Covering lattice points by subspaces and counting point-hyperplane incidences

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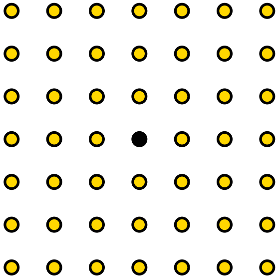

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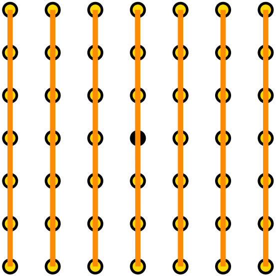

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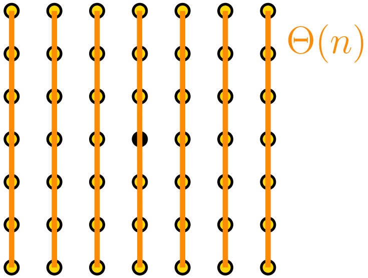

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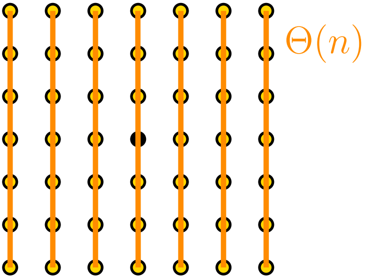

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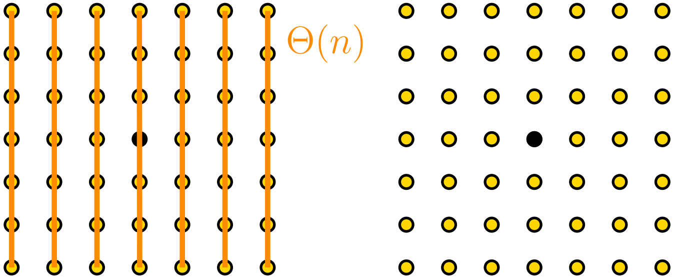

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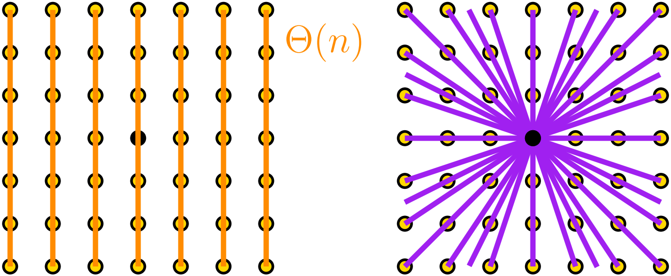

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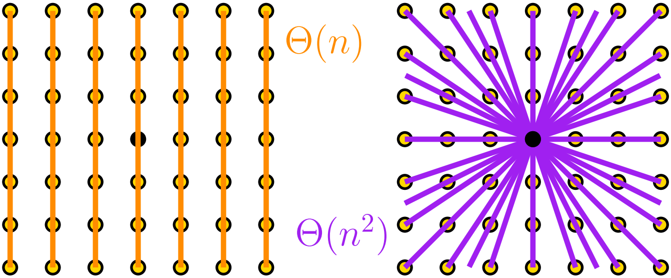

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- They showed that the answer is $\Theta(n^{d/(d-1)})$.
- Their proof works in the following more general setting.

Lattices and symmetric convex bodies

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- For linearly independent vectors $b_1, \ldots, b_d \in \mathbb{R}^d$, the d-dimensional lattice Λ with basis $\{b_1, \ldots, b_d\}$ is the set

$$\Lambda = \{a_1 b_1 + \cdots + a_d b_d \colon a_1, \ldots, a_d \in \mathbb{Z}\}.$$

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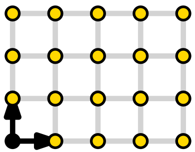

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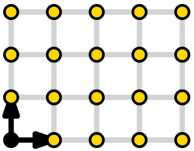

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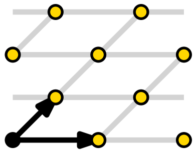

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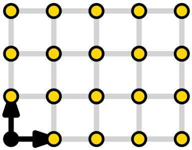

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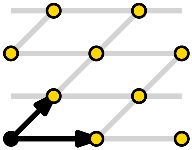

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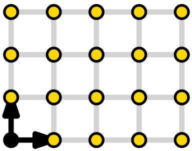

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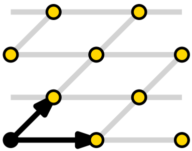

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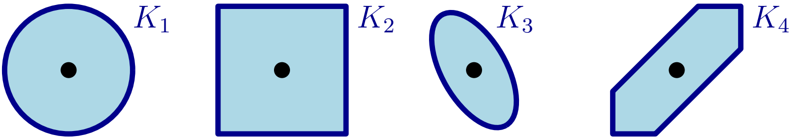

