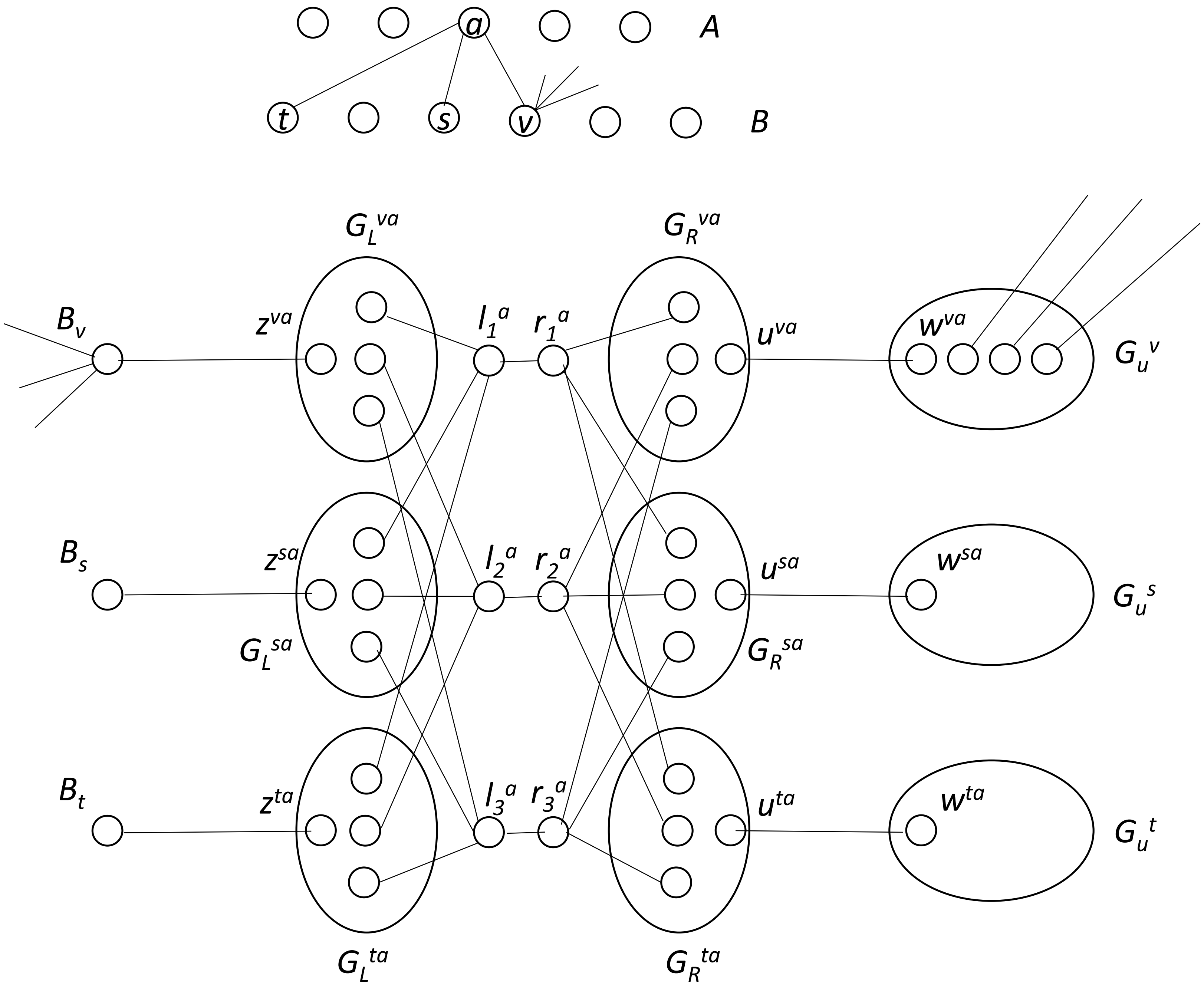
Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



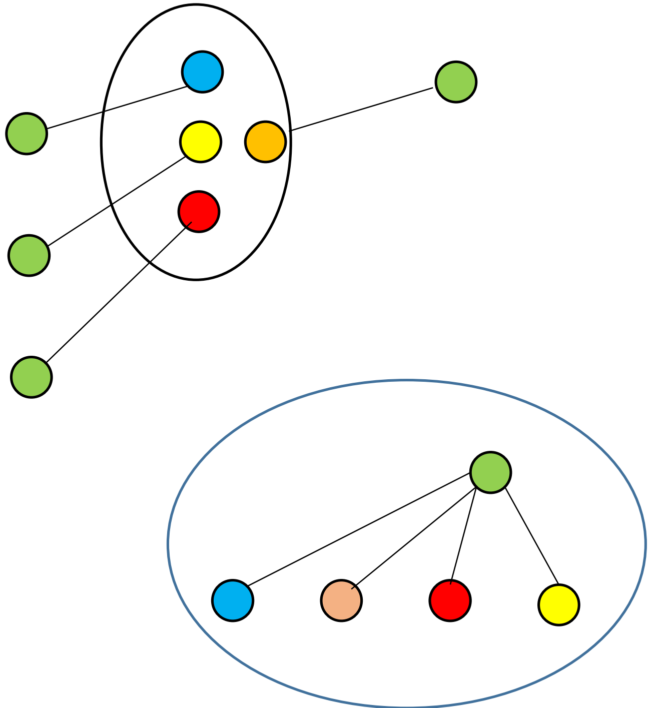
Multicover gadget:

