A superlinear lower bound on the number of 5-holes

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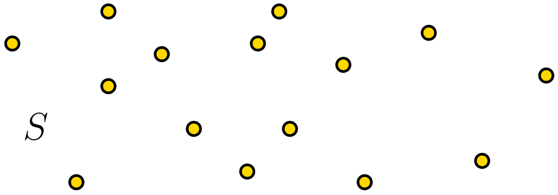

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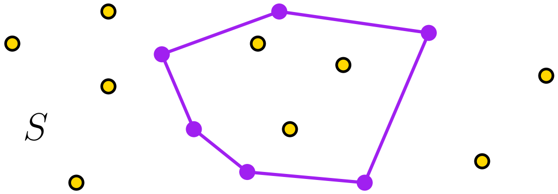

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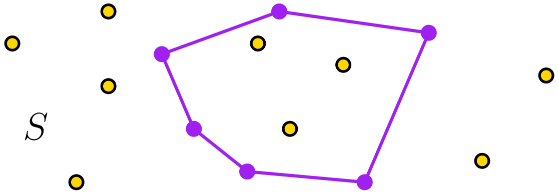

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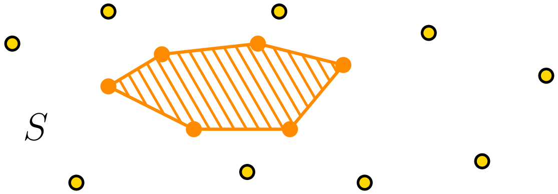

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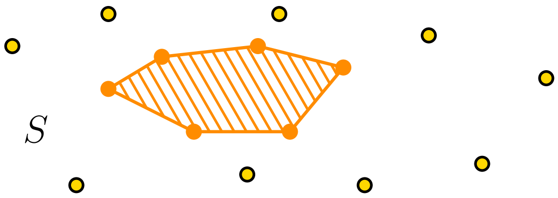

- A k-hole in a point set S is a convex polygon with k vertices from S and with no points of S in its interior.
- Every set of 3 points contains a 3-hole. Also, 5 points $\rightarrow$ 4-hole and 10 points $\rightarrow$ 5-hole (Harborth, 1978).

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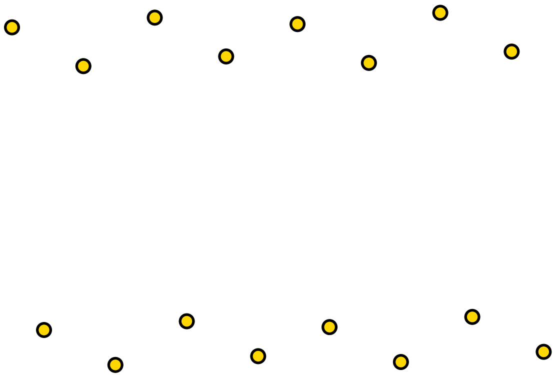

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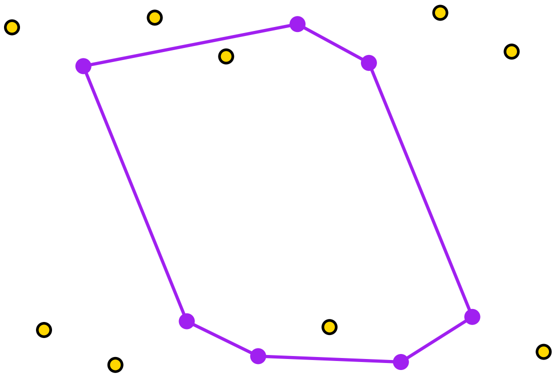

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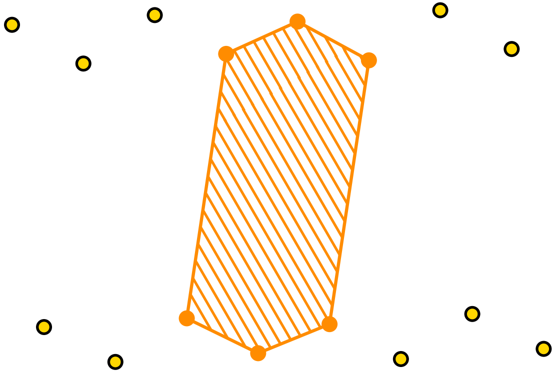

