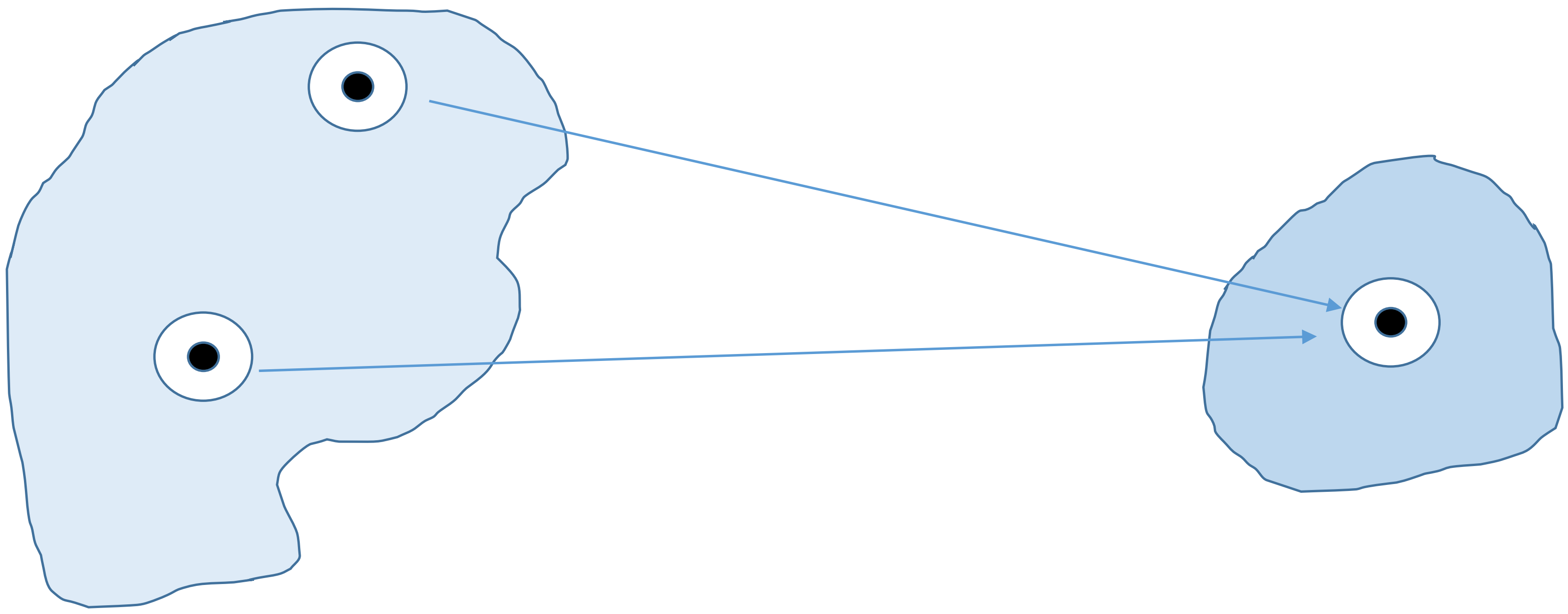
Covers of Graphs

Jan Kratochvíl, Charles University

Based on joint work with

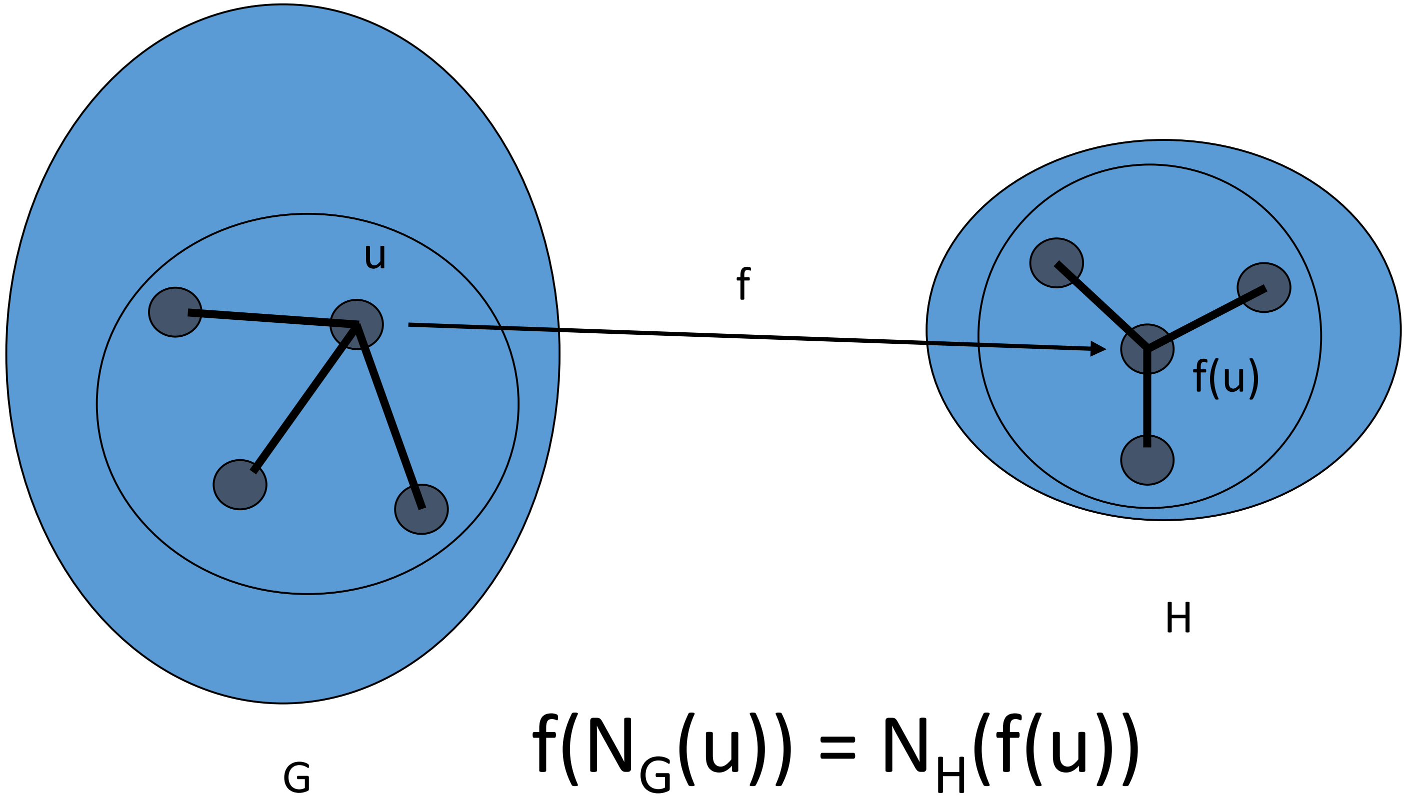
J. Bok, J. Fiala, P. Hliněný, N. Jedličková, and M. Seifertová