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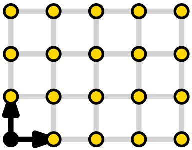

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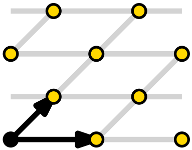

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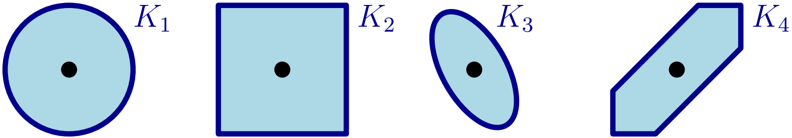

- Let $\mathcal{L}^d$ be the set of d-dimensional lattices and $\mathcal{K}^d$ be the set of d-dimensional compact convex bodies in $\mathbb{R}^d$ that are symmetric about 0.

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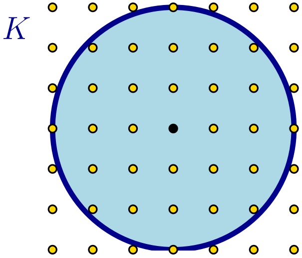

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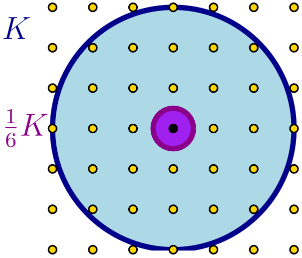

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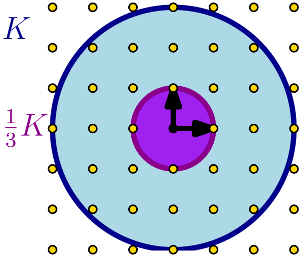

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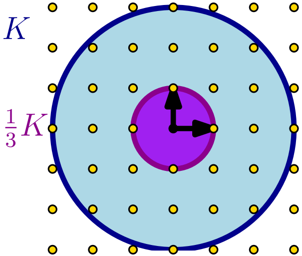

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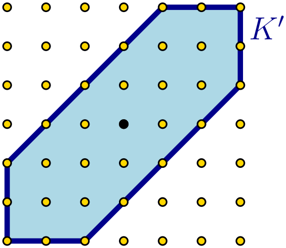

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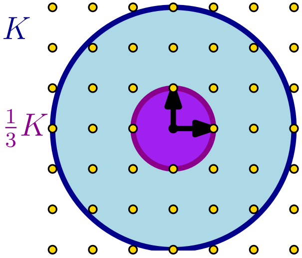

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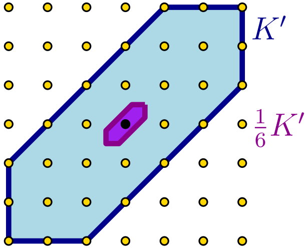

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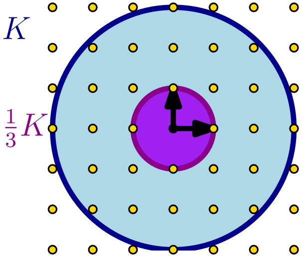

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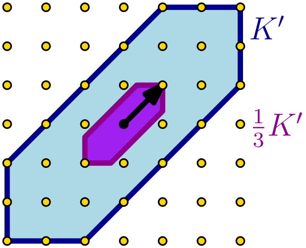

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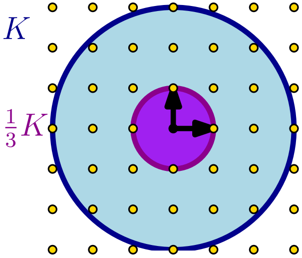

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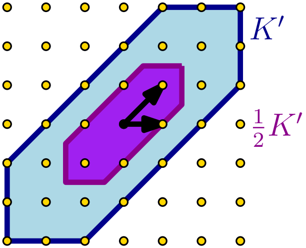

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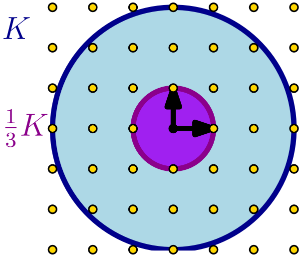

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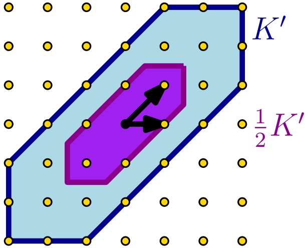

- The successive minima are achieved and $0 < \lambda_1 \leq \cdots \leq \lambda_d$.

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For $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, the set $\Lambda \cap K$ can be covered with at most

$$O\left(\min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}\right)$$

$(d-1)$-dimensional linear subspaces and this is tight if λ_d is not close to 1.