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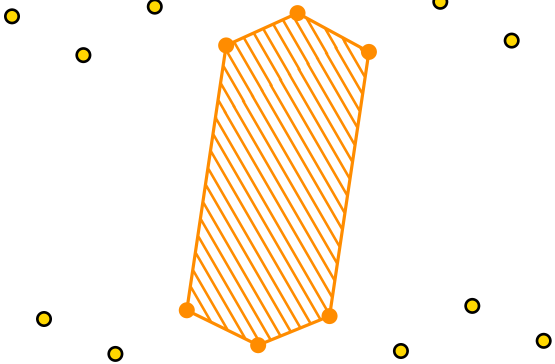

- Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).

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  - $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ (García, 2012)
  - $h_5(n) \geq \lceil 3/7(n-11) \rceil$ (Aichholzer, Hackl, Vogtenhuber, 2012)
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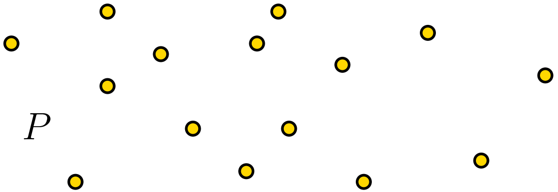

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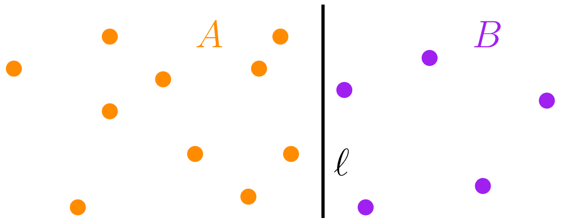

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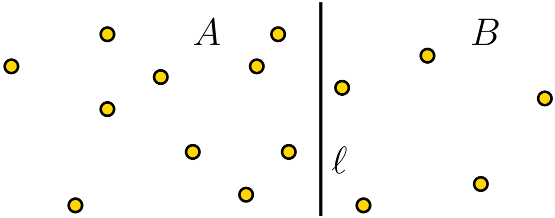

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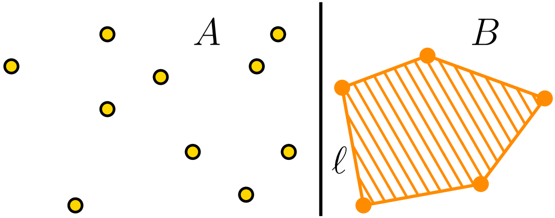

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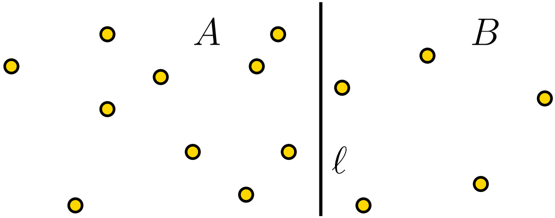

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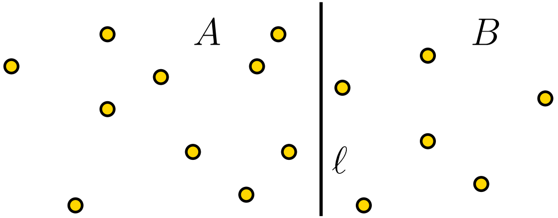

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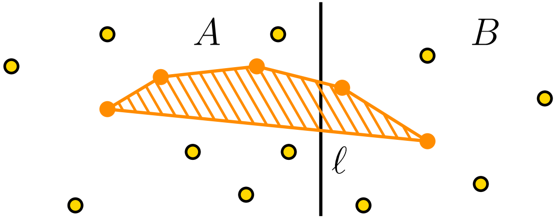

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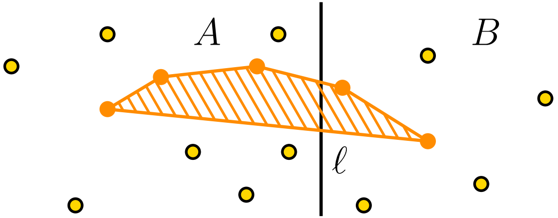

- The proof is computer assisted and quite complicated.