Motivation from topology

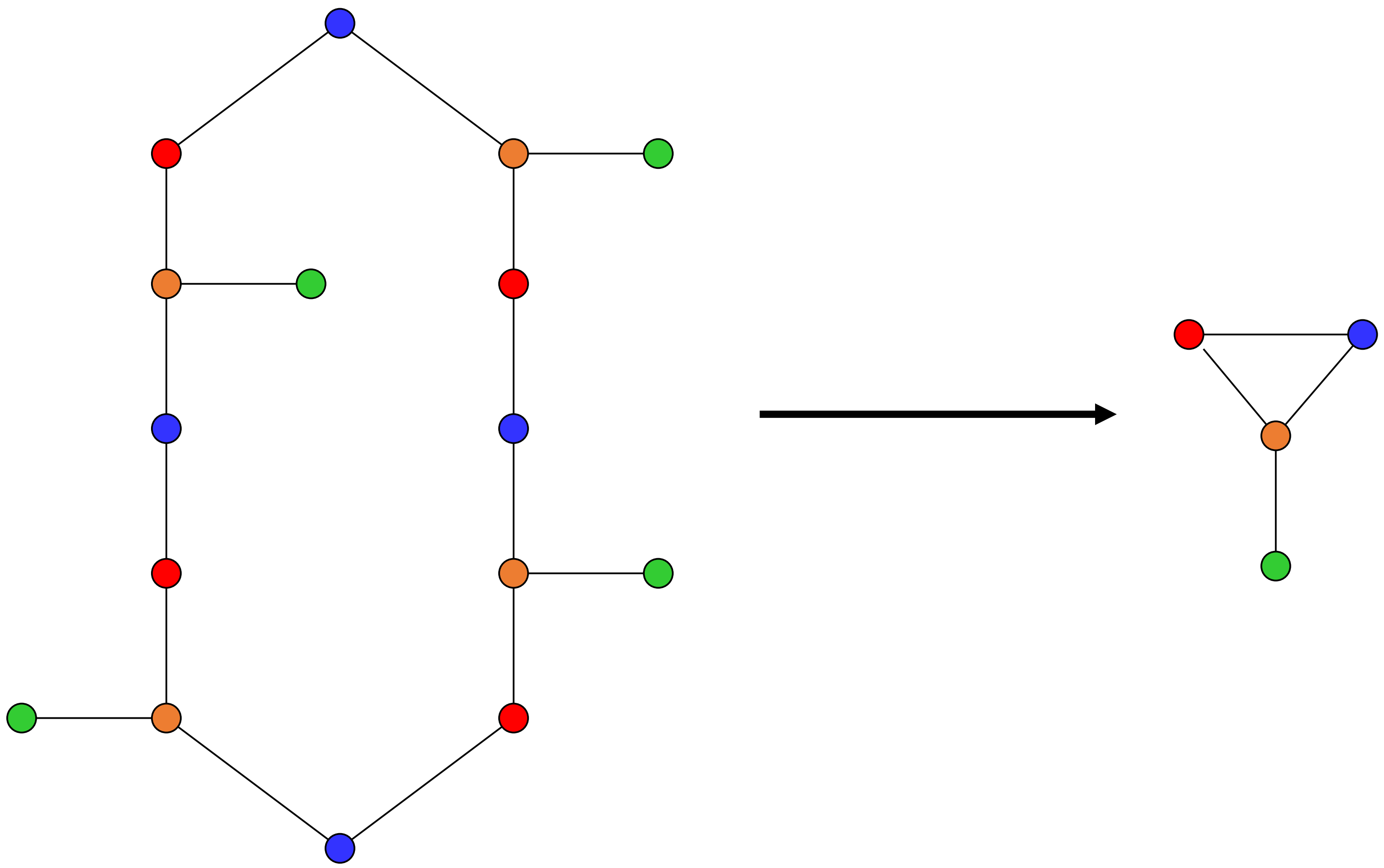


Definition of graph covering

$f: V(G) \to V(H)$ is a graph covering projection if for every $u \in V(G)$,
$f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$



$$f(N_G(u)) = N_H(f(u))$$



Definition of graph covering

f: $V(G) \to V(H)$ is a graph covering projection if for every $u \in V(G)$,

$f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$

$\Rightarrow$

f is a locally bijective homomorphism

A bit of the history

- Topological graph theory, construction of highly symmetric graphs (Biggs 1974)
- Local computation (Angluin 1980, Courcelle 1994, Chalopin 2006)
- Common covers (Leighton 1982)
- Finite planar covers (Negami's conjecture 1988, Hliněný 1998)
- Computational complexity of graph covers (Bodlaender 1989, Abello, Fellows, Stilwell 1991)