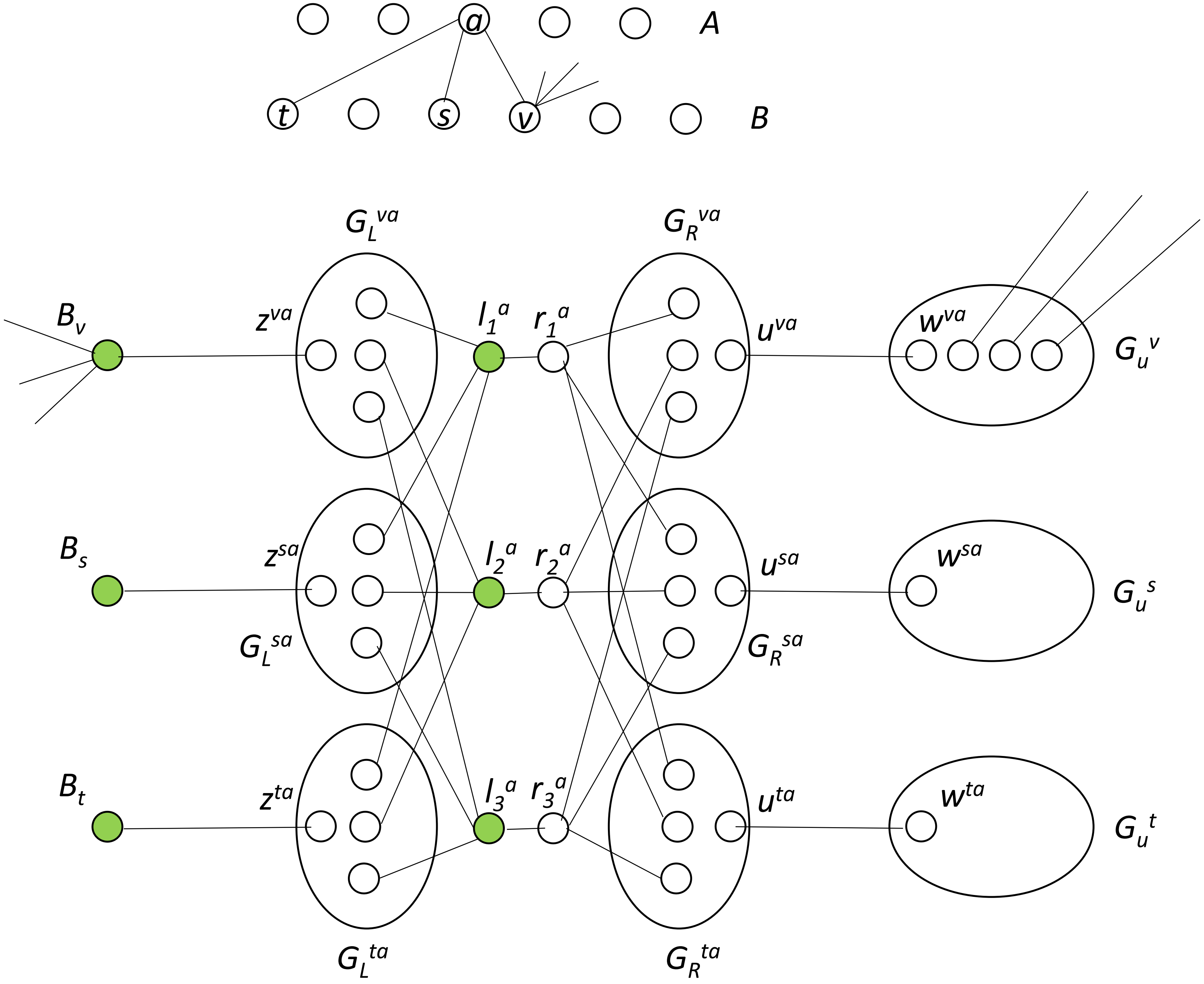


Reduction from *k*-edge colorability *of k*-regular (*k*-1)-uniform hypergraphs.