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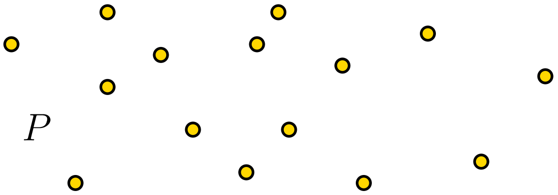

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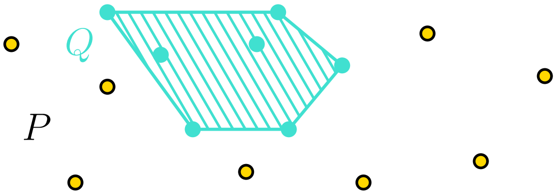

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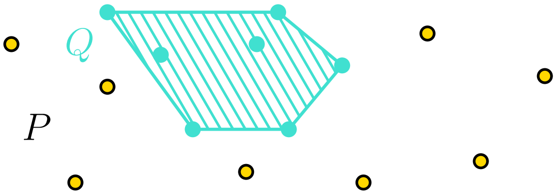

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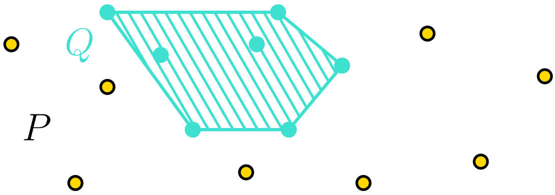

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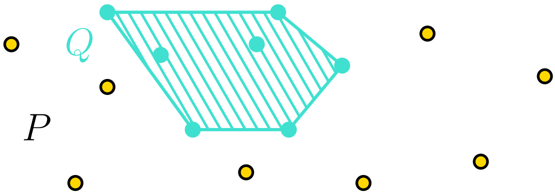

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- Base case: For $t = 5^5$, we have $n = 2^t > 10$ and $h_5(10) = 1$ gives at least $c \cdot n \log_2^{4/5} n$ 5-holes in P for c small enough.

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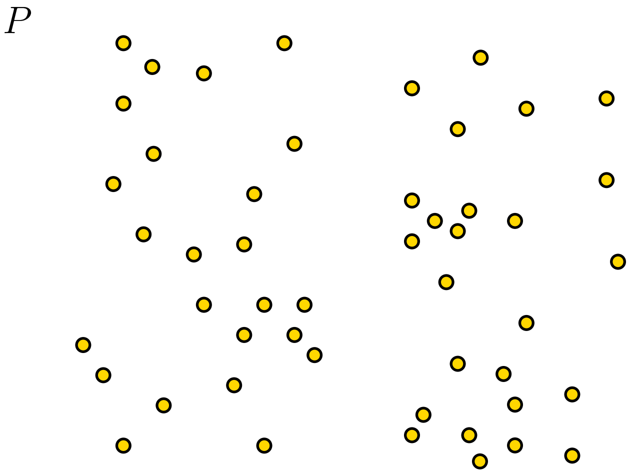

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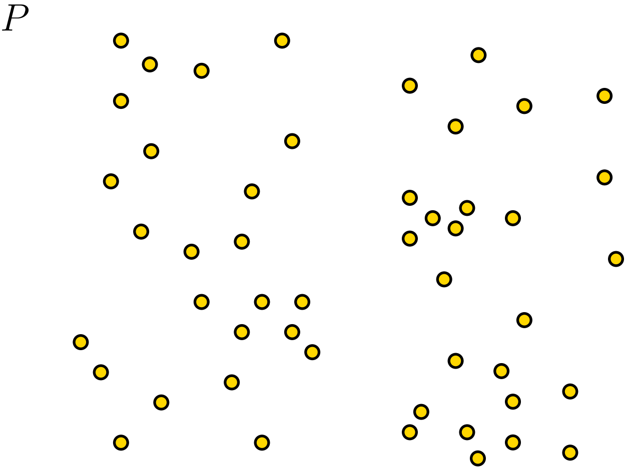