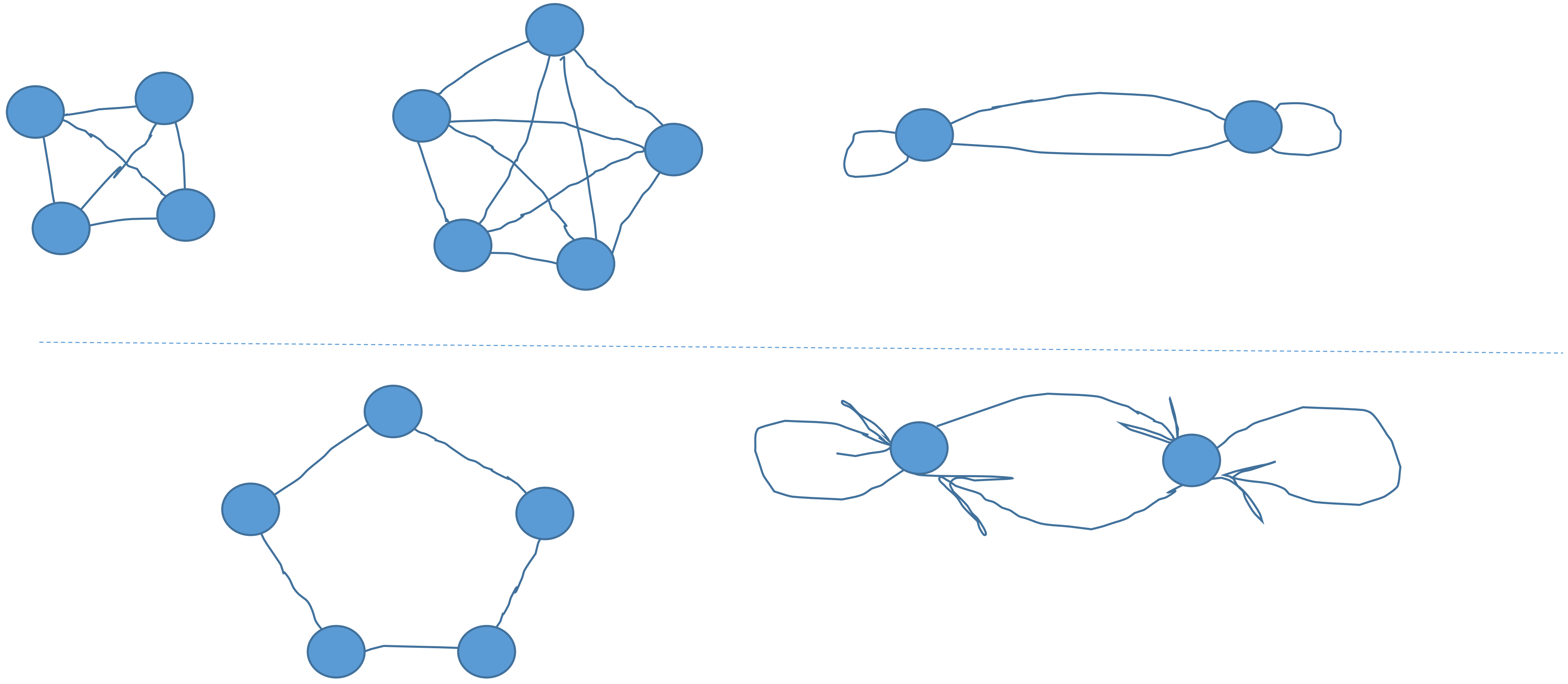
Computational complexity of graph covers

H-COVER

Input: A graph G

Question: Does G cover H?

Computational complexity of graph covers



H-COVER as CSP

Homomorphism – binary relation for preserving edges

Local injectivity – binary relation “not equal” for pairs of vertices with common neighbors

Local bijectivity – unary relations for preserving degrees

H-COVER as CSP

Homomorphism – binary relation for preserving edges

Local injectivity – binary relation “not equal” for pairs of vertices with common neighbors

Local bijectivity – unary relations for preserving degrees

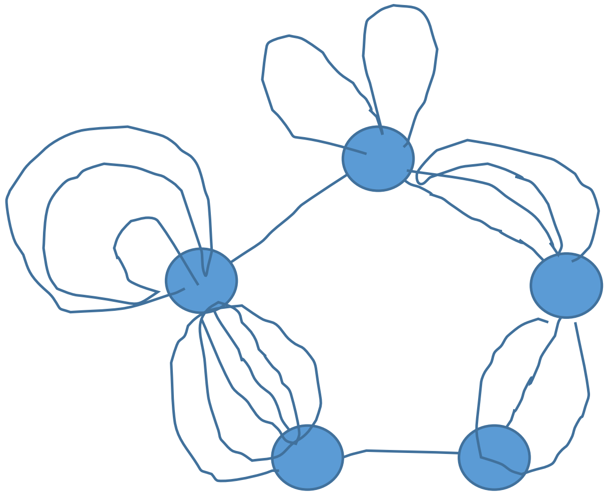
The CSP Dichotomy theorem is of no help

Computational complexity of graph covers

Thm (Fiala, Kratochvil, Proskurowski, Telle 1998): H-COVER is NP-complete for every simple regular graph of valency at least 3.