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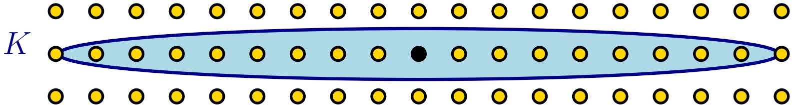

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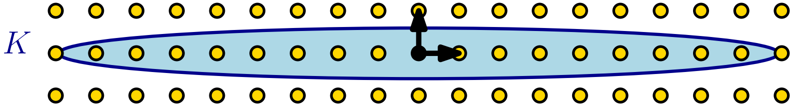

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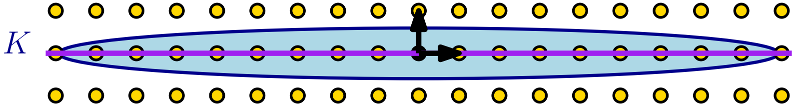

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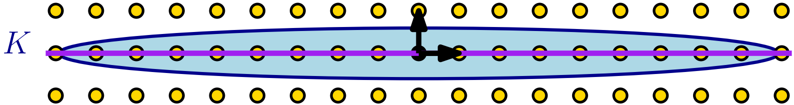

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- We consider Generalized problem 1 for general k.

Our results – covering by linear subspaces

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For k with $1 \leq k \leq d-1$, $\Lambda \in \mathcal{L}^d$, and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, we can cover $\Lambda \cap K$ with $O(\alpha^{d-k})$ k-dimensional linear subspaces, where

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For k with $1 \leq k \leq d-1$, $\Lambda \in \mathcal{L}^d$, $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, and $\varepsilon \in (0,1)$, we need at least $\Omega(((1-\lambda_d)\beta)^{d-k-\varepsilon})$ k-dimensional linear subspaces to cover $\Lambda \cap K$, where

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- The bounds are not tight. The lower bound can be improved?

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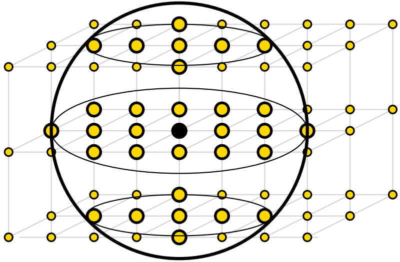

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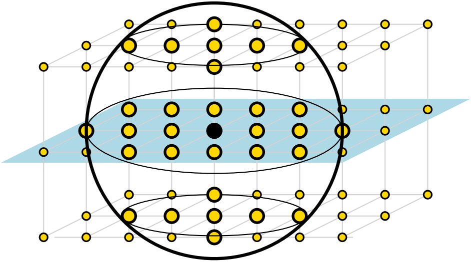

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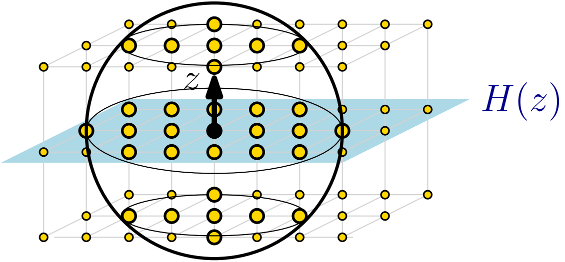

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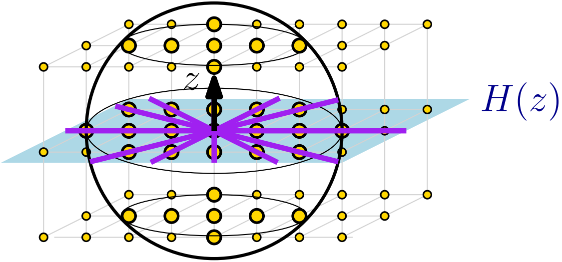

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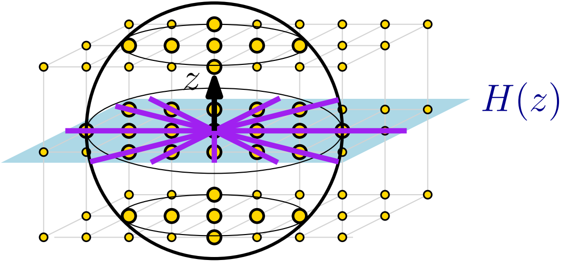

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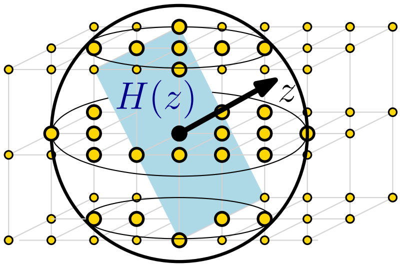