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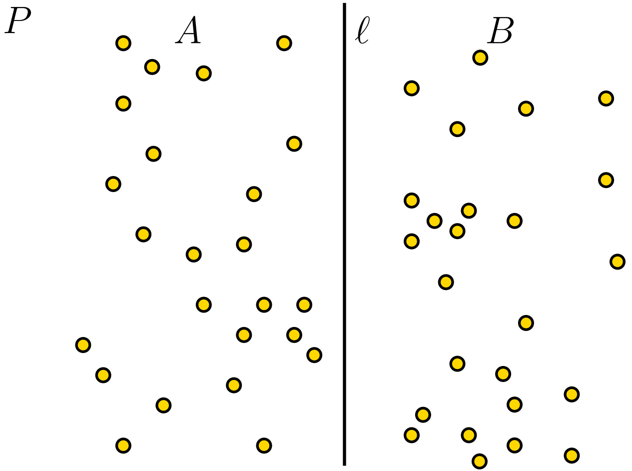

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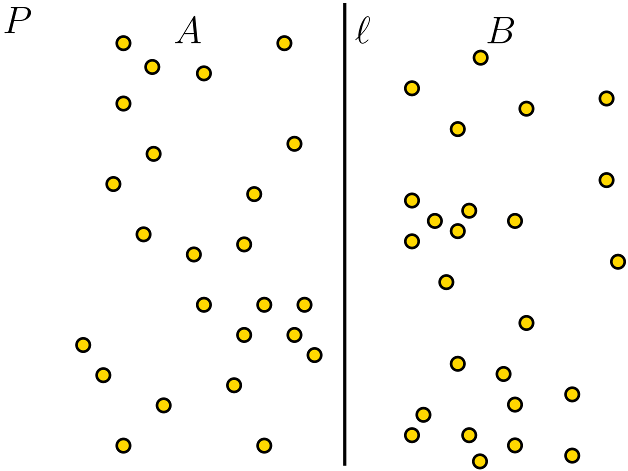

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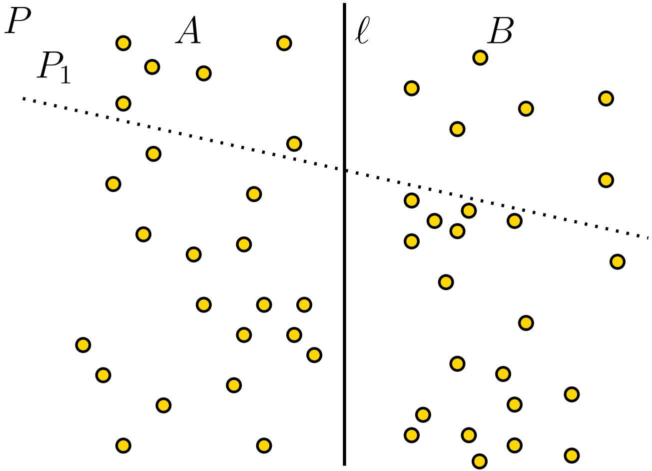

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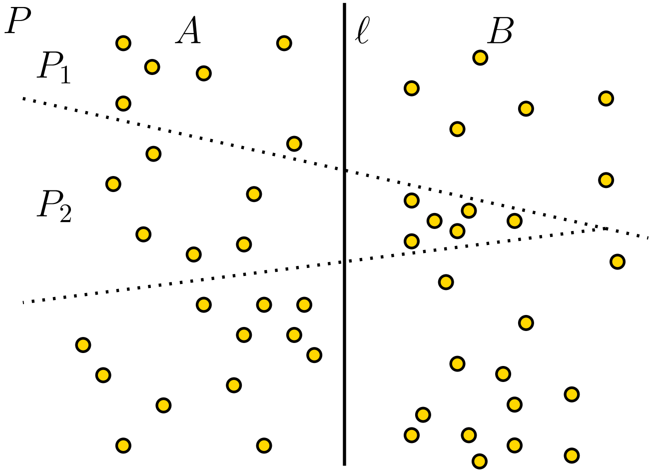