Thm (Kratochvil, Proskurowski, Telle 1994): H-COVER is in P for every simple graph with at most 2 vertices per equivalence class in the degree partition

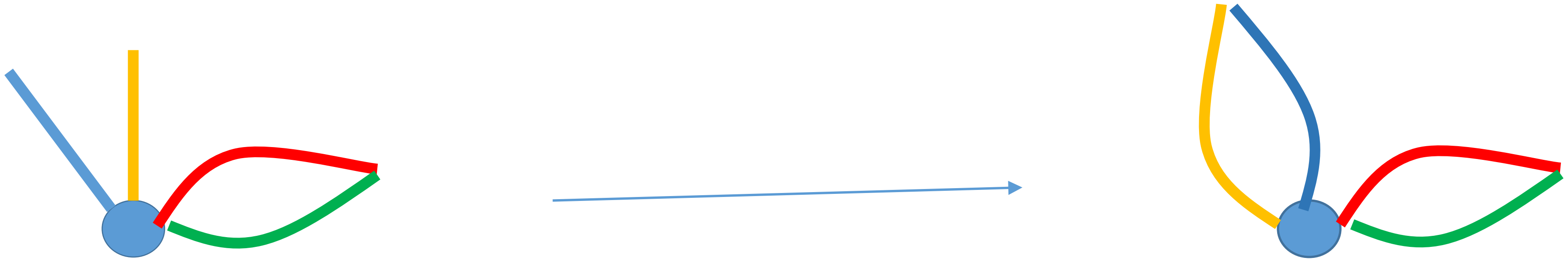
Covers of multigraphs



Covers of multigraphs

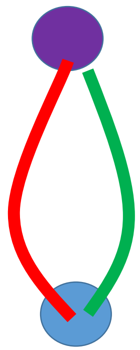
$f=(f_V,f_E): G \rightarrow H$ is a graph covering projection if

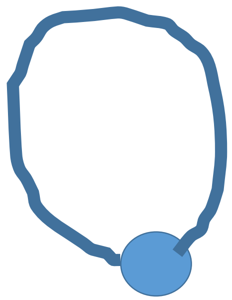
- $f_V:V(G) \rightarrow V(H)$ is a homomorphism,
- $f_E:E(G) \rightarrow E(H)$ is a bijection of {edges incident to u} onto {edges incident to $f_V(u)$} for every $u \in V(G)$



Covers of multigraphs

What is the preimage
- of a multiple edge?
- of a loop?





Covers of multigraphs

What is the preimage

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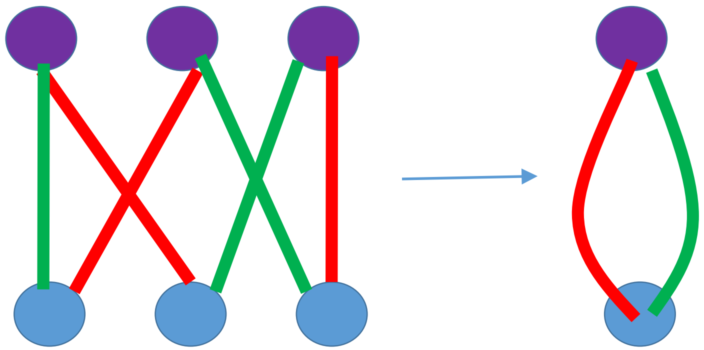
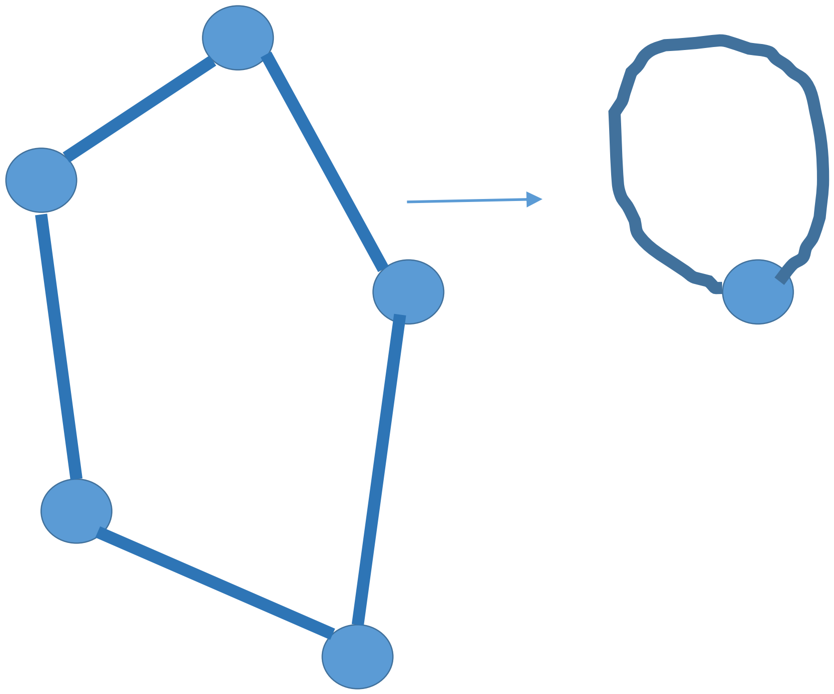




Covers of multigraphs

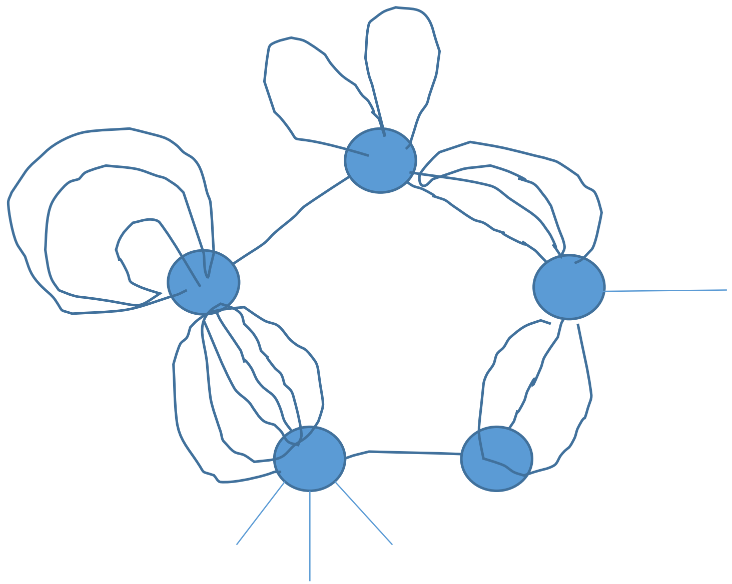
Observation: A degree obedient vertex mapping always extends to a graph covering projection.