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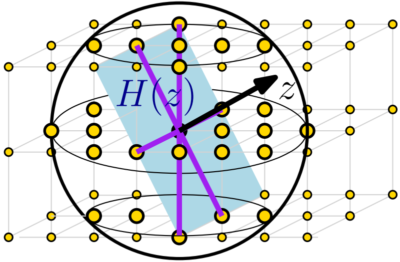

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Application: bounds for point-hyperplane incidences

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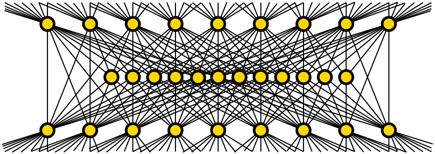

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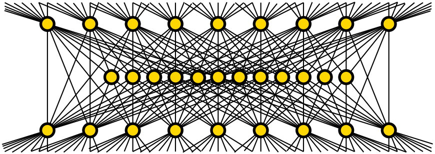

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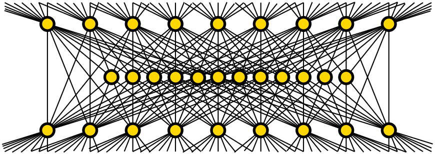

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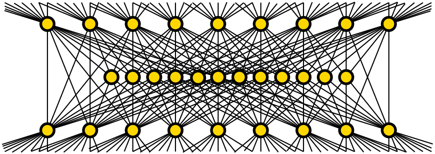

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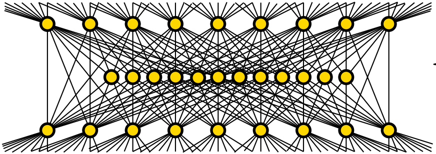

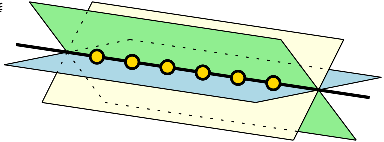

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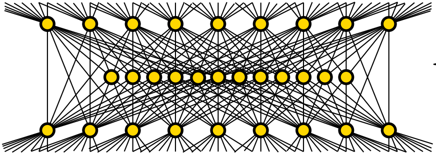
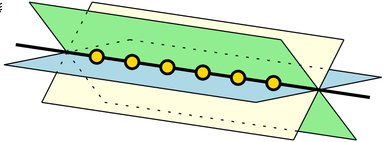

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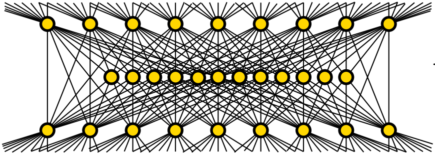
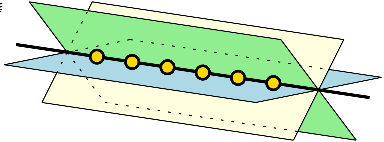
- To avoid this, we forbid $K_{r,r}$ for some fixed r in the incidence graph.

Application: bounds for point-hyperplane incidences

- An incidence between an n-point set $P \subseteq \mathbb{R}^d$ and a set of m hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ is a pair (p, H) such that $p \in P$, $H \in \mathcal{H}$, and $p \in H$.
- What is the maximum number of incidences between P and $\mathcal{H}$ in $\mathbb{R}^d$?





- In the plane, the Szemerédi–Trotter Theorem says that it is at most $O((mn)^{2/3} + m + n)$ for all P and $\mathcal{H}$. Moreover, this is tight.
- For $d \geq 3$ it is trivially at most mn and this is tight!
- To avoid this, we forbid $K_{r,r}$ for some fixed r in the incidence graph.
- Then the maximum number of incidences is at most $O\left((mn)^{1-1/(d+1)} + m + n\right)$ (Chazelle, 1993).

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Theorem (Brass and Knauer, 2003)

For $d \geq 3$, $\varepsilon > 0$ there is an r such that for all n and m there is a set P of n points in $\mathbb{R}^d$ and a set $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ with no $K_{r,r}$ in the incidence graph and with the number of incidences at least

$$\begin{cases} \Omega\left((mn)^{1-2/(d+3)-\varepsilon}\right) & \text{if } d \text{ is odd and } d > 3, \\ \Omega\left((mn)^{1-2(d+1)/(d+2)^2-\varepsilon}\right) & \text{if } d \text{ is even,} \\ \Omega\left((mn)^{7/10}\right) & \text{if } d = 3. \end{cases}$$

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- For $d \geq 4$, we improve these bounds to

$$\begin{aligned} &\Omega\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right) && \text{if } d \text{ is odd},\\ &\Omega\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right) && \text{if } d \text{ is even}. \end{aligned}$$

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Thank you.