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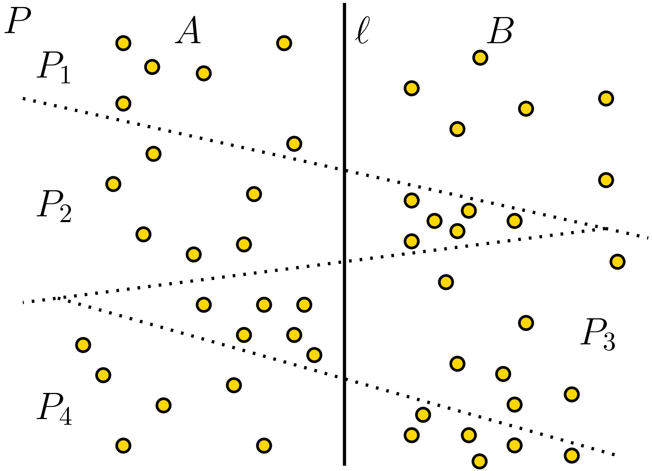

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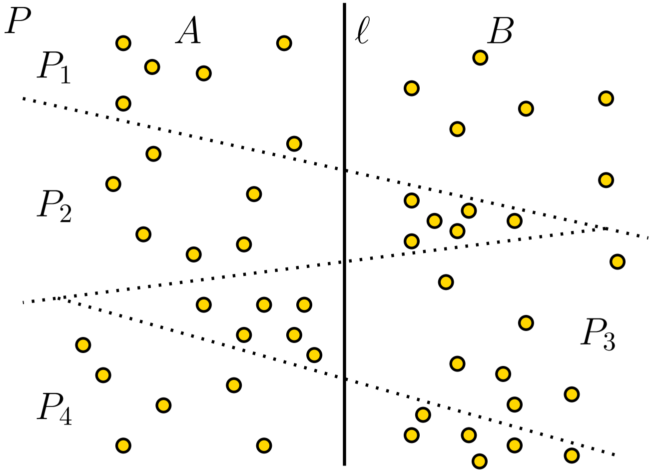

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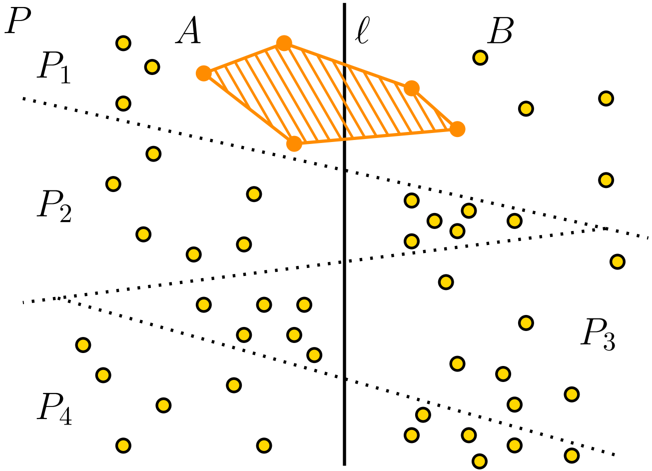

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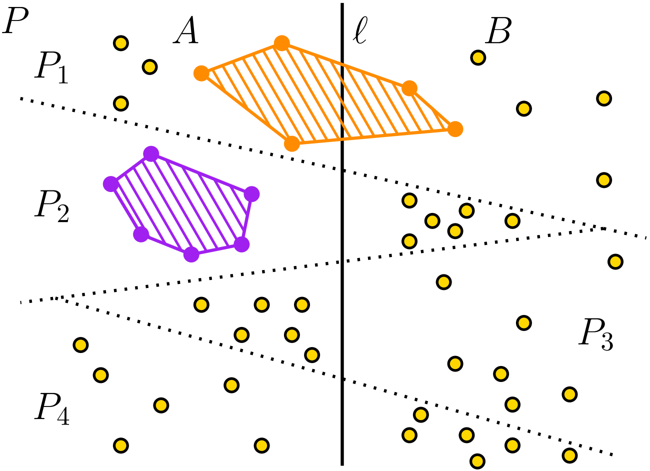