Complexity of covering multigraphs

Thm (Kratochvil, Proskurowski, Telle 1997): Complete characterization of the computational complexity of H-COVER for 2-vertex multigraphs H.

Thm (Kratochvil, Telle, Tesař 2116): Complete characterization of the computational complexity of H-COVER for 3-vertex multigraphs H.

(Multi)graphs with semi-edges



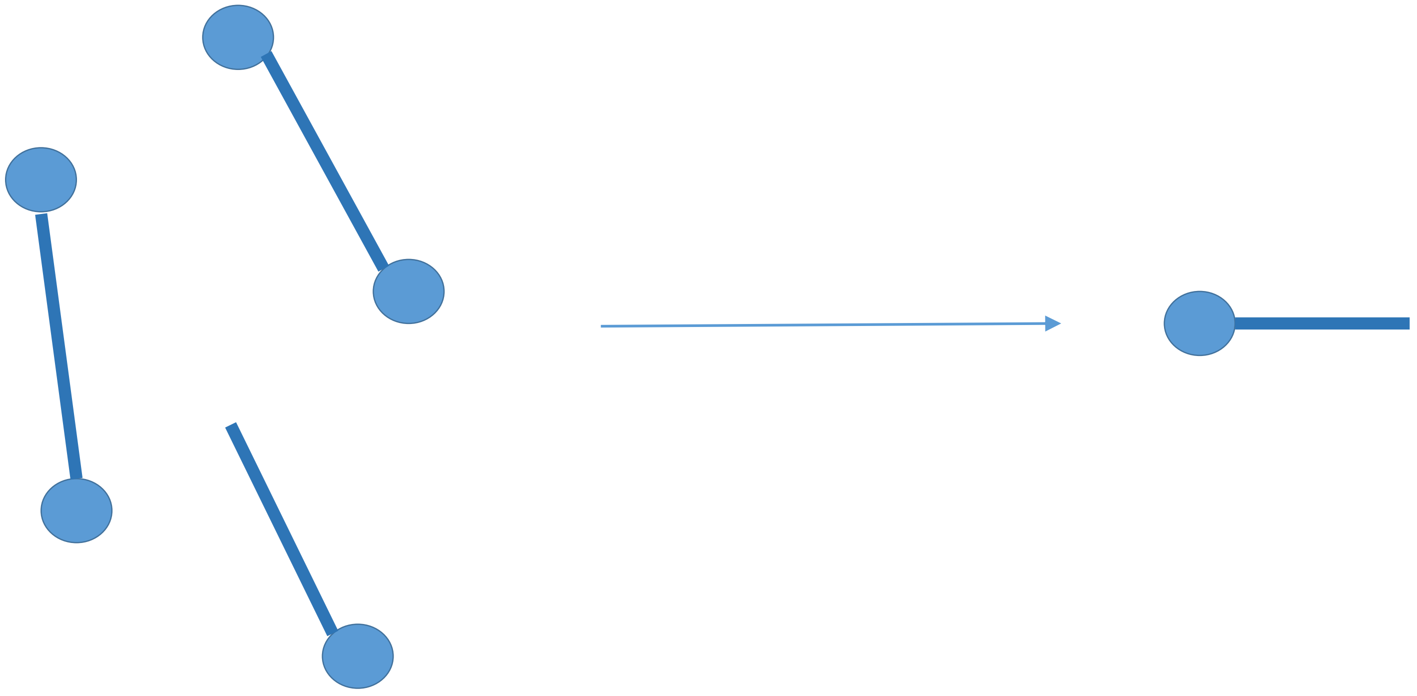
Loops are edges incident with one vertex only, and they add 2 to the degree of the vertex.

Semi-edges are edges incident with one vertex only, which add only 1 to the degree of the vertex.

Covers of graphs with semi-edges

What is the preimage
- of a semi-edge?