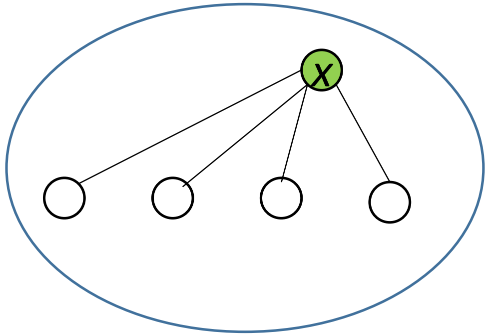


Multicover gadget:

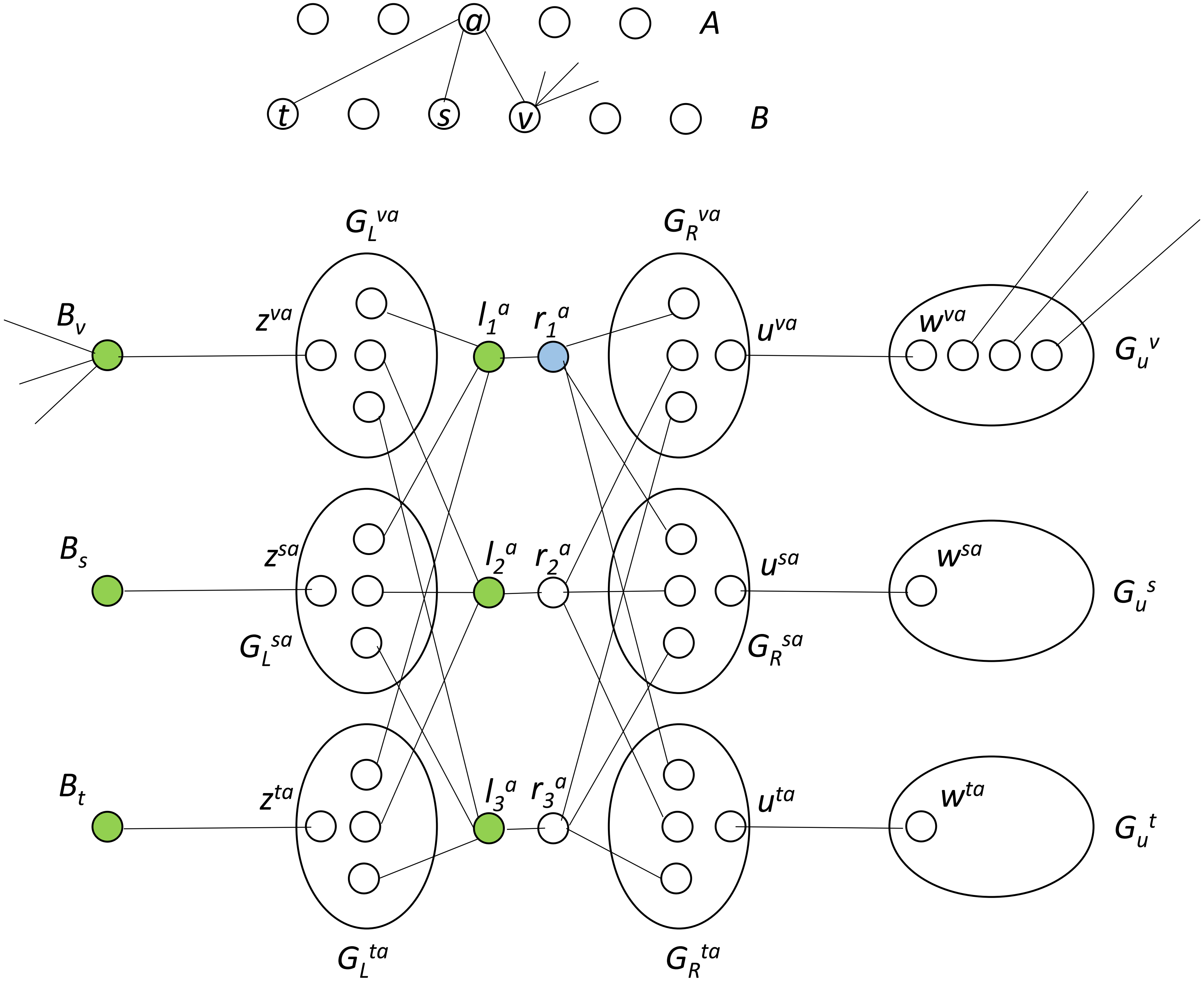


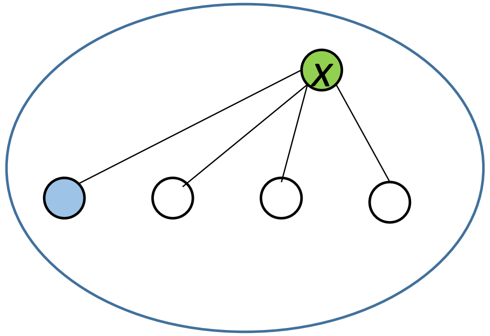


Reduction from k-edge colorability *of* k-regular (k-1)-uniform hypergraphs.