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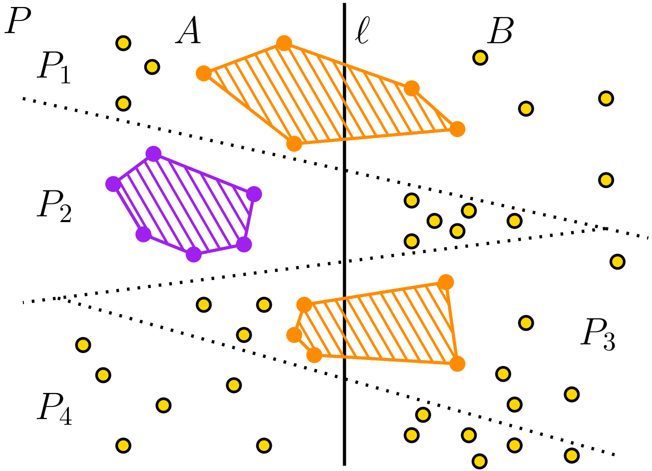

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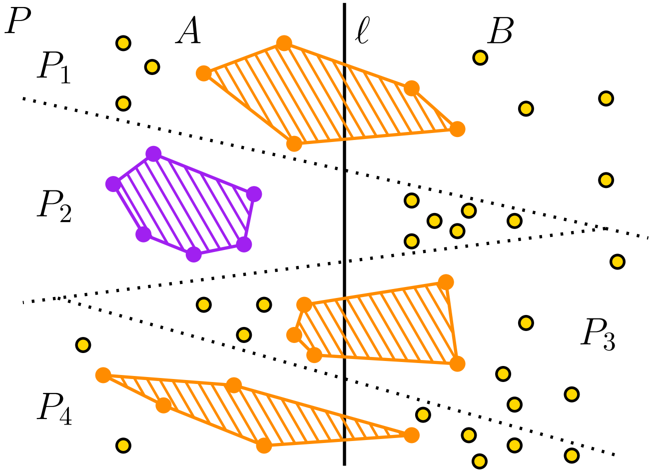

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  - We proceed inductively on A and B and obtain at least

$$h_5(n/2) + h_5(n/2) + n/(4r) \geq c \cdot n \log_2^{4/5} n$$

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Sketch of the proof of Theorem 2 – peeling off extremal points

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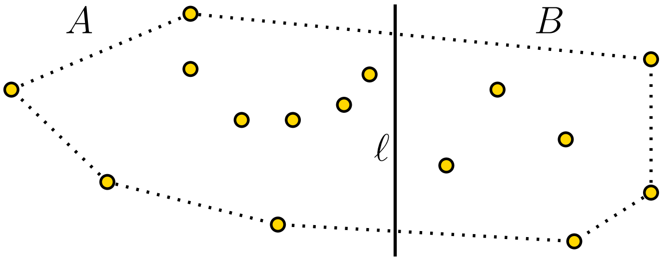

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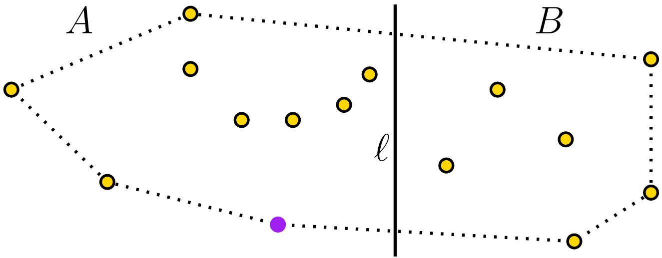