A matching plus a collection of semi-edges

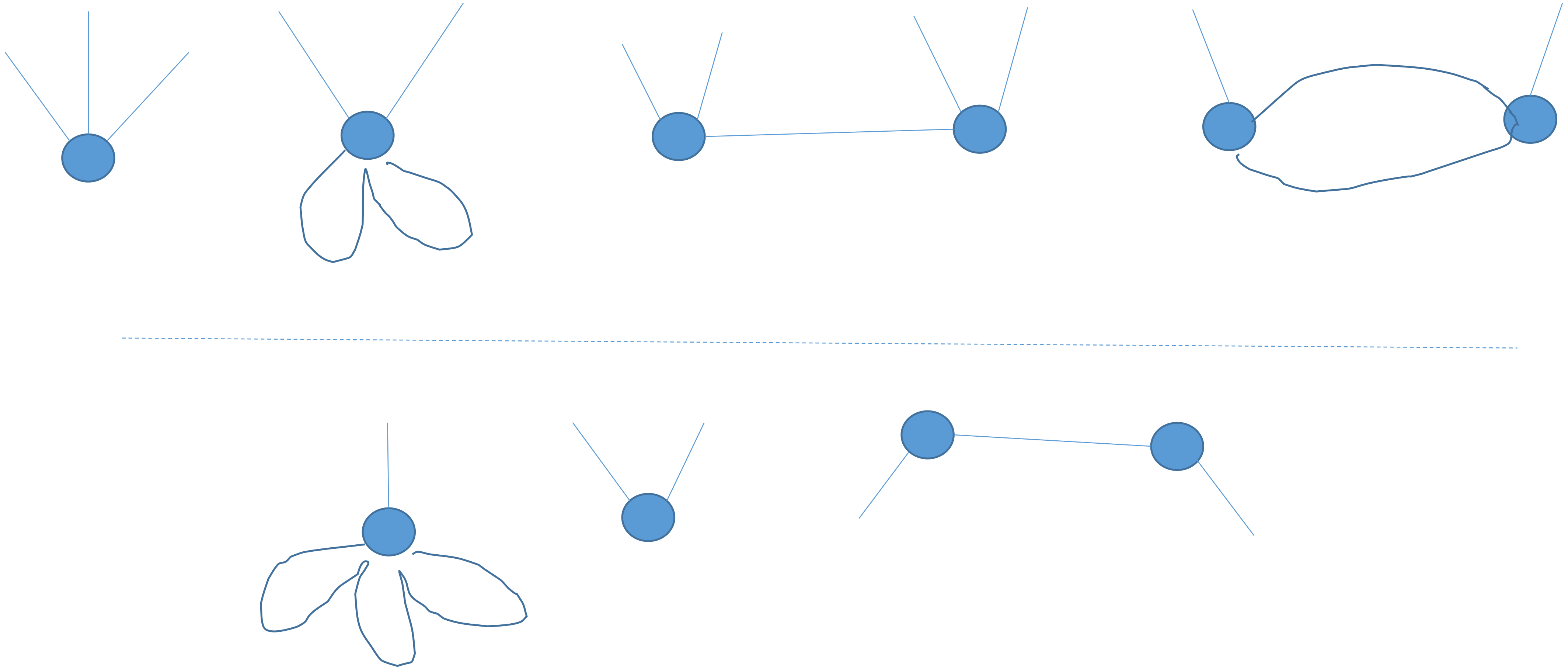


Graphs with semi-edges

- Algebraic graph theory (action of groups of automorphisms) (Nedela, Malnic, Marusic, Potoznik)
- Mathematical physics
- Common covers (Woodhouse 2018)

Computational complexity of covering graphs with semi-edges

Bok, Fiala, Hlineny, Jedlickova, Kratochvil MFCS 2021



Covers of disconnected graphs

"For a disconnected graph H, the H-COVER problem is polynomially solvable (NP-complete) if and only if the H_i-COVER problem is polynomially solvable (NP-complete) for every (for some) connected component H_i of H" (1994)

Covers of disconnected graphs

“For a disconnected graph H, the H-COVER problem is polynomially solvable (NP-complete) if and only if the H_i-COVER problem is polynomially solvable (NP-complete) for every (for some) connected component H_i of H” (WG 1994)

But what is a cover of a disconnected graph?

Covers of disconnected graphs

Complexity of Graph Covering Problems

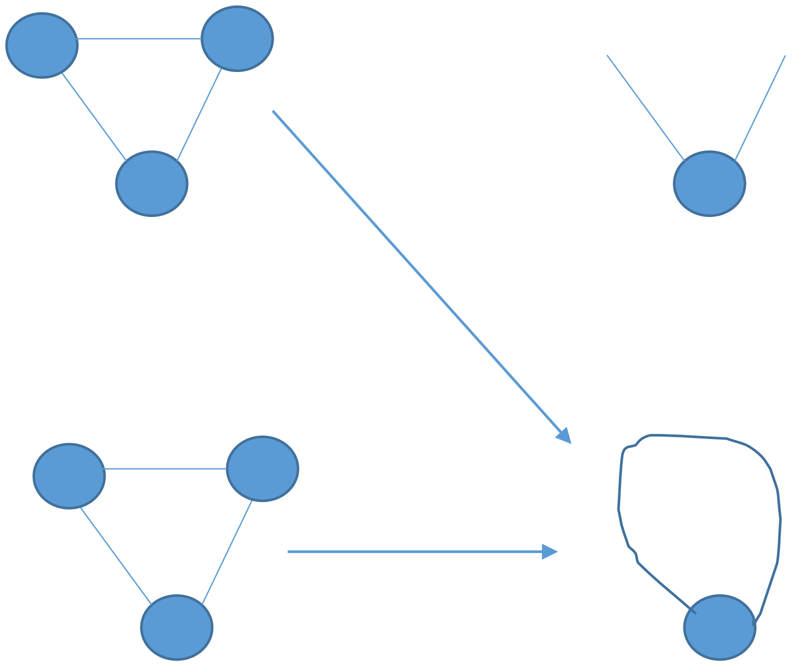
Jan Kratochvíl[1], Andrzej Proskurowski[2] and Jan Arne Telle[2]

[1] Charles University, Prague, Czech Republic
[2] University of Oregon, Eugene, Oregon

Abstract. Given a fixed graph H, the H-cover problem asks whether an input graph G allows a degree preserving mapping $f : V(G) \rightarrow V(H)$ such that for every $v \in V(G)$, $f(N_G(v)) = N_H(f(v))$. In this paper, we design efficient algorithms for certain graph covering problems according to two basic techniques. The first one is a reduction to the 2-SAT problem. The second technique exploits necessary and sufficient conditions for the existence of regular factors in graphs. For other infinite classes of graph covering problems we derive $\mathcal{NP}$-completeness results by reductions from graph coloring problems. We illustrate this methodology by classifying all graph covering problems defined by simple graphs with at most 6 vertices.