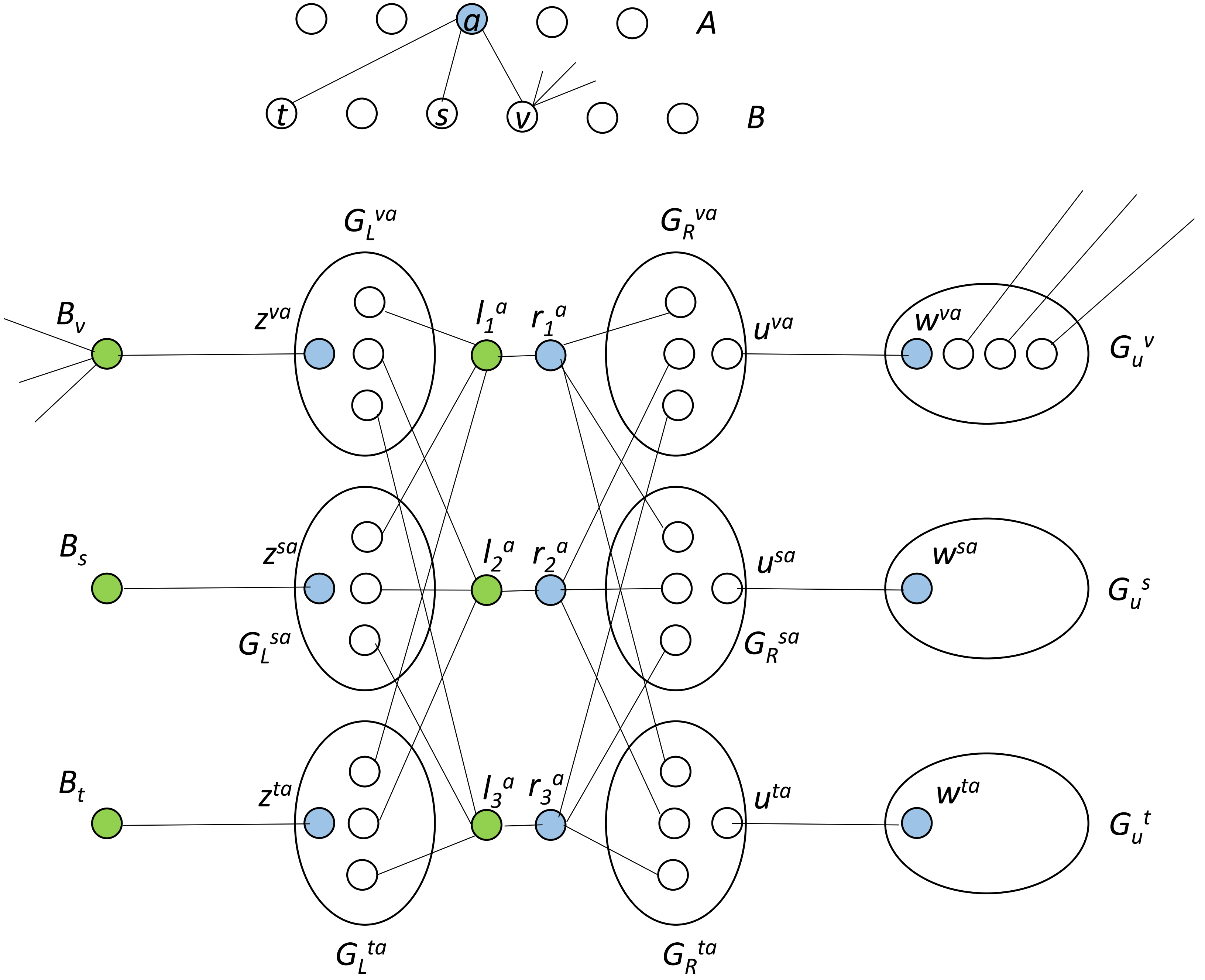


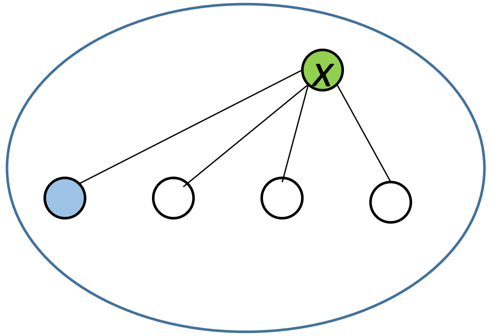
Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



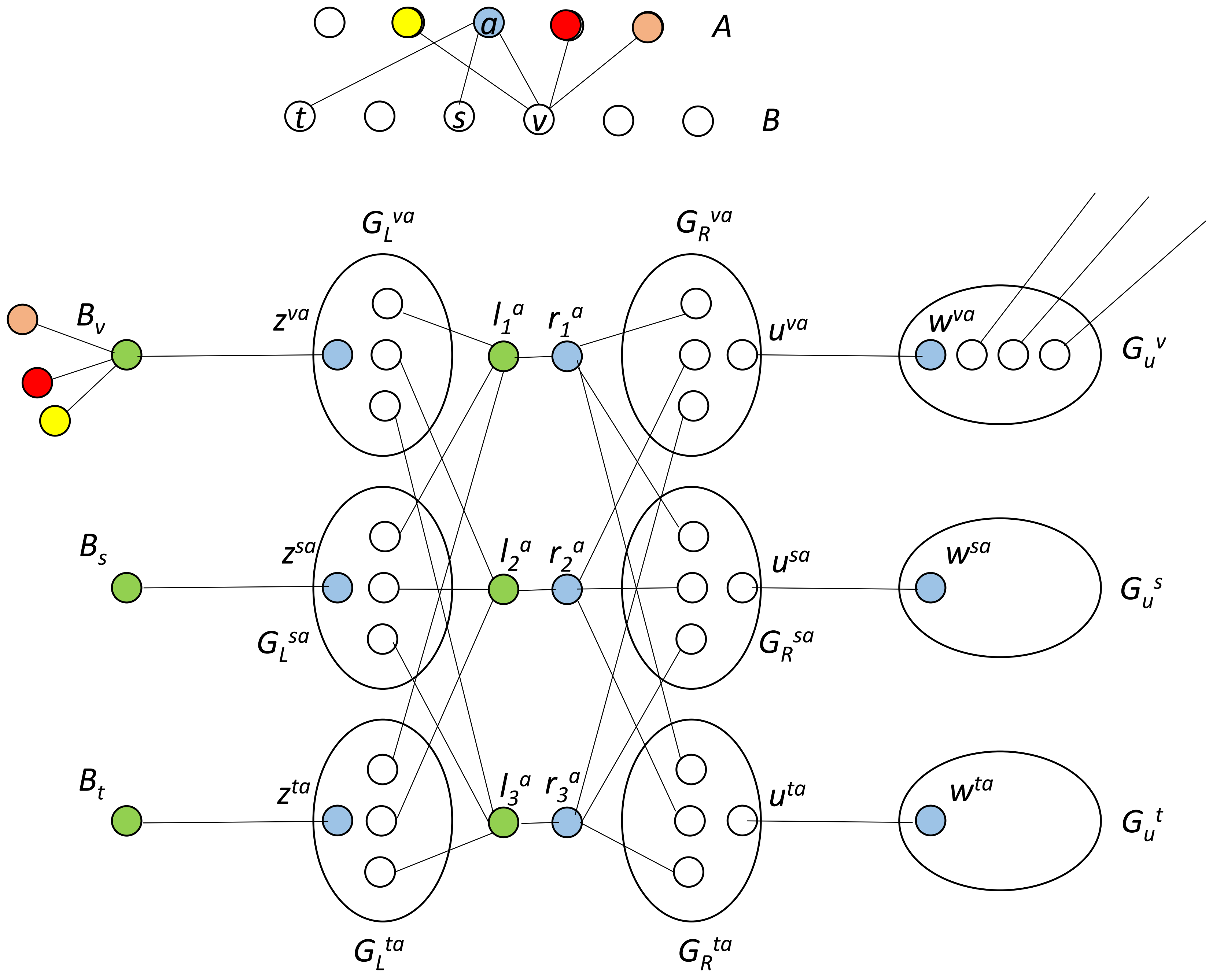


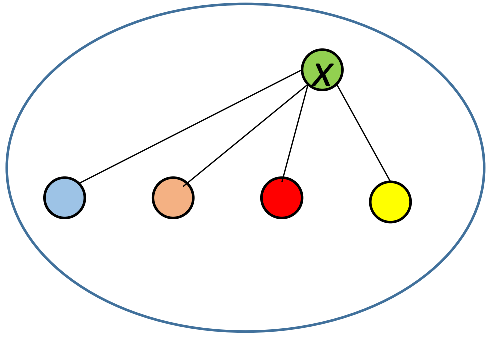
Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.



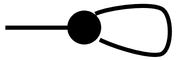


Reduction from *k*-edge colorability *of* *k*-regular (*k*-1)-uniform hypergraphs.