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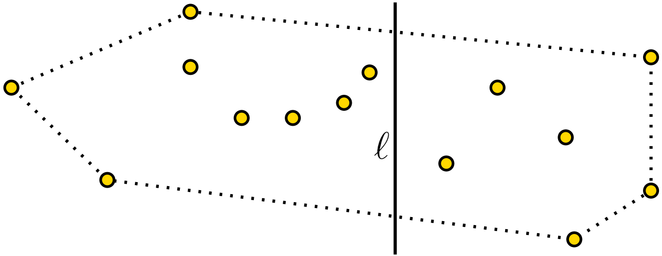

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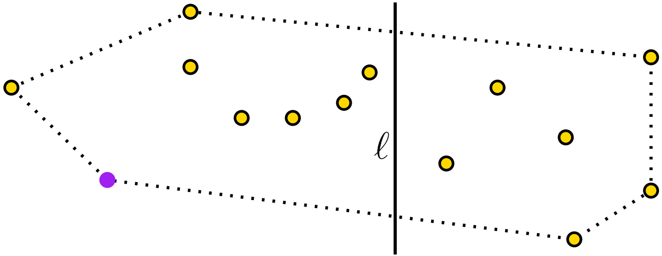

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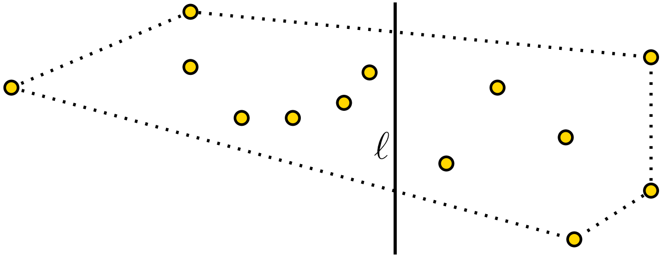

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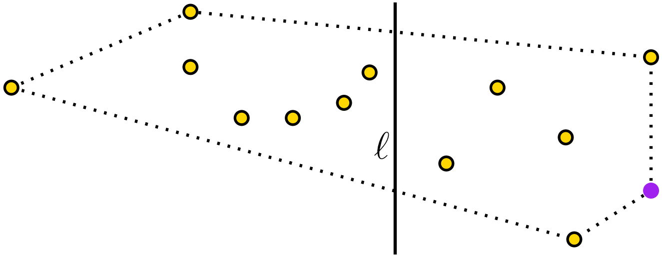

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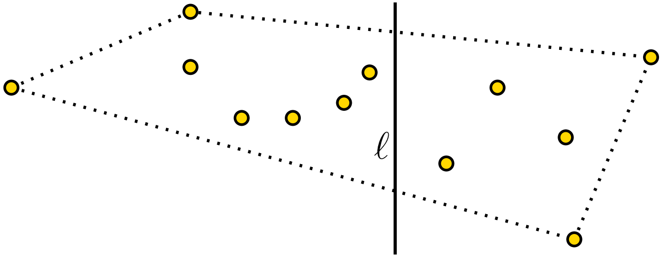

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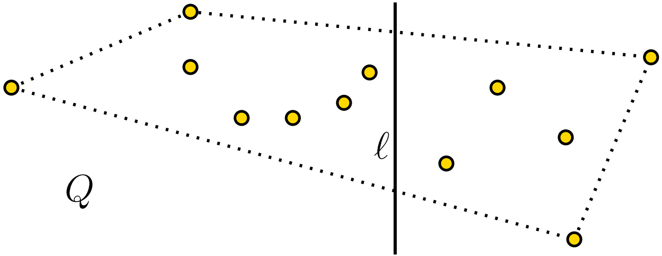

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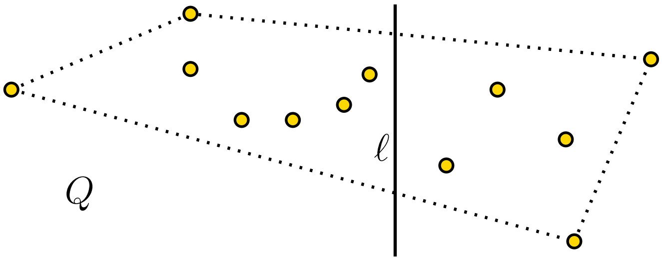