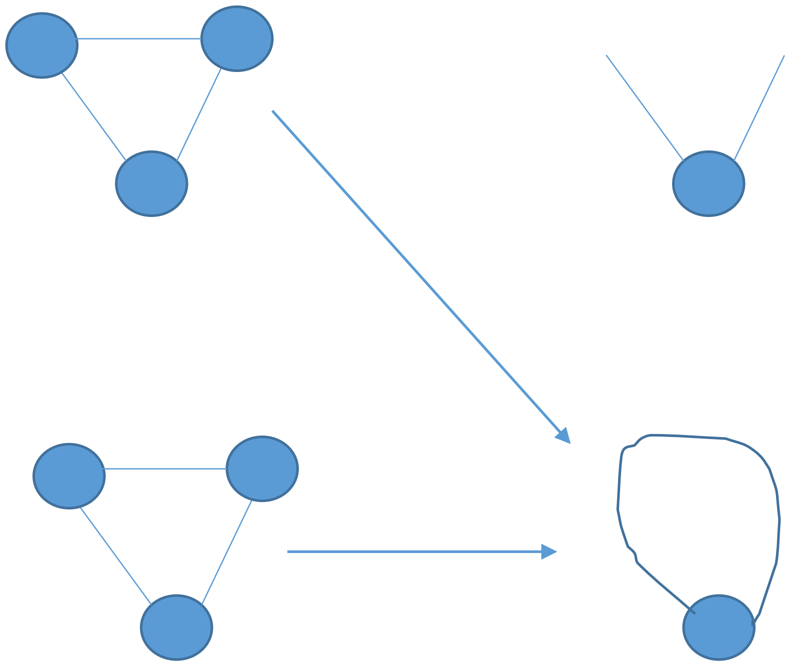
Covers of disconnected graphs

Locally bijective homomorphism

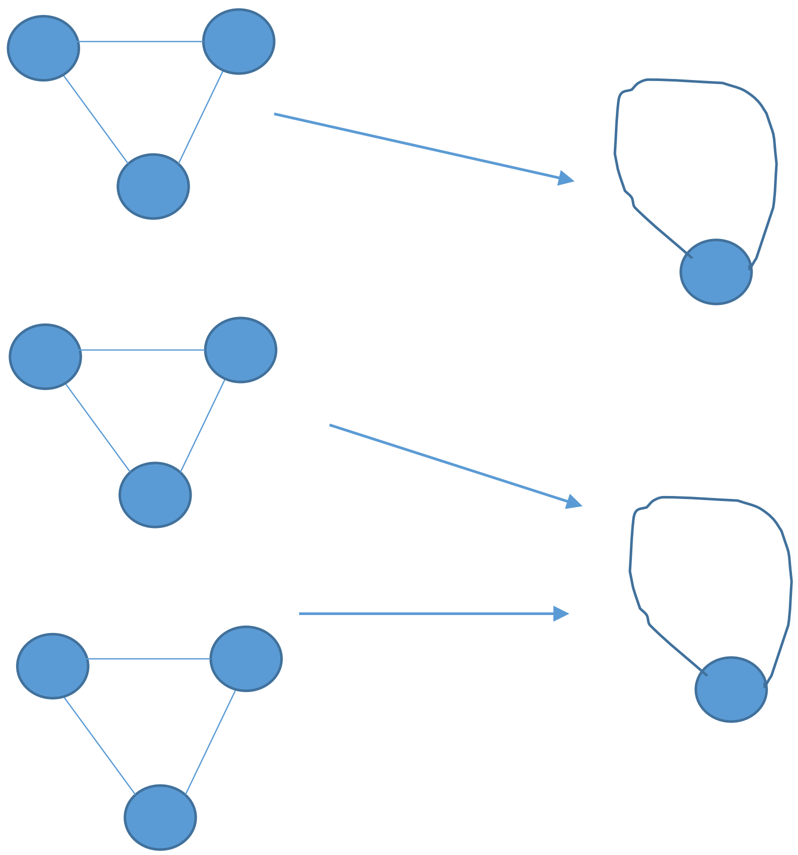


Covers of disconnected graphs

Locally bijective homomorphism

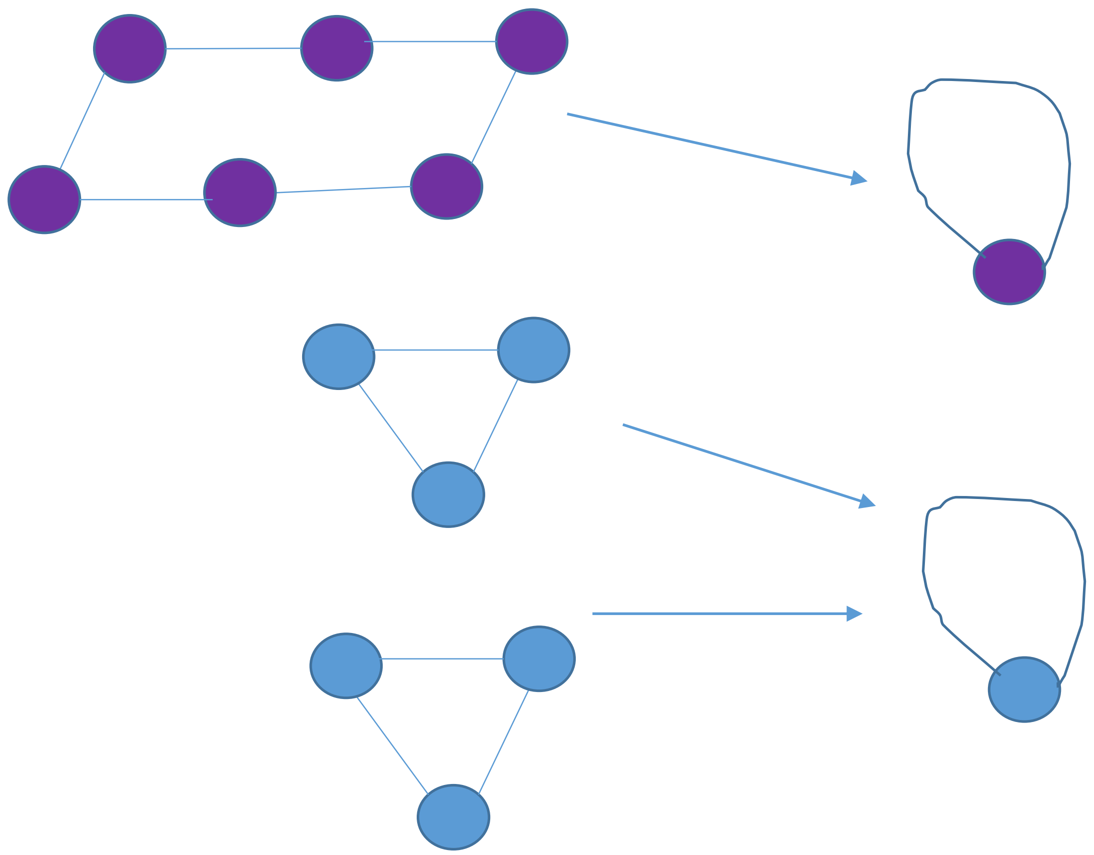


Surjective cover



Covers of disconnected graphs

Equitable cover



Computational complexity of covering disconnected graphs

Thm (Bok, Fiala, Jedlickova, Kratochvil, Seifrtova FCT2021):
For a disconnected graph H,

- both the H-SURJECTIVE-COVER and H-EQUITABLE-COVER problems are polynomially solvable if the H_i-COVER problem is polynomially solvable for every connected component H_i of H, and
- both the H-SURJECTIVE-COVER and H-EQUITABLE-COVER problems are NP-complete for simple input graphs if the H_i-COVER problem is NP-complete for simple input graphs for some connected component H_i of H.

Intermezzo

For all connected graphs H, if H-COVER is known to be NP-complete, it is NP-complete for simple graphs (i.e., no loops, no multiple edges, no semi-edges) on the input.

Open problem: Is this always true? Or does there exist a connected graph H (loops, multiple edges and semi-edges allowed) such that the H-COVER problem is NP-complete for general inputs, but polynomial time solvable for simple graphs on the input?

Computational complexity of covering disconnected graphs

Thm (Bok, Fiala, Jedlickova, Kratochvil, Seifrtova FCT2021):
For a disconnected graph H,

- both the H-SURJECTIVE-COVER and H-EQUITABLE-COVER problems are polynomially solvable if the H_i-COVER problem is polynomially solvable for every connected component H_i of H, and
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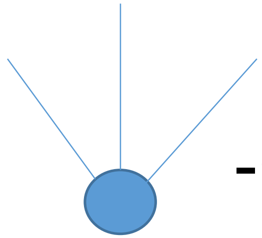
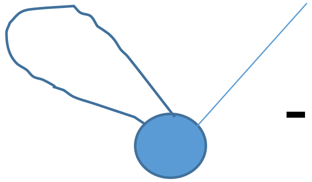
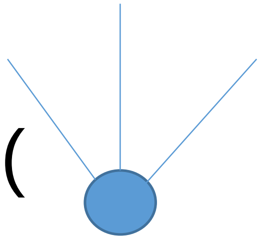
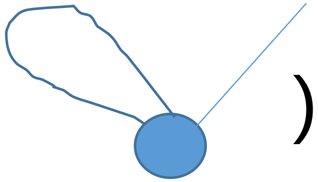
Not true for Locally bijective homomorphisms !!

Computational complexity of covering disconnected graphs

 -COVER is NP-complete

 -COVER is in P

( )-LBHOM is in P

Computational complexity of covering disconnected graphs

Proof of “the H-SURJECTIVE-COVER problem is NP-complete for simple input graphs if the H_i-COVER problem is NP-complete for simple input graphs for some connected component H_i of H.“

Computational complexity of covering disconnected graphs

Proof of “the H-SURJECTIVE-COVER problem is NP-complete for simple input graphs if the H_i-COVER problem is NP-complete for simple input graphs for some connected component H_i of H.“

Let H=H1+H2+…+Hk. Suppose that H1-COVER is NP-complete for simple input graphs, and let G1 be a simple graph whose covering of H1 is to be tested. For each j=2,3,…,k, fix a simple graph Gj such that Gj covers Hj, and moreover Gj does not cover H1, unless Hj is such that every simple graph that covers Hj also covers H1.

Then G=G1+G2+…+Gk surjectively covers H is and only if G1 covers H1.

⊦-relation on connected graphs

Definition: Given connected graphs A and B, we say that A ⊦B if for every simple graph G, it is true that G covers B whenever G covers A.

⊢-relation on connected graphs

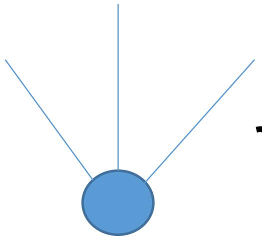
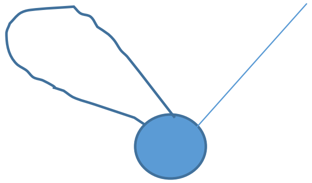
Definition: Given connected graphs A and B, we say that A ⊢B if for every simple graph G, it is true that G covers B whenever G covers A.

Example 1: If A $\rightarrow$ B, then A ⊢ B.

ⱶ-relation on connected graphs

Definition: Given connected graphs A and B, we say that A ⱶB if for every simple graph G, it is true that G covers B whenever G covers A.

Example 1: If A $\rightarrow$ B, then A ⱶ B.

Example 2:  ⱶ 

Ⱶ-relation on connected graphs

Open problem: Describe all pairs of connected graphs A and B such that AⱶB and A does not cover B.

Thank you