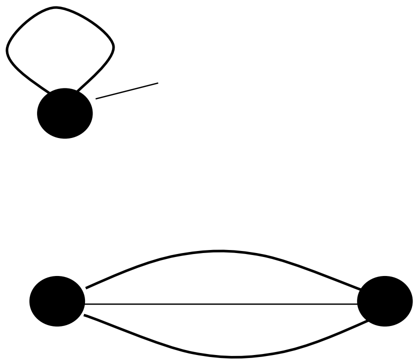
4. Strong Dichotomy for cubic graphs

Theorem (strong dichotomy for 3-regular target graphs): List-*H*-COVER is polynomial time solvable for  and NP-complete for all other target graphs *H*, even for simple inputs.

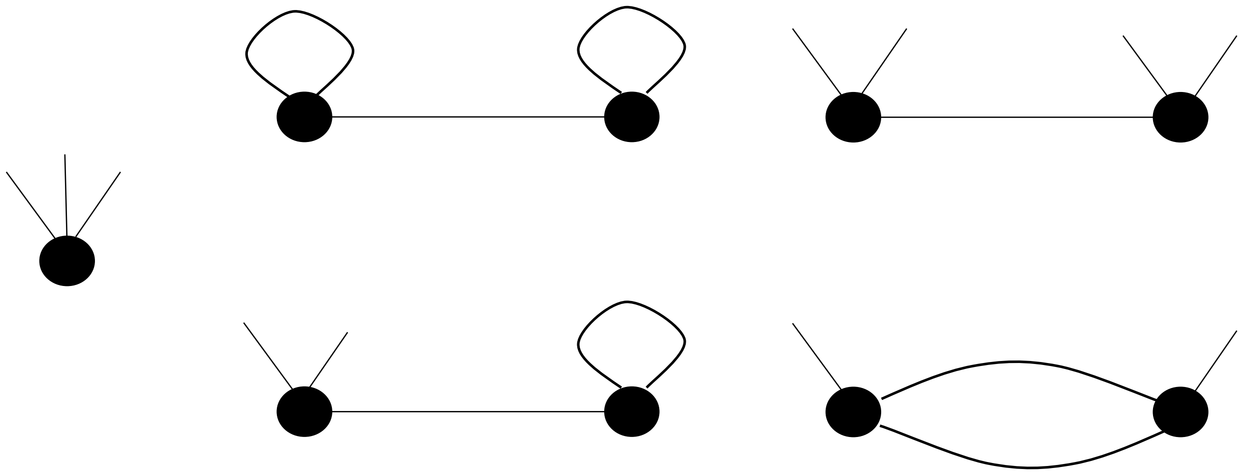
Strong Dichotomy for cubic graphs

If *H* has at most 2 vertices, the following is known from Bok, Fiala, Hlineny, Jedlickova, Kratochvil 2021 for the *H*-COVER problem:

Polynomial

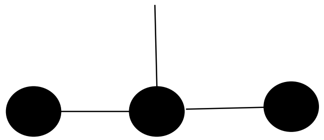


NP-complete without lists

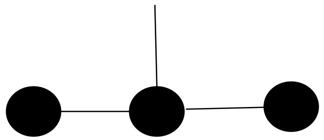


Fact: For every graph *H*, *H*-COVER α List-*H*-COVER

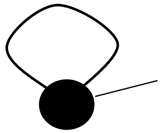
Strong Dichotomy for cubic graphs

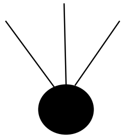
Case A: *H* has a vertex with 3 different neighbors  this is a semi-simple vertex and List-*H*-COVER is NP-complete for simple input graphs by the Theorem.

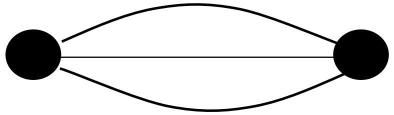
Strong Dichotomy for cubic graphs

Case A: *H* has a vertex with 3 different neighbors  this is a semi-simple vertex and List-*H*-COVER is NP-complete for simple input graphs by the Theorem.

Case B: H has a vertex whose all 3 neighbors are the same vertex

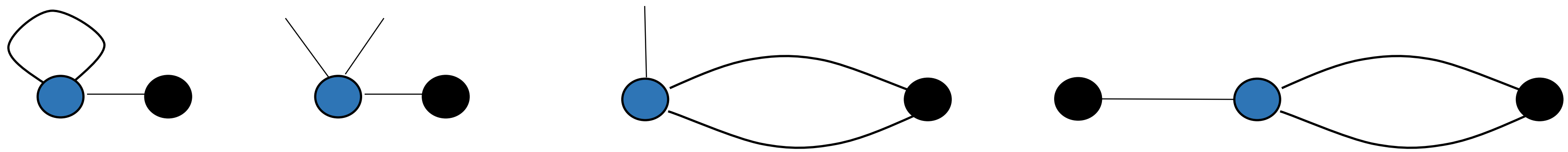
Case B1:  List-*H*-COVER is polynomial time solvable via perfect matching

Case B2:  *H*-COVER is NP-complete for simple inputs (3-edge-colorability)

Case B3:  List-*H*-COVER is NP-complete for simple inputs (via Precoloring extension for line graphs of cubic bipartite graphs, Fiala 1998)

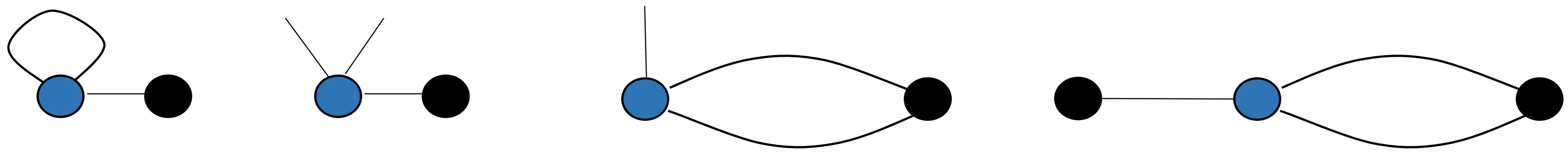
Strong Dichotomy for cubic graphs

Case C: Every vertex of *H* has exactly 2 neighbors, one adjacent via a double edge or via a loop or via 2 semi-edges, and the other one via a single edge or via a semi-edge.

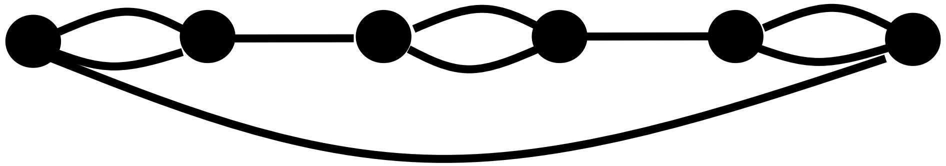


Strong Dichotomy for cubic graphs

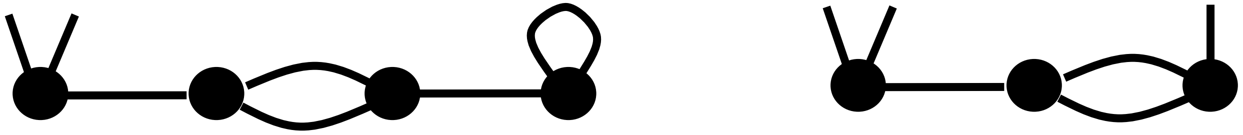
Case C: Every vertex of *H* has exactly 2 neighbors, one adjacent via a double edge or via a loop or via 2 semi-edges, and the other one via a single edge or via a semi-edge.



Case C1: *H* is a ring



Case C2: *H* is a sausage graph



Strong Dichotomy for cubic graphs

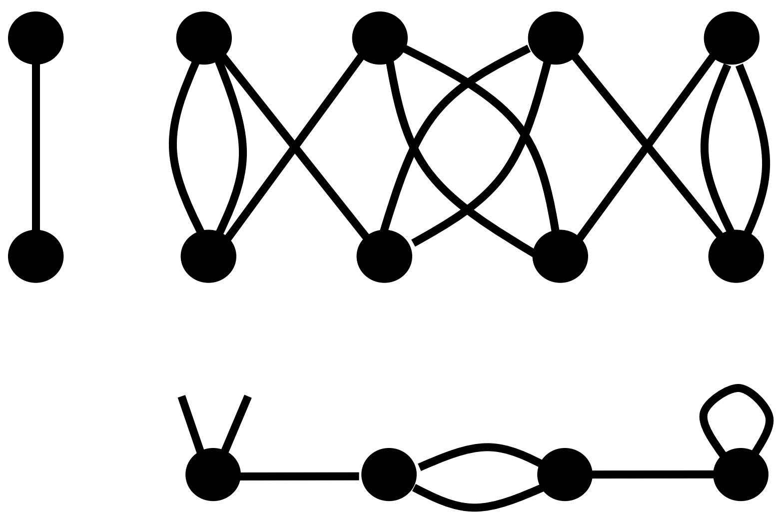
Cases C1 and C2:

Lemma 1: If *H* is a sausage graph with *k* vertices, then *k*-Ring-COVER α *H*-COVER

Lemma 2: For every k $\geq$ 2, the *k*-Ring-COVER problem is NP-complete for simple input graphs

Strong Dichotomy for cubic graphs

Proof of Lemma 1: For every non-bipartite graph *H* and for every graph *G*, *G* covers *H* x K_2 if and only if *G* is bipartite and covers *H* (Fiala 1998). And if *H* is a sausage with *k* vertices, then *H* x K_2 is a *k*-Ring.



Research questions

Problem 1: Full characterization and strong dichotomy for List-H-COVER for k-regular target graphs H for $k \geq 4$?

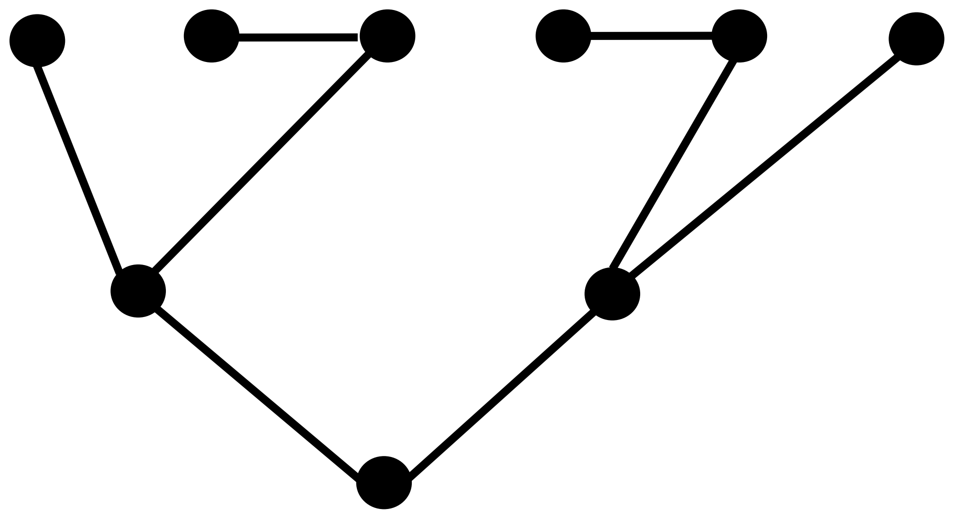
Problem 2: Can we do without lists?

Problem 3: Can we do without semi-simple vertices?

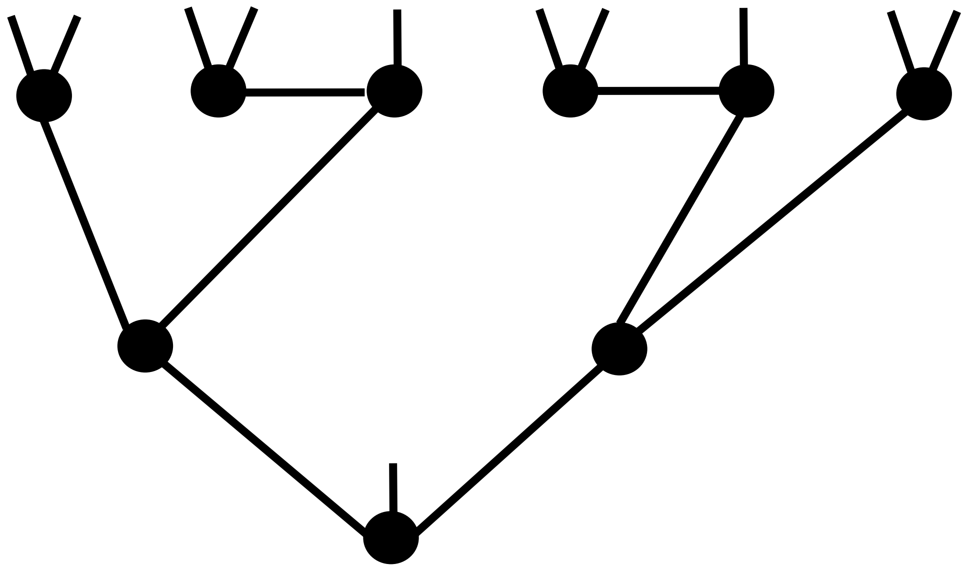
Conjecture: Let H be a connected k-regular graph (loops, multiple edges and semi-edges allowed), with $k \geq 3$. Then both H-COVER and List-H-COVER are polynomial time solvable if H is a single-vertex graph with at most one semi-edge, H-COVER is solvable in polynomial time if H is a two-vertex graph with k parallel edges between its vertices, and both problems are NP-complete for simple input graphs otherwise.

Research questions

Partial result (unpublished): Let *T*’ be a regular graph obtained from a tree *T* by adding semi-edges to its vertices. Then *T*’-COVER is NP-complete (even for simple input graphs).



T



T‘


Thank you!