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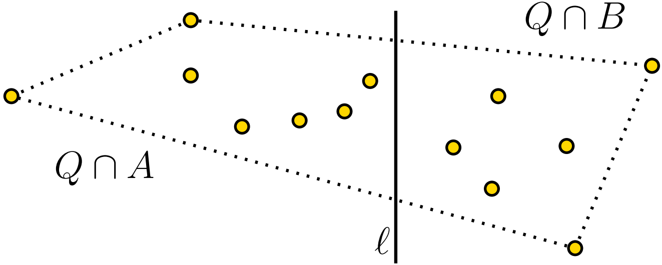

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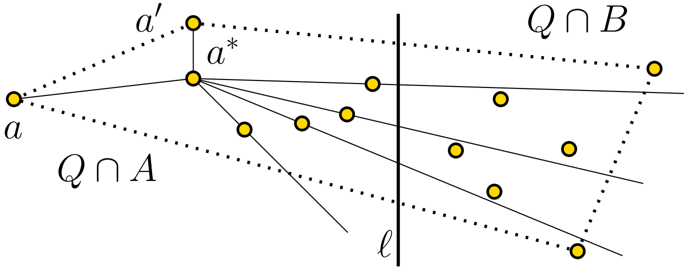

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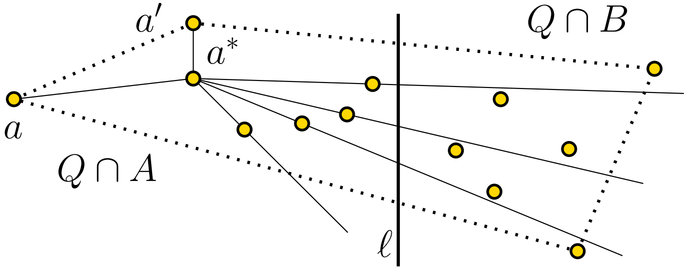

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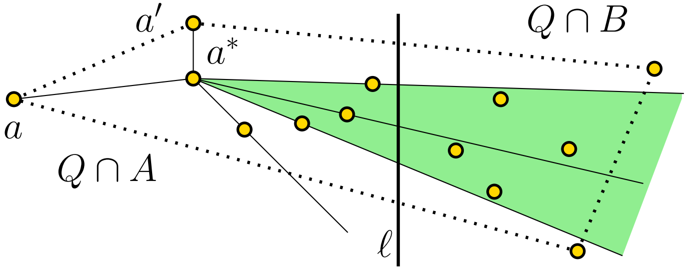

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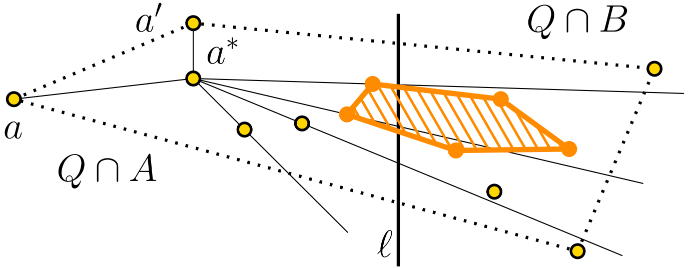
- Rays from a^* to $(Q \cap A) \setminus \{a^*\}$ partition the plane into a^*-wedges.
- Since Q is ℓ-critical, it has a special structure:
  - There are at most two extremal points of Q in $Q \cap A$.
  - If there are two, then a^* is the unique interior point in $Q \cap A$.
  - By symmetry, analogous statements hold for $Q \cap B$.



- No ℓ-divided 5-hole in Q forces several restrictions on numbers of points from $Q \cap B$ in a^*-wedges.

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- Propositions 1 and 2 thus give $|Q \cap B| < |Q \cap A| \leq |Q \cap B|$, a contradiction